# SMARANDACHE-GROUP SEMIRING

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**الملخص**: كحالة استثنائية يقدم هذا البحث فكرة جديدة في علم الرياضيات والتي تم تطويرها في السنوات الحالية ومعرفة جيداً ببناءات سمارأنداش، نسبة إلى أول من قدمها العالم فلورنتين سمارنداش وباديلا رال سنة 1998 هذه البناءات قدمت حديثاً في عام 2002 من خلال العالم فاسنثا كأنداسامي.

**abstract**

As a special case, we present the new notion of mathematics that has been developed in recent year and it is well known as Smarandache structures which first introduced by Florentine Smarandache and Padilla Raul in year 1998 . Which was recently introduced in 2002 by W.B. Vasantha Kandasamy .

**Keywords**: Smarandache semiring, commutative semiring, Smarandache idempotent, Smarandache zero-divisor .

## INTRODUCTION

First, we will introduce some basic definitions to explain the structure of Smarandache –group semiring. Based on some Theories regarding this structure that important results have been reached of Smarandache-zero divisor and Smarandache-idempotent .

## PRELIMINARIES

**Definition 1.**[**1**] A Smarandache semiring ( S-semiring ) is defined to be a semiring $S$ such that a non-empty proper subset $B$ of $S$ is a semifield (with respect to the same induced operation) .

**Definition 2.**[**2**] Let $S$ be a commutative semiring with unit. Let $G$ be a group with multiplication. The group semiring $SG$ of the group $G$ over the semiring $S$ consists of all finite formal sums of the form $\sum_{i=1}^{n} s_i g_i$ ($i$ –runs over a finite number $n$) with $s_i t_i \in S$ and $g_i \in G$ satisfying the following properties :

1. $\sum_i s_i g_i = \sum_i t_i g_i \Leftrightarrow s_i = t_i$

2. $(\sum_i s_i g_i) + (\sum_i t_i g_i) = \sum_i (s_i + t_i) g_i$

3. $(\sum_i s_i g_i) \cdot (\sum_i t_i g_i) = \sum m_k g_k$ where $m_k = \sum_i s_i t_i \, , g_k = g_i h_i \, , h_i \in G$

4. $s_i g_i = g_i s_i$

$\sum_i s_i g_i \in SG$ . and For $s s_i \in S$ 5. $s(\sum_i s_i g_i) = \sum_i (s s_i)\, g_i$

6. As $1 \in G$ and $1 \in S$ we have $1.G = G \subseteq SG$ and $S.1 = S \subseteq SG$ .

**Definition 3.**[**3**] Let $S$ be a semiring. We say $a, b \in S \backslash \{0\}$ are a Smarandache zero-divisor (S-zero-divisor) if $a.b = 0$ and there exists $x, y \in S \backslash \{a, b, 0\}, x \neq y$ with (i) $ax = 0$ or $xa = 0$ (ii) $by = 0$ or $yb = 0$ and (iii) $xy \neq 0$ or $yx \neq 0$ .

**Definition 4.**[**4**] An element $a \in S\backslash\{0\}$ is a Smarandache weak zero-divisor (S-weak zero-divisor) if $a$ is a zero-divisor, i.e. $a.b = 0$ for $b \in S\backslash\{0, a\}$

and there exist $x, y \in S\backslash\{0, a, b\}$ such that

(i) $ax = 0$ or $xa = 0$

(ii) $by = 0$ or $yb = 0$

(iii) $xy = 0$ or $yx = 0$.

**Definition 5.**[**5**] Let S be a semiring. An element $a \in S\backslash\{0\}$ is a Smarandache idempotent (S-idempotent) of $S$ if :

(1) $a^2 = a$ .

(2) There exists $b \in S\backslash\{0, a\}$ such that

(i) $b^2 = a$ and

(ii) $ab = a$ $(ba = a)$ $or$ $ab = b$ $(ba = b)$.

Note that if S is a lattice then S has no S- idempotent

**Corollary 1**: Let A be a set such that$|A| > 2$, and G be the cyclic group of order n. Then the group semiring $P(A)G$ has a S-zero-divisor .

**Proof :**

Let , $y \in P(A)G$ .

Take $x = \sum_{i=1}^{n} ag^i$ and $y = \sum_{i=1}^{n} bg^i$ where $a, b \in P(A)$ such that $a.b = \emptyset$

And take $w = \sum_{i=1}^{n} a^c g^i$ and $z = \sum_{i=1}^{n} b^c g^i$ where ($a^c \neq b$and $b^c \neq a$)

Now we have $x.y = \emptyset$ and $x.w = \emptyset, y.z = \emptyset$

Finally $w.z \neq \emptyset$.

Hence $x$ and $y$ are a S-zero-divisor in $(A)G$ .

**Corollary 2**: If $|A| = 2$ and G be the cyclic group of order n., then the group lattice $P(A)G$ has a S-weak zero-divisor.

**Proof :**

Let $x, y \in LG$

Take $x = \sum_{i=1}^{n} ag^i$ and $y = \sum_{i=1}^{n} a^c g^i$ where $a \in P(A)$

And take $z = a^c$ and $w = a$

Now we have $x.y = 0$ are zero-divisor

, $y.w = 0$ And $x.z = 0$

Finally $w.z = 0$

Then $x$ and $y$ are S-weak zero-divisor in $G$ .

