Matrix Representation of Non-Nested Real Algebraic Number

التعبير المصفوفي عن العدد الجبري الحقيقي الغير متداخل

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الملخص: نحن عبرنا عن الأعداد الجبرية في $Q(\sqrt[n]{s})$, $Q(\sqrt[n]{s}, \sqrt[m]{t})$ و $Q(\sqrt[n]{s}, \sqrt[m]{t}, \sqrt[r]{u})$

بواسطة كتل مصفوفية . وبالتالي يمكن حساب هذا العدد الجبري بعكس المصفوفة المناظرة .

الكلمات المفتاحية : العدد الجبري، المجال العددي، الجبر التنسيقي

Abstract: We represent an algebraic number in $(\sqrt[n]{s})$, $Q(\sqrt[n]{s}, \sqrt[m]{t})$ and $Q(\sqrt[n]{s}, \sqrt[m]{t}, \sqrt[r]{u})$ by a block matrix . The inverse of such an algebraic number can thus be computed by the inverse of the corresponding matrix .

Keywords : algebraic number, number field, associative algebra .

INTRODUCTION

The number fields of the $Q(\sqrt[n]{s})$, $Q(\sqrt[n]{s}, \sqrt[m]{t})$ and $Q(\sqrt[n]{s}, \sqrt[m]{t}, \sqrt[r]{u})$ associative algebras over Q , where s , t, u are respectively n^{th} , m^{th} , r^{th} power free relatively prime positive integers . Monomorphisms from these number fields into the matrix algebras $M_i(Q)$ where $i = n, nm, nmr$ respectively are constructed. This enables us to represent the algebraic number in $Q(\sqrt[n]{s})$, $Q(\sqrt[n]{s}, \sqrt[m]{t})$ and $Q(\sqrt[n]{s}, \sqrt[m]{t}, \sqrt[r]{u})$ by matrices of related blocks .

PRELIMINARIES

A complex number α is called an algebraic number[3] if it satisfies some monic polynomial equation $f(x) = b_0x^n + b_1x^{n-1} + \cdots + b_n$ where $f(x) \in Q[x]$. If α satisfies some polynomial equation of degree n , but none of lower degree we say that α is an algebraic number of degree n . Non-algebraic number are called transcendental numbers .

Examples:

1) Every rational number is an algebraic .
2) $\sqrt{1 + 2\sqrt{3}}$ is an algebraic number of degree 4 .
3) π is transcendental number [1] .

The minimal polynomial f of an algebraic number α is the monic polynomial in $Q[x]$ of smallest degree such that $(\alpha) = 0$.
Note that the minimal polynomial is irreducible over .

Theorem 1[3]: The set of all algebraic number is a field .
The number field is any subfield of this field .

Let F be a field . We say A is an associative algebra over F [2] if A is a ring with identity which is an F -vector space , such that the F -action is compatible with multiplication in A in the sense that $(x.a).b = x.(ab) = a(x.b)$ for all $a, b \in A$, $x \in F$.

The ring $M_n(F)$ of all nxn-matrices over the field F is an F-associative algebra of dimension n^2 .

The number field induced by α is given by :

$Q(\alpha) = \{a_1 + a_2\alpha^2 + \cdots + a_n\alpha^{n-1}: a_i \epsilon Q\}\ modulo\ f(\alpha).$

$Q(\alpha)$ is n-dimensional vector space over Q with basis $\{1, \alpha, \dots, \alpha^{n-1}\}$.

$Q(\alpha)$ is also associative algebra over Q .

If α and β are two algebraic number of degree m and n respectively , we defined $Q(\alpha, \beta) = Q(\alpha)Q(\beta)$.

$Q(\alpha, \beta)$ is an associative algebra over Q of dimension mn .

The basis of $Q(\alpha, \beta)$ is :

$\{1, \alpha, \dots, \alpha^{n-1}, \beta, \alpha\beta, \dots, \alpha^{n-1}\beta, \dots, \beta^{m-1}, \alpha\beta^{m-1}, \dots, \alpha^{n-1}\beta^{m-1}\}$.

