Approximations of maximal and principal prime Ideal

التقريبات في المثالي الأعظمي والمثالي الرئيسي

د. فرج ارخيص عبدالنبي. محاضر بقسم بكلية العلوم. جامعة اجدابيا

Dr: Faraj A. Abdulnabi. Lecturer, Department of the Faculty of Science. Ajdabiya University.
Email: faraj.a.abunabi@uoa.edu.ly

الملخص: تهدف الدراسة إلى معرفة العلاقة بين النظرية الخشنة ونظرية الحلقة في الجبر وتحديدا في المثالي الأعظمي، وكذلك المثالي الرئيسي، حثث سنجد أن هناك مثالي أعظمي خشن، و ومثالي رئيسي خشن، وهذه المفاهيم للمثالين الخشنيين (الأعظمي، الرئيسي) يعتبران كتوسيع لمفهوم المثاليين (الاعظمي والرئيسي) على التوالي المعرفين في الحلقات. كما تمت دراسة بعض خصائص التقريبات من أعلي ومن أسفل علي هذين المثاليين. حيث نقدمهما كنتيجة جديدة. ونأمل أن يكون مفيدًا جدًا من الناحيتين النظرية والتطبيقية.

الكلمات الداله: تقريب أقل ، تقريب علوي ، حلقة ، مثالي

Abstract

In this paper, we will delving deeper into the connection between the rough theory and the ring theory precisely in the maximal ideal. We will find that there is a rough maximal and principal ideal as an extension of the notion of a maximal and principle ideal respectively. Some properties of approximations of the maximal and the principal are studied.

Keywords: lower approximation, upper approximation, ring, ideal

1-Introduction

The rough set theory has shown by Pawlak [1] in 1982. It is a good formal tool for modeling and processing incomplete information in information system. In recently 40 years, some researchers develop this theory and use it in many areas. The upper approximation of a given set is the union of all the equivalence classes, which are subsets of the set, and the lower approximation is the union of all the equivalence classes, which are intersection with set non-empty. Many researchers develop and use the rough theory in the group and ring theory. For example, the notation of rough subring with respect ideal has presented by B.Davvaz[2]. Algebraic properties of rough sets have been studied by Bonikowaski [3]. John N. Mordeson[4], he used covers of the universal set to defined approximation operators on the power set of the given set. Some concept lattice in Rough set theory has studied by Y.Y. Yao[5]. Ronnason Chinram, [6] , he study rough prime ideas and Rough fuzzy prime ideals in gamma-semigroups. Some other substitute an algebraic structure instead of the universe set. Like Biswas and Nanda [7], they make some notions of rough subgroups. Kuroki in [8], introduced the notion of a rough ideal in a semi group. Some properties of the Also, Kuroki and Mordeson in [9] studied the structure of rough sets and rough groups. S.B Hosseinin[10], he introduced and discussed the concept of T-rough (prime, primary) ideal and T-rough fuzzy (prime, primary) ideal in a commutative ring In addition, B.Davvaz [11] applied the concept of approximation spaces in the theory of algebraic hyperstructures, and in investigated the similarity between rough membership functions and condi-tional probability. Nurettin Ba˘gırmaz[14] introduced the notion of rough prime ideals. In this paper, we shall introduce the maximal ideal. Our result will introduce the rough maximal ideal as an extended notion of a classic maximal ideal and we study some properties of the lower and the upper approximations a maximal ideal.

2- Preliminaries

Suppose that ***U*** (*universe*) be a nonempty finite set. Let ~ *an equivalence relation* (reflexive, symmetric, and transitive) on an *U*. Some authors say ~ is *indiscernibility relation*. The discernibility relation represents our lack of knowledge about elements of ***U***. For simplicity, they assume ~ *an equivalence relation*. We use ***U/~*** to denote the family of all equivalent classes of ~ (or classifications of ***U***), and we use $[\boldsymbol{x}]_{\sim}$ to denote an equivalence class in ***R*** containing an element $\boldsymbol{x}\in\boldsymbol{U}$. The pair (***U,~***) is called an approximation space. The empty set $\varnothing$ and the elements of ***U/~*** are call elementary sets. For any $X \subseteq \boldsymbol{U}$, we write X^c to denote the complementation of X in ***U***.