**Theorem 1**: Let $S_n(n > 2)$ be the symmetric group of order $n$ and $c_2$ be the chain lattice of order2, then the group semiring $c_2S_n$ has a S-idempotent.

**Proof:**

Let $a = \sum_{i=1}^{n} p_i$ where $p_i = \begin{pmatrix} 1 & 2 & \ldots & k & \ldots & n \\ i & i+1 & \ldots & k+1+i & \ldots & i+n-1 \end{pmatrix}$.

Not that if $(k+1)+i > n$ then $(k+1)+i \coloneqq (k+1)+i-n$.

We have

$$a^2 = (\sum_{i=1}^{n} p_i)^2 = (e + p_2 + \ldots + p_n)^2 = e + p_1 + p_2 + \ldots + p_n + p_2 p_2 +$$

$$p_2 p_3 + \ldots + p_2 p_n + \ldots + p_n p_2 + p_n p_3 + \ldots + p_n p_n .$$

$$= e + p_2 + p_3 + \ldots p_{n-1} + p_n$$

Where

$$p_2 p_2 = p_3 \ , p_2 p_3 = p_4 \ , \ldots , \ p_2 p_n = e \ , \ p_n p_2 = e \ , p_n p_3 = p_2 \ , \ldots , p_n p_n = p_{n-1} .$$

Choose $b = \sum_{i=1}^{n} q_i$ where $q_i = \begin{pmatrix} 1 & 2 & \ldots & n \\ i+n-1 & i+n-2 & \ldots & i \end{pmatrix}$

Not that if $i+n-k > n$ then $i+n-k \coloneqq (i+n-k)-n$ where $k = 1,2, \ldots$

$$b^2 = \left(\sum_{i=1}^{n} q_i\right)^2 = (q_1 + q_2 + \cdots + q_n)^2$$

$$= q_1 q_1 + q_1 q_2 + \cdots + q_1 q_n + \cdots + q_n q_1 + q_n q_2 + \cdots + q_n q_n$$

where

$$q_1 q_1 = e , \ldots , q_1 q_n = p_n \ , q_2 q_1 = p_n \ , \ldots , q_2 q_n = p_{n-1} \ , \ldots , q_n q_1 = p_2 \ , q_n q_n = e$$

Then $b^2 = e + p_2 + p_3 + \cdots + p_n = a$

Finally $a.b = (\sum_{i=1}^{n} p_i)(\sum_{i=1}^{n} q_i) = \sum_{i=1}^{n} q_i = b$

Hence $a$ is a S-idempotent of $C_2S_n$.

**Theorem 2** : Let $D_{2n}$be the dihedral group of order 2n, i.e. $D_{2n} = \{a, b: a^n = e = b^2, ba = a^{n-1}b\}$ and $C_2$ be the chain lattice of order 2. Then the group semiring $C_2D_{2n}$ has a S-idempotent.

**Proof** : (i) If $(n - even)$

Let $x \in C_2D_{2n}$

Take $x = e + a^{n/2+1}b$ we have $x^2 = x$ idempotent.

Choose $y = a^{n/2} + ab \in C_2D_{2n}$ such that $y^2 = x$

Now we have $x.y = y$.

Hence $x$ is a S-idempotent in$C_2D_{2n}$.

(ii) If $(n - 0dd)$

Let $x = \sum_{i=0}^{n-1} a^i \in C_2D_{2n}$ such that $x^2 = x$ idempotent.

(1) Let $n = 3$ then $x = e + a + a^2 \Longrightarrow x^2 = x$

(2) Suppose that for $n = k$ then $\left(\sum_{i=0}^{k-1} a^i\right)^2 = \sum_{i=0}^{k-1} a^i$

(3) For $n = k + 2$ we have

$$(\sum_{i=0}^{k+1} a^i)^2 = (\sum_{i=0}^{k-1} a^i + a^k + a^{k+1})^2$$

$$= \sum_{i=0}^{k-1} a^i (\sum_{i=0}^{k-1} a^i + a^k + a^{k+1}) + a^k (\sum_{i=0}^{k-1} a^i + a^k + a^{k+1}) + a^{k+1} (\sum_{i=0}^{k-1} a^i + a^k + a^{k+1})$$

$$= \sum_{i=0}^{k-1} a^i + a^k \sum_{i=0}^{k-1} a^i + a^{k+1} \sum_{i=0}^{k-1} a^i + a^{2k} + a^{2k+2}$$

$$= \sum_{i=0}^{k-1} a^i + (a^k + a^{k+1} + a^{k+2} + \ldots + a^{2k-1}) + (a^{k+1} + a^{k+2} + \ldots + a^{2k}) + a^{2k} + a^{2k+2}$$

$$= \sum_{i=0}^{k-1} a^i + a^k + a^{k+1} = \sum_{i=0}^{k+1} a^i$$

Choose $y = \sum_{i=0}^{n-1/2} a^i \in C_2D_{2n}$ .

Clearly $y^2 = x$ and $x.y = x$. Hence $x$ is a S-idempotent in $C_2D_{2n}$.

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