Examples :

Let s and t be n^{th} power free and m^{th} power free relatively prime positive integer . Let $= \sqrt[n]{s}\ \ and\ \ \beta = \sqrt[m]{t}$. We form the following two number fields :

1) $Q(\alpha) = \{a_1 + a_2\alpha + a_3\alpha^2 + \cdots + a_n\alpha^{n-1}: a_1, \dots, a_n \epsilon Q\}$

It is basis is $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$

2) $(\alpha, \beta) = \{a_1 + a_2\alpha + \cdots + a_n\alpha^{n-1} + \cdots + a_{n+m}\alpha^{n-1}\beta^{m-1}: a_i \epsilon Q\}$. It is basis is $\{1, \alpha, \dots, \alpha^{n-1}, \beta, \alpha\beta, \dots, \alpha^{n-1}\beta^{m-1}\}$.

The general elements are :

$(a_1, \dots, a_n; s) = a_1 + a_2\alpha + \cdots + a_n\alpha^{n-1}\ in\ Q(\alpha)\ (a_{11}, \dots, a_{n1}; a_{12}, \dots, a_{n2}; \dots; a_{1m}, \dots, a_{nm}; s, t) = a_{11} + a_{21}\alpha + \cdots + a_{n1}\alpha^{n-1} + a_{12}\beta + a_{22}\alpha\beta + \cdots + a_{n2}\alpha^{n-1}\beta + \cdots + a_{1m}\beta^{m-1} + a_{2m}\alpha^{m-1}\beta + \cdots + a_{nm}\alpha^{n-1}\beta^{m-1}\ in\ Q(\alpha, \beta)$.

If R is an algebra over a field F , subrings and ideal of the ring R that are also F-subspaces of R are called subalgebras and algebra ideals respectively .

If S is another F-algebra ,a ring homomorphism then $\theta: R \to S$ is called an algebra homomorphism if it also is F-linear,that is, if $\theta(ar) = a.\theta(r)$for all $a \in F and\ r \in R$.

Theorem 2 [2]: let R be an n-dimensional algebra over a field F . Then there exists a one-to-one algebra homomorphism $R \to M_n(F)$.

Proof .

Let $\{u_1, u_2, \dots, u_n\}$ be a basis of R . Given $\in R$, write $u_i r = \sum_{j=1}^{n} r_{ij} u_j$, $r_{ij} \in F$.

Then define $\theta: R \to M_n(F)\ \ by\ \theta(r) = [r_{ij}]^T$

$\theta(ar) = [ar_{ij}]^T = a[r_{ij}]^T\ \ \forall\ \ a \in F\ and\ \ \ r \in R$

Then θ is an F- linear homomorphism of additive groups .

Let $\theta(r) = [r_{ij}]^T = [r_{ji}]$ so that $u_i s = \sum_{j=1}^{n} s_{ij} u_j$. Then :

$u_i rs = (\sum_k r_{ik} u_k)s\ = \sum_k r_{ik}(\sum_j s_{kj} u_j) = \sum_j(\sum_k r_{ik} s_{kj})u_j$.

Thus $\theta(rs) = [\sum_k r_{ik} s_{kj}]^T = [\sum_k r_{ki} s_{jk}] = [r_{ji}][s_{ji}] = \theta(r).\theta(s)$.

Then θ is homomorphism .

Suppose that $\theta(r) = 0\ that\ is\ u_i r = 0\ \forall\ i$

If $1 = \sum_i a_i u_i$, $then\ r = 1.r = \sum_i a_i u_i\, r = 0$.

$So\ ker\theta = 0\ and\ \theta\ is\ one\ to\ one$.

RESULTS

If $(a_1, \dots, a_n; s)$ is the algebraic number in $(\sqrt[n]{s}\ )$, where s is n^{th} power free positive integer , then the basis is given as follows : $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$ where $\alpha = \sqrt[n]{s}$.