Definition 2.1:

[1] For an approximation space (***U,*** ~), we define the upper approximation of X by $\overline{RX} = \{x \in U : [x]_{\sim} \cap X \neq \emptyset\}$;i.e. $\overline{RX}$ the set of all objects which can be only classified as *possible* members of X with respect to ~ is called the *~-upper approximation* of a set X with respect to ~. And the lower approximation of X by $\underline{RX} = \{x \in U : [x]_{\sim} \subseteq X\}$. i.e $\underline{RX}$ is the set of all objects which can be with *certainty* classified as members of X with respect to ~ is called the *~lower approximation* of a set X with respect to~.

Definition 2.2:

[1] For an approximation space (***U,*** ~) ,we define the *boundary region* by $BX_{\sim} = \overline{RX} - \underline{RX}$. *i.e* . $BX_{\sim}$ is the set of all objects which can be decisively classified neither as members of X nor as the members of X^c with respect to ~. If $BX_{\sim} = \emptyset$, we say X is exact (*crisp)* set . But if $BX_{\sim} \neq \emptyset$, we say X *Rough set* (inexact).

Proposition 2.3

[1] Let $X, \subset$ U, where U is a universe. Then, the approximations have the following properties :

1) $\underline{\sim X} \subseteq X \subseteq \overline{\sim X}$
2) $\underline{\sim \emptyset} = \overline{\sim \emptyset}\,, \underline{\sim U} = \overline{\sim U}\,,$
3) $\sim\underline{(X \cup Y)} \supseteq \underline{\sim(X)} \cup \underline{\sim(Y)},$
4) $\underline{\sim(X \cap Y)} = \underline{\sim(X)} \cap \underline{\sim(Y)},$
5) $\overline{\sim(X \cup Y)} = \overline{\sim(X)} \cup \overline{\sim(Y)}$
6) $\overline{\sim(X \cap Y)} \subseteq \overline{\sim(X)} \cap \overline{\sim(Y)}$
7) $\overline{\sim X^C} = (\underline{\sim X})^c$
8) $\underline{\sim X^C} = (\overline{\sim X})^c$
9) $\underline{\sim(\underline{\sim X})} = \overline{\sim(\underline{\sim X})} = \underline{\sim X}$
10) $\overline{(\sim(\overline{\sim X})} = \underline{\sim(\overline{\sim X})} = \overline{\sim X}$

Example 2.4

Let us consider set of objects $U =\{ x_1, x_2, x_3, x_4, x_5, x_6, x_7 \}$, and the Equivalence relation $\sim = \{\{ x_1\}, \{ x_2\}, \{ x_3, x_4 \}, \{ x_5, x_7 \}, \{ x_6 \}\}$, and Let $X= \{ x_1, x_4, x_6 \}$. Now, the upper approximations of X: $\overline{\sim X} = \{x \in U : [x]_{\sim} \cap X \neq \emptyset\}$.So $\overline{\sim X} = \{ x_1, x_3, x_4, x_6\}$ and the lower approximation of X :$\underline{\sim X} = \{x \in U : [x]_{\sim} \subseteq X\}$.So $\underline{\sim X} = \{ x_1, x_6 \}$. The *boundary region* by $BX_{\sim} = \overline{\sim X} - \underline{\sim X}$ .So, $BX_{\sim} = \{ x_3, x_4 \}$. Then $BX_{\sim} \neq \emptyset$, **so** X is *Rough set* .

Now**,** we define the ideal and maximal idea. Then we will study the upper and lower approximations ideal. We suppose we have a ring $\mathcal{R}$ and I be an Ideal of a ring $\mathcal{R}$, and X be a non-empty subset of $\mathcal{R}$.