$(a_1, \dots, a_n; s)1 = a_1 + a_2\alpha + a_3\alpha^2 + \cdots + a_n\alpha^{n-1}$

$(a_1, \dots, a_n; s)\alpha = a_n s + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1}$

$(a_1, \dots, a_n; s)\alpha^2 = a_{n-1}s + a_n s\alpha + a_1\alpha^2 + \cdots + a_{n-2}\alpha^{n-1}$

$(a_1, \dots, a_n; s)\alpha^{n-1} = a_2 s + a_3 s\alpha + a_n s\alpha^2 + \cdots + a_1\alpha^{n-1}$

$$M(a_1, a_2, ..., a_n; s) = \begin{bmatrix} a_1 & sa_n & sa_{n-1} & . & . & . & sa_2 \\ a_2 & a_1 & sa_n & . & . & . & sa_3 \\ a_3 & a_2 & a_1 & . & . & . & sa_4 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_n & a_{n-1} & a_{n-2} & . & . & . & a_1 \end{bmatrix}$$

Thus we have :

Corollary 1.

The matrix $M(a_1, ..., a_n; s)$ corresponds to the algebraic number $(a_1, ..., a_n; s)$, where s is n^{th} power free positive integer .

If $\alpha = \sqrt[n]{s}$ and $\beta = \sqrt[m]{t}$ are two algebraic numbers of degree n and m respectively where s and t is n^{th} power free and m^{th} power free relatively prime positive integer.$Q(\alpha, \beta) = Q(\alpha)Q(\beta)$,then the basis is :

$\{1, \alpha, \alpha^2, ..., \alpha^{n-1}\} \times \{1, \beta, \beta^2, ..., \beta^{m-1}\} =$

$1, \alpha, \alpha^2, ..., \alpha^{n-1}, \beta, \alpha\beta, ..., \alpha^{n-1}\beta, ..., \beta^{m-1}, ..., \alpha^{n-1}\beta^{m-1}$.

The general element is :

$(a_{11}, ..., a_{n1}; a_{12}, ..., a_{n2}; ...; a_{1m}, ..., a_{nm}; s, t)$

The algebraic number :

$(a_{11}, ..., a_{n1}; s); (a_{12}, ..., a_{n2}; s); ...; (a_{1m}, ..., a_{nm}; s)$ correspond to

$M(a_{11}, ..., a_{n1}; s); M(a_{12}, ..., a_{n2}; s); ...; M(a_{1m}, ..., a_{nm}; s)$.

By using Corollary 1 we can represent the algebraic number by the blocks as follows :

$$B_1 = \begin{bmatrix} a_{11} & sa_{n1} & sa_{(n-1)1} & . & . & . & sa_{21} \\ a_{21} & a_{11} & sa_{n1} & . & . & . & sa_{31} \\ a_{31} & a_{21} & a_{11} & . & . & . & sa_{41} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n1} & a_{(n-1)1} & a_{(n-2)1} & . & . & . & a_{11} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} a_{12} & sa_{n2} & sa_{(n-1)2} & . & . & . & sa_{22} \\ a_{22} & a_{12} & sa_{n2} & . & . & . & sa_{32} \\ a_{32} & a_{22} & a_{12} & . & . & . & sa_{42} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n2} & a_{(n-1)2} & a_{(n-2)2} & . & . & . & a_{12} \end{bmatrix},...,$$

$$B_m = \begin{bmatrix} a_{1m} & sa_{nm} & sa_{(n-1)m} & . & . & . & sa_{2m} \\ a_{2m} & a_{1m} & sa_{nm} & . & . & . & sa_{3m} \\ a_{3m} & a_{2m} & a_{1m} & . & . & . & sa_{4m} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{nm} & a_{(n-1)m} & a_{(n-2)m} & . & . & . & a_{1m} \end{bmatrix}$$

Now we get a monomorphism from $Q(\alpha,\beta)$ $into$ $M_n(Q)$ where :

$$M(B_1, B_2, ..., B_n; t) = \begin{bmatrix} B_1 & tB_n & tB_{n-1} & . & . & . & tB_2 \\ B_2 & B_1 & tB_n & . & . & . & tB_3 \\ B_3 & B_2 & B_1 & . & . & . & tB_4 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ B_n & B_{n-1} & B_{n-2} & . & . & . & B_1 \end{bmatrix}$$

Thus we get :

Corollary 2 .