Definition 2.5

[13]: Let I be an Ideal of $\mathcal{R}$; For $a,b \in \mathcal{R}$ we say a is congruent of b *mod I*, we express this fact in symbols as $a \equiv b(mod\ I) if\ a - b \in I$ …………….(1)

Not that, it easy to see the relation (1) is an equivalents relation. Therefore, when we let ***U***= $\mathcal{R}$ and we suppose a relation ~ is the equivalents relation (1), so we can defined the upper and lower approximation of X with respect of I as:$\overline{I(X)} = \cup\{ x \in \mathcal{R} : (x + I) \cap X \neq \emptyset\}$, $\underline{I(X)} = \cup\{ x \in \mathcal{R} : x + I \subseteq X\}$, respectively. Moreover, the boundary of X with respect of I is $BX = \overline{I(X)} - \underline{I(X)}$. If BX=∅ we say X is *Rough set with respect I*. For any approximation space (U, ~) by rough approximation on (U, ~), we mean a mapping *Apr*(X): $p(U) \rightarrow P(U) \times P(U)$ defined by for all $x \in P(U)$, *Apr*(X)= $(\overline{I(X)}, \underline{I(X)})$, where

$\overline{I(X)} = \{ x \in \mathcal{R} : (x+I) \cap X \neq \emptyset\}$, $\underline{I(X)} = \{ x \in \mathcal{R} : x+I \subseteq X\}$.

Definition 2.6

[14] Let $(U, \sim)$ be an approximation space and $(\cdot)$ be a binary operation defined on U. An rough ideal H is called rough prime ideal of a rough semigroup S if for $a, b \in S$, $ab \in R(H)$ implies $a \in H$ *or* $b \in H$.

Definition 2.7

[13]: An ideal M in a ring $\mathcal{R}$ we called maximal if $M \neq \mathcal{R}$ and the only ideal strictly containing M is $\mathcal{R}$.

Definitions 2.8

[12]: Let $\mathcal{R}$ be a commutative ring with identity. Let S be a subset of $\mathcal{R}$. The ideal generated by S is the subset $< S > = \{r_1s_1 + r_2s_2 + \ldots + r_ks_k \in \mathcal{R} \mid r_1, r_2, \ldots \in \mathcal{R}, s_1, s_2, \ldots \in S, k \in N\}$.In particular, if S has a single element s this is called the principal ideal generated by s.That is, $< s > = \{rs \mid r \in \mathcal{R}\}$.

Examples 2.9

The ideal $2\mathbb{Z}$ of $\mathbb{Z}$ is the principal ideal $< 2 >$.

Proposition 2.10

[12] Let $\mathcal{R}$ be a commutative ring with identity. Then every maximal ideal of $\mathcal{R}$ is prime.

Proof: suppose that $ab \in J$ where $a,b \in A$and J is an idealof A, Let $a \notin J$ to show $b \in J$.

Since $a \notin J$ *and* $a+J \neq 0+J$ *Hence* $a+J$ *has an inverse in* A/J, *So* $\exists (c+J) \in A/J$ *such that* $(a+J)(c+J)=1+J \Rightarrow ac+J=1+J \Rightarrow ac-1 \in J \Rightarrow bac-b \in J \Rightarrow abc-b \in J$. *Again* $ab \in J \Rightarrow abc \in J$. *Thus* $b=b-abc+abc \in J$.

Example 2.11

For $\mathcal{R} = \mathbb{Z}_{12}$ there are only two maximal ideals $M_1 = \{0, 2, 4, 6, 8, 10\}$ and $M_2 = \{0, 3, 6, 9\}$. Two other ideals, which are not maximal are $\{0, 4, 8\}$ and $\{0, 6\}$.

3- Upper and lower maximal ideals

In this section, we introduce the approximations of a non-empty subset of a ring $\mathcal{R}$ with respect of *the maximal ideal of* $\mathcal{R}$ and we study some properties of these approximations. Let consider the example2-4:

Example 3.1

Let consider the example2-4 . Suppose $\mathcal{R} = \mathbb{Z}_{12}$, we take $M = \{0, 3, 6, 9\}$. Let $A=\{1,2,6,7,9\}$ For $\in \mathcal{R} : x + M$, we get $\{0,3,6,9\}$, $\{1,4,7,10\}$,$\{2,5,8,11\}$. Now, the upper approximations of A with respect of M: $\overline{M(A)} = \cup\{ x \in \mathcal{R} : (x+M) \cap A \neq \emptyset\}$, $= \{0,1,2,3,4,5,6,7,8,9,10,11\}$ and lower approximation of A with respect of M= $\underline{M(X)} = \cup\{ x \in \mathcal{R} : x+M \subseteq A\}$, So, $\underline{M(A)} = \emptyset$ because no element satisfy the definition of $\underline{M(A)}$. Moreover, $BA = \overline{I(A)} - \underline{I(A)} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$. Then, A is rough set with respect M.

Example 3.2

Let us consider the ring $\mathcal{R} = \mathbb{Z}_6$. Suppose the maximal ideals is $M=\{0,2,4\}$ and $X=\{1,2,3,4,5\}$. For $x \in \mathcal{R} : x + M$, we get $\{0,2,4\}$, $\{1,3,5\}$. The upper approximations of X with respect of M: $\{0, 2, 4\} \cup \{1, 3, 5\}$.*so*, $\overline{M(X)} = \{0,1,2,3,4,5\}$ and the lower approximation of X with respect of M :$\underline{M(X)} = \{1,3,5\}$. $BX = \overline{M(X)} - \underline{M(X)} = \{0,2,4\}$. Then X is rough set with respect maximal ideal M.

Example 3.3

Let us consider the ring $\mathcal{R} = \mathbb{Z}_6$. Suppose the maximal ideals is $M=\{0,3\}$ and $X=\{1,2,3,4,5\}$. For $x \in \mathcal{R} : x + M$, we get $\{0,3\}$, $\{1,4\}$,$\{2,5\}$. The upper approximations of X with respect of M: $\{0, 3\} \cup \{1, 4\} \cup \{2,5\}$.*so*, $\overline{M(X)} = \{0,1,2,3,4,5\}$ and the lower approximation of X with respect of M :$\underline{M(X)} = \{1,2,4,5\}$. $BX = \overline{M(X)} - \underline{M(X)} = \{0,3\}$. Then X is rough set with respect maximal ideal M.

Example 3.4

Let us consider the ring $\mathcal{R} = \mathbb{Z}_6$. Suppose the maximal ideals is $M=\{0,3\}$ and $A=\{4,5\}$. For $x \in \mathcal{R} : x + M$, we get $\{0,3\}$, $\{1,4\}$,$\{2,5\}$. The upper approximations of X with respect of M: $\{1, 4\} \cup \{2,5\}$.*so*, $\overline{M(A)} = \{1,2,4,5\}$ and the lower approximation of X with respect of M :$\underline{M(A)} = \emptyset$.

$BA = \overline{M(A)} - \underline{M(A)} = \{1,2,4,5\}$. Then A is rough set with respect maximal ideal M.

Example 3.5

Let us consider the ring $\mathcal{R} = \mathbb{Z}_6$. Suppose the maximal ideals is M={0,3} and B={1,2,3}. For $x \in \mathcal{R} : x + M$, we get {0,3}, {1,4},{2,5}. The upper approximations of B with respect of M: {0,3}{1,4}∪{2,5}.$So, \overline{M(B)} =$ {0,1,2,3,4,5} and the lower approximation of B with respect of M :$\underline{M(B)} = \varnothing$. $BB = \overline{M(B)} - \underline{M(B)} =$ {0,1,2,3,4,5}. Then B is rough set with respect maximal ideal M.