The matrix corresponds to algebraic number $(a_{11}, \dots, a_{n1}; a_{12}, \dots, a_{n2}; \dots; a_{1m}, \dots, a_{nm}; s, t)$ is computed by the block matrix $M(B_1, B_2, ..., B_n; t)$.

Application 1.

a)In $Q(\sqrt[3]{s})$: where s is - 3^{th}power free

by corollary 1 $M(a,b,c;s) = \begin{bmatrix} a & sc & sb \\ b & a & sc \\ c & b & a \end{bmatrix}$

b) In $(\sqrt[3]{s}, \sqrt{t})$: where s is 3^{th} -power free, t is 2^{th}- power free and s , t are relatively prime positive integers

by corollary 2 $M(a,b,c,d,e,f;s,t) = \left[\begin{array}{ccc|ccc} a & sc & sb & td & tsf & tse \\ b & a & sc & te & td & tsf \\ c & b & a & tf & te & td \\ \hline d & sf & se & a & sc & sb \\ e & d & sf & b & a & sc \\ f & e & d & c & b & a \end{array}\right]$

c) In $(\sqrt[4]{s}, \sqrt{t})$: where s is 4^{th} -power free, t is 2^{th}- power free and s , t are relatively prime positive integers by corollary 2

$$M(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8;s,t)=\begin{bmatrix} a_1 & sa_4 & sa_3 & sa_2 & | & ta_5 & tsa_8 & tsa_7 & tsa_6 \\ a_2 & a_1 & sa_4 & sa_3 & | & ta_6 & ta_5 & tsa_8 & tsa_7 \\ a_3 & a_2 & a_1 & sa_4 & | & ta_7 & ta_6 & ta_5 & tsa_8 \\ a_4 & a_3 & a_2 & a_1 & | & ta_8 & ta_7 & ta_6 & ta_5 \\ - & - & - & - & | & - & - & - & - \\ a_5 & sa_8 & sa_7 & sa_6 & | & a_1 & sa_4 & sa_3 & sa_2 \\ a_6 & a_5 & sa_8 & sa_7 & | & a_2 & a_1 & sa_4 & sa_3 \\ a_7 & a_6 & a_5 & sa_8 & | & a_3 & a_2 & a_1 & sa_4 \\ a_8 & a_7 & a_6 & a_5 & | & a_4 & a_3 & a_2 & a_1 \end{bmatrix}$$

Now , if $\alpha = \sqrt[n]{s}$, $\beta = \sqrt[m]{t}$ and $\gamma = \sqrt[r]{u}$ are there algebraic number of degrees n,m and r respectively and s ,t ,u are n^{th} power free , m^{th} power free and r^{th} power free respectively and they are relatively prime positive integer and $Q(\alpha,\beta,\gamma) = Q(\alpha,\beta)(\gamma)$, then the basis is :

$\{1,\alpha,\alpha^2,\dots,\alpha^{n-1}\} \times \{1,\beta,\beta^2,\dots,\beta^{m-1}\} \times \{1,\gamma,\gamma^2,\dots,\gamma^{r-1}\} =$

$\{1,\alpha,\dots,\alpha^{n-1},\beta,\alpha\beta,\dots,\alpha^{n-1}\beta,\dots,\beta^{m-1},\dots,\alpha^{n-1}\beta^{m-1}\}x\{1,\gamma,\dots,\gamma^{r-1}\}$

$=\{1,\alpha,\dots,\alpha^{n-1},\beta,\alpha\beta,\dots,\alpha^{n-1}\beta,\dots,\beta^{m-1},\alpha\beta^{m-1},\dots,\alpha^{n-1}\beta^{m-1}$