Example 3.6

Let us consider the ring $\mathcal{R} = \mathbb{Z}_6$. Suppose the maximal ideals is M={0,3} and B={0,4,5}. For $x \in \mathcal{R} : x + M$, we get {0,3}, {1,4},{2,5}. The upper approximations of B with respect of M: {0,3}{1,4}∪{2,5}.$So, \overline{M(B)} =$ {0,1,2,3,4,5} and the lower approximation of B with respect of M :$\underline{M(B)} = \varnothing$. $BB = \overline{M(B)} - \underline{M(B)} =$ {0,1,2,3,4,5}. Then B is rough set with respect maximal ideal M.

From Example 3,3 , example 3.4, and example 3.5, we can see that A={4,5} and B={1,2,3}. And X=A∪B.

$\overline{M(A)} = \{1,2,4,5\} + \overline{M(B)} = \{0,1,2,3,4,5\} = \overline{M(X = A \cup B)}$;

$\underline{M(A)} = \varnothing \cup \underline{M(B)} = \varnothing \subseteq \underline{M(X)} = \{1,2,4,5\}$.

We can get the properties of maximal ideal in following proposition:

Proposition 3-7

Suppose $\mathcal{R}$ is the ring. Let M be the maximal ideal of $\mathcal{R}$. For every subset $A,B \subseteq \mathcal{R}$ we have:

1) $\underline{M(A)} \subseteq A \subseteq \overline{M(A)}$;
2) $\underline{M(\emptyset)} = \emptyset = \overline{M(\emptyset)}$;
3) $\underline{M(\mathcal{R})} = \mathcal{R} = \overline{I(\mathcal{R})}$;
4) $\underline{M(A \cap B)} = \underline{M(A)} \cap \underline{M(B)}$;
5) $\overline{M(A \cup B)} = \overline{M(A)} \cup \overline{M(B)}$
6) If A ⊆B , then $\underline{M(A)} \subseteq \underline{M(B)}, and\ \overline{M(A)} \subseteq \overline{M(B)}$;
7) $\underline{M(A \cup B)} \supseteq \underline{M(A)} \cup \underline{M(B)}$;
8) $\overline{M(A \cap B)} \subseteq \overline{M(A)} \cap \overline{M(B)}$
9) $\underline{M(\underline{I(A)}} = \overline{M(\underline{M(A)})} = \underline{M(A)}$
10) $\overline{(M(\overline{M(A)}} = \underline{(\overline{M(A)}} = \overline{M(A)}$

Proof: If $x \in \underline{M(A)}$, then $x \in \underline{M(A)} = \{x \in \mathcal{R} : x + M \subseteq A$, then $x \in$A,Hence $\underline{M(A)} \subseteq A$, next if $x \in$ A, $\overline{M(A)} = \{x \in \mathcal{R} : (x + M) \cap A \neq \emptyset\}$,, then $x \in \overline{M(A)}$ then A $\subseteq \overline{M(A)}$.

1) And 3) it easy to see that.
4) If $x \in \underline{M(A \cap B)}$, then $x \in \underline{M(A \cap B)} = \{x \in \mathcal{R} : x + M \subseteq A \cap B\}$, then $x \in \mathcal{R} : x + M \subseteq A\ and\ x + MB$, then then $x \in \underline{M(A)} \cap \underline{M(B)}$.
5) It say way in 4)
6) Since $A \subseteq B$, then $A \cap B$=A, by 4) then $\underline{M(A)} \cap \underline{M(B)}$; It implies$\overline{M(A)} \subseteq \overline{M(B)}$, also, by 5) we get $\overline{M(A)} \subseteq \overline{M(B)}$.
7) Since $A \subseteq A \cup B$, $B \subseteq A \cup B$, by 6) we get $\underline{M(A)} \subseteq \underline{M(A \cup B)}, and\ \overline{M(B)} \subseteq \overline{M(A \cup B)};\ wich\ yields\ \underline{M(A \cup B)} \supseteq \underline{M(A)} \cup \underline{M(B)}$;
8) It say way in 7)
9) -10) it is easy to see that by using of definition of upper and lower approximations of A with respect M.

Now, suppose A is any subset of a ring $\mathcal{R}$ and $(\mathcal{R}, \theta)$ be a rough approximation space. let $\boldsymbol{U}= \mathcal{R}$, we can defined the upper and lower approximation of X with respect of M as:$\overline{M(X)} = \cup \{x \in \mathcal{R} : (x + M) \cap X \neq \emptyset\}$, $\underline{M(X)} = \cup \{x \in \mathcal{R} : x + I \subseteq X\}$, respectively. Moreover, the boundary of X with respect of M is $BX = \overline{M(X)} - \underline{M(X)}$. If B$X$=∅ we say X is *Rough set with respect M.* For any approximation space (U, ~) by rough approximation on (U, ~), we mean a mapping $Apr(X)$: $p(U) \rightarrow P(U) \times P(U)$ defined by for all $x \in P(U)$, $Apr(X) = (\overline{M(X)}, \underline{M(X)})$, where $\overline{M(X)} = \{x \in \mathcal{R} : (x + M) \cap X \neq \emptyset\}$, $\underline{M(X)} = \{x \in \mathcal{R} : x + M \subseteq X\}$. If $\underline{\theta(A)}$ and $\overline{\theta(A)}$ are ideals , then

$\overline{\theta(A)}$ is called a lower and upper rough maximal ideal R, respectively. $(A) = (\overline{\theta(A)}, \underline{\theta(A)})$ is called rough maximal ideal of $\mathcal{R}$.

Theorem 3.8

Let M be an Idea of a ring $\mathcal{R}$, $a \equiv b(mod\ I) if\ a - b \in I$, If A maximal ideal of $\mathcal{R}$, then $\overline{\theta(A)}$ is a is ideal of ring $\mathcal{R}$.

Proof.: Let $a, b \in \overline{\theta(A)}$. Then $[a] \cap A \neq \emptyset$,$[b]_\theta \cap A \neq \emptyset$. So there exist $x \in [a] \cap A$ and $y \in [b] \cap A$. Since $x, y \in A$, $x + y \in A$. Now $x + y \in [a] + [b] \subseteq [a + b]$. Therefore $[a + b] \cap A \neq \emptyset$ and this means that $a + b \in \overline{\theta(A)}$. Again let $x \in \overline{\theta(A)}$ and $r \in \overline{\theta(A)}$. Then there exists $y \in [a] \cap A$ and $(y, x) \in \theta$. Since θ is congruence relation, (xr, yr) and $(rx, ry) \in \theta$. This means that $xr, rx \in \overline{\theta(A)}$. Thus $\overline{\theta(A)}$ is an ideal of $\mathcal{R}$.

Lemma 3.9

Suppose $\mathcal{R}$ is a ring and M is a maximal ideal of $\mathcal{R}$, For any $A \subseteq \mathcal{R}$ and $\underline{M(A)} \cap \overline{M(A)} \neq \emptyset$, then $(\overline{M(A)}, \underline{M(A)})$ is a rough maximal ideal of $\mathcal{R}$.

Lemma 3.10

Suppose $\mathcal{R}$ is a ring and M is a maximal ideal of $\mathcal{R}$. and A $\subseteq R$, the following hold:

(i) $\underline{M\big((\mathcal{R}\backslash A)\big)} = \mathcal{R} \backslash \overline{M(A)}$);

(ii) $\overline{M(\mathcal{R} \backslash A)} = \mathcal{R} \backslash \underline{M(A)}$;

(iii) $\overline{M(A)} = (\underline{M(A)}^c)^c$; $\underline{M(A)} = (\overline{M(A^c}))^c$

Proof: The proof is obvious and hence omitted by using of definition of upper and lower approximations of A with respect M.

4- Upper and lower principal ideal:

In this section, we using a same way in section 3 to find a rough principle ideal.