$\gamma,\alpha\gamma,\dots,\alpha^{n-1}\gamma,\beta\gamma,\alpha\beta\gamma,\dots,\alpha^{n-1}\beta\gamma,\dots,\beta^{m-1}\gamma,\alpha\beta^{m-1}\gamma,\dots,\alpha^{n-1}\beta^{m-1}\gamma,\dots,$

$\gamma^{r-1},\alpha\gamma^{r-1},\dots,\alpha^{n-1}\gamma^{r-1},\beta\gamma^{r-1},\alpha\beta\gamma^{r-1},\dots,\alpha^{n-1}\beta\gamma^{r-1},\dots,\beta^{m-1}\gamma^{r-1},$

$\alpha\beta^{m-1}\gamma^{r-1},\dots,\alpha^{n-1}\beta^{m-1}\gamma^{r-1}\}$.

Briefly.

1)$M_1 = M(a_1,\dots,a_n;s)$

For algebraic numbers in (α) .

2) $M_2 = M((M(a_{11},\dots,a_{n1};s),\dots,M(a_{1m},\dots,a_{nm};s));t)$

$= M(M_1,\dots,M_1;t)$

For algebraic numbers in (α,β) .

Then we have :

Corollary3.

The matrix representation of an algebraic number in $Q(\alpha,\beta,\gamma)$ is given by:

$M_3 = M(M_2,\dots,M_2;u)$.

Application 2.

Consider $Q(\sqrt{s},\sqrt{t},\sqrt{u})$ where s , t , u are 2^{th}- power free and they are relatively prime positive integers

By corollary 3

$$M_3 = \begin{bmatrix} a_{11} & sa_{21} & | & ta_{12} & tsa_{22} & | & ub_{11} & usb_{21} & | & utb_{12} & utsb_{22} \\ a_{21} & a_{11} & | & ta_{22} & ta_{12} & | & ub_{21} & ub_{11} & | & utb_{22} & utb_{12} \\ - & - & | & - & - & | & - & - & | & - & - \\ a_{12} & sa_{22} & | & a_{11} & sa_{21} & | & ub_{12} & usb_{22} & | & ub_{11} & usb_{21} \\ a_{22} & a_{12} & | & a_{21} & a_{11} & | & ub_{22} & ub_{12} & | & ub_{21} & ub_{11} \\ - & - & | & - & - & | & - & - & | & - & - \\ b_{11} & sb_{21} & | & tb_{12} & tsb_{22} & | & a_{11} & sa_{21} & | & ta_{12} & tsa_{22} \\ b_{21} & b_{11} & | & tb_{22} & tb_{12} & | & a_{21} & a_{11} & | & ta_{22} & ta_{12} \\ - & - & | & - & - & | & - & - & | & - & - \\ b_{12} & sb_{22} & | & b_{11} & sb_{21} & | & a_{12} & sa_{22} & | & a_{11} & sa_{21} \\ b_{22} & b_{12} & | & b_{21} & b_{11} & | & a_{22} & a_{12} & | & a_{21} & a_{11} \end{bmatrix}$$

Application 3.

To find the inverse of $a + b\sqrt[3]{s} + c\sqrt[3]{s^2}$ by matrix representation .

$$M_1 = M(a,b,c;s) = \begin{bmatrix} a & sc & sb \\ b & a & sc \\ c & b & a \end{bmatrix}$$

$$M_1^{-1} = \frac{1}{a^3+b^3s+c^3s-3abcs} \begin{bmatrix} a^2 - sbc & s(b^2 - ac) & s(sc^2 - ab) \\ sc^2 - ab & a^2 - sbc & s(b^2 - ac) \\ b^2 - ac & sc^2 - ab & a^2 - sbc \end{bmatrix}$$

Then $(a + b\sqrt[3]{s} + c\sqrt[3]{s^2})^{-1} =$

$$\frac{1}{a^3+b^3s+c^3s-3abcs} \left(a^2 - sbc + (sc^2 - ab)\sqrt[3]{s} + (b^2 - ac)\sqrt[3]{s^2}\right) .$$

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