Example 4.1

Let us consider the ring $\mathcal{R} = \mathbb{Z}_6$, J={0,2,4} and X={1,2,3,4,5}. For $x \in \mathcal{R} : x + J$, we get {0,2,4}, {1,3,5}. The upper approximations of X with respect of J: {0, 2, 4} ∪{1, 3, 5}. $so, \overline{J(X)} = \{0,1,2,3,4,5\}$. And the lower approximation of X with respect of J:$\underline{J(X)} = \{1,3,5\}$,$BX = \overline{J(X)} - \underline{J(X)} = \{0,2,4\}$. Then X is rough set with respect J.

we study the properties of principle ideals in a ring using congruence and complete congruence relations. We have obtained characterizations of principle ideals and complete prime ideals in terms of the lower and upper approximations.

Theorem 4.2

[16] If P is an ideal in the ring R, then the following propositions are equivalent:

(1) P is a prime ideal;

(2) If $< a >, < b >$ are principal ideals, and $< a >< b > \subseteq P$ then $a \in P$ or $b \in P$

(3) For every $i, \in \mathcal{R}$, $i \notin P$ and $j \notin P$ implies $(i)(j) \nsubseteq P$.

In the following theorem, we can study the properties of the principle rough.

Theorem 4-3

suppose $\mathcal{R}$ commtive ring and I principle ideal. For every approximation ($\mathcal{R}$,I) and Every subset $A,B \subseteq \mathcal{R}$ we have:

1) $\underline{I(A)} \subseteq A \subseteq \overline{I(A)}$;

2) $\underline{I(\emptyset)} = \emptyset = \overline{I(\emptyset)}$;

3) $\underline{I(\mathcal{R})} = \mathcal{R} = \overline{I(\mathcal{R})}$;

4) $\underline{I(A \cap B)} = \underline{I(A)} \cap \underline{I(B)}$;

5) $\overline{I(A \cup B)} = \overline{I(A)} \cup \overline{I(B)}$

6) If A $\subseteq$B , then $\underline{I(A)} \subseteq \underline{I(B)}, and\ \overline{I(A)} \subseteq \overline{I(B)}$;

7) $\underline{I(A \cup B)} \supseteq \underline{I(A)} \cup \underline{I(B)}$;

8) $\overline{I(A \cap B)} \subseteq \overline{I(A)} \cap \overline{I(B)}$
9) $\overline{I(\mathrm{A})} = (\underline{I(A^C)})^c$
10) $\underline{I(A)} = (\overline{I(A^C)})^c$

Proof : we use the same technique in proposition 3-1.

Theorem 4.4

Let M be an Idea of a ring $\mathcal{R}$, $a \equiv b(mod\ M) if\ a - b \in M$, If A principle ideal of $\mathcal{R}$, if and only if $\underline{I(\mathrm{M})}$ principle ideal of $\mathcal{R}$.

Proof.:

Let M be a principle ideal of $\mathcal{R}$. Let $a, b \in \mathcal{R}$ such that $ab \in \overline{I(\mathrm{M})}$. Suppose $a, b \notin \overline{I(\mathrm{M})}$. Then $[a]_{\sim} \cap M = \emptyset$ and $[b]_{\sim} \cap \mathrm{M} = \emptyset$. This implies that $a, b \notin M$. Then $ab \notin M$, otherwise $ab \in M$ and since M is principle ideal, $a \in M$ or $b \in M$, a contradiction. Now $ab \notin M \Rightarrow [ab]_{\sim} \cap M = \emptyset$ and $ab \notin \overline{I(\mathrm{M})}$, contradicts to assumption. Thus $\overline{I(\mathrm{M})}$ is principle. Conversely, we assume that $\overline{I(\mathrm{M})}$ principle ideal of a ring $\mathcal{R}$. Suppose that $a, b \in \mathcal{R}$ such that $ab \in M$. Let $a, b \notin M$. we get $[a]_{\sim} \cap M = \emptyset$ and $[b]_{\sim} \cap M = \emptyset$. This implies that $a, b \notin \overline{I(\mathrm{M})}$. This leads to $ab \notin \overline{I(\mathrm{M})}$. Otherwise $ab \in \overline{I(\mathrm{M})}$ implies, $\overline{I(\mathrm{M})}$ being principle, $a, b \in \overline{I(\mathrm{M})}$ which a contradiction with assumption. Thus $ab \notin \overline{I(\mathrm{M})}$. Now, suppose that $ab \notin \overline{I(\mathrm{M})}$. Then $[a]_{\sim} \cap \mathrm{M} = \emptyset$ and $ab \notin \overline{I(\mathrm{M})}$, contradiction. Thus, M is a principle ideal of $\mathcal{R}$.

Conclusion: The theory rings has wide applications in several areas such as optimization theory, discrete event dynamical systems, automata theory, formal language theory and parallel computing. The theory of rough sets also has many applications in the above areas. In this paper, we introduce the rough maximal ideal and principle ideal and study some properties of approximations of them. Moreover, we find for any a non-empty sets of a ring $\mathcal{R}$ if the intersection between upper and lower maximal ideal of a ring $\mathcal{R}$ is non empty, then the boundary will maximal ideal of a ring $\mathcal{R}$ as a new result . We certainly hope that our work will be very useful both in the theoretical and application aspect.

ACKNOWLEDGMENTS

I would like to thank Tracy Tian Editorial Assistant of OJDM suggestion on this paper.

References

- Z. Pawlak, Rough sets, Int. J. Comput. Inform. Sci., 11 (1982), 341–356.
- http://dx.doi.org/10.1007/bf01001956
- B. Davvaz, (2004),Roughness in Rings , Inform. Sci. 164 147-163.
- Z. Bonikowaski,(1995),Algebraic structures of rough sets, in: W.P. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer-Verlag, Berlin, pp. 242–247.
- John N. Mordeson, Elsevier, (2001) ,Rough set theory applied to (fuzzy) ideal theory, Fuzzy Sets and Systems 121 315–324.
- Y.Y. Yao , Department of Computer Science, University of Regina, Regina, Saskatchewan, Canada S4S 0A2 .
- Ronnason Chinram, ((2009),Rough prime ideas and Rough fuzzy prime ideals in Gamma-semigroups, Commun. Korean Math. Soc. 24, No. 3, pp. 341{351
- Biswas, S. Nanda, (1994) Rough groups and rough subgroups, Bull. Polish Acad. Sci. Math. 42 251–254.
- N. Kuroki,((1997), Rough ideals in semigroups, Inform. Sci. 100 139–163.
- N. Kuroki, J.N. Mordeson, (1997), Structure of rough sets and rough groups, J. Fuzzy Math. 5 (1) 183–191.
- S.B. Hosseini,(2012) T- Rough(Prime, Primary) ideal and t- Rough fuzzy 9Prime, Primary) ideal om commutative rings, int.J, contemp. Math science, Vol 7, n.4, 337-350.
- B. Davvaz, (1998) Rough sets in a fundamental ring, Bull. Iranian Math. Soc. 24 (2) 49–
- Oystaeyen, Freddy M.J. van (Ed.)(1986), Springer-Verlag Berlin Heidelberg, first edition
- Allenby R. B. J. T.(1986), Rings, Fields and Groups. An introduction to abstract algebra, London, Arnold.
- Nurettin Ba˘gırmaz, (2016), Rough Prime Ideals in Rough Semigroups International Mathematical Forum, Vol. 11, no. 8, 369 - 377
- Z. Pawlak and A. Skowron, (2007) Rough Sets: Some Extensions, Information Sciences, 177 28-40.

- 5. K. Iseki, (1958),Ideals In Semirings, Proceedings of the Japan Academy, 34(1) 29-31.
- D. D. ANDERSON ، and SANGMIN CHUN(2011), SOME REMARKS ON PRINCIPAL PRIME IDEALS, Bull. Aust. Math. Soc. 83 (2011), 130–137 doi:10.1017/S000497271000170X