New Properties On NG-Groups

خصائص جديدة في الزمر NG

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الملخص: الهدف من هذه الورقة هو تقديم زمر NG التي تتكون من تحويلات غير أحادية أو فوقية والتي لا يمكن أن تكون فئة جزئية من الزمر المتماثلة. وتقدم هذه الزمر بمفهوم جديد كزمر شبه مضادة للانعكاس. علاوة على ذلك ، فقد تمت دراسة الزمر بالمفهوم الجديد ودراسة بعض خصائصها.

الكلمات الداله: الزمر المتماثلة، فئة التكافؤ، الزمر NG، الزمر شبه المضادة للانعكاس

Abstract: The aim of this paper is to present NG groups that consist of non-bijective transformations that can not be a subset of symmetric groups as anti-reflective semigroups. Moreover, some properties are studied.

Keywords: Symmetric group, Equivalence class, NG groups, anti-reflective-semigroups.

1. Introduction

The theory of the transformation group is one of the parts of Mathematics [1]. NG group presented by Abdunbai [2] as a group consisting of bijections transformations from X to itself with respect to compositions mapping on anon empty set X. Theses groups was as problem 1.4 in [3]. In [4], Y. Wu, X.Wei present the conditions of the groups generated by nonobjective transformations on a set. Authors in [5], introduce the study regularity of these groups such as new results. In this paper, we introduce the new definition of these groups are anti-reflective NG groups. Some properties of our result are studied.

2-Preliminary:

In this section, we recall and study some basic notations and properties of anon empty and finitgroupsup that are used in our paper. For more detail in lots of abstract algebra and finite group theory can see [6],[7] would be good supplementary sources for the theory needed here. Through this paper $P(X)$ is denoted to the set of all its transforms and the image of f is Im(f) for any $f \in P$(X).

Definition 2.1. Suppose that an A is an anon-empty set. A binary relation ~ is an equivalence relation if it satisfy the following:

1) $a \sim a$, for all $a \in A$;
2) $If\ a \sim b\ then\ b \sim a$ for all $a, b \in A$;
3) $If\ a \sim b\ , b \sim c\ then\ a \sim c. for\ all\ \ a, b, c \in A.$

Definition 2.2. The equivalence class of an equivalence relation on X $([x]_{\sim}) = \{x \in X | x\sim\}\ and\ X/\sim = \{[x]_{\sim} | x \in X\}$ is said to be the quotient set of X relative to the equivalence relation.

Propstion2.1 [2]. Suppose that NG is a group. For any $f \in$NG and the e the identity element of NG, then $\sim e\ =\ \sim f$.

Proof. We suppose that X= NG, for any $x \in X$, our goal is to show that $[x]_f = [x]_e$. On one hand, if $a \in [x]_f, i.e. f(a) = f(x)$. Since X is a group with identity element e, then there is a transformation $f' \in X$ such that $f'f=e=f'f$. Therefore,

$e(a)\ =\ f'(f(a))\ =\ f'(f(x)) = e(x)$, Which yields that $a \in [x]_e$.

($\Leftarrow$) If $y \in [x]_e\ i.e.\ e(a)e(y)$.Hence,$f(a) = (fe)(a) = f(e(y)) = (fe)(y) = f(y) \Rightarrow y \in [x]_f$. It follows that $[x]_e .=\ [x]_f$ for any $x \in X$=NG, as wanted.

Remark 2.1. For proposition 2.1, $\sim f = \sim g$ for any element $f,g \in NG$.

Propstion2.2.[2] Suppose that f is an element ($P(X)$) and $\check{f}$ be the induced transformation of f on $X/\sim_f$, $i.e\ \check{f}: X/\sim_f \longrightarrow X/\sim_f$, $[x]_f \mapsto [f(x)]_f$. Then there exists a group $NG \subseteq$P(A) containing f as the identity element iff $f^2=f$. Moreover, there is a group NG$\subseteq$P(X) containing f as the identity element iff $\check{f}$ is bijective on $X/\sim_f$.

Proposition 2.3. Suppose that X is a non-empty set and NG$\subseteq$P(X) is a group this is a not subset of S_n. Set $NG = \{\hat{}f \mid f \in NG\}$; then $\hat{}NG$ is a symmetric group on $X/\sim$ and ρ: $NG \longrightarrow \hat{}NG, f \mapsto \hat{}f$,is an isomorphism.

Proof. Suppose that f, $g \in NG$ and for any$[x] \in X/\sim$, we have $\rho(fg)([x]) = [(fg)(x)] = [f(g(x))] = \rho(f)([g(x)]) = (\rho(f)\rho(g))([x]) \Rightarrow \rho(fg) = \rho(f)\ \rho(g) \Rightarrow \rho$ is a homomorphism. By the definition of ˆNG, it is obvious that ρ is surjective.

Now, for any two elements f, $g \in NG$ we suppose that $\rho(f) = \rho(g), i.e. [f(x)] = [g(x)], \forall x \in X$ .Suppose that e is the identity element of NG, then we have $[f(x)]e = [g(x)]e;\ \forall x \in X.$ It follows that $e(f(x)) = e(g(x));\ \forall x \in X\ . Hence, f(x) = (ef)(x) = e(f(x)) = e(g(x)) = g(x), \forall x \in X,$ and therefore $f=g$. We conclude that ρ is injective. As a consequence, ρ is an isomorphism.

3-New results

In this section, we introduce the new concepts in NG groups, particularly on a set that has three or four elements.

Examples3-1: There are 27 transformations maps from A ={1,2,3} to itself.

Trans(A) as: {(1,1,1), (1,1,2), (1,1,3), (1,2,1), (1,2,2), (1,2,3), (1,3,1), (1,3,2), (1,3,3), (2,1,1), (2,1,2), (2,1,3), (2,2,1), (2,2,2), (2,2,3), (2,3,1), (2,3,2), (2,3,3), (3,1,1), (3,1,2), (3,1,3), (3,2,1), (3,2,2), (3,2,3), (3,3,1), (3,3,2), (3,3,3)} and S3 ={(1,2,3), (2,3,1), (3,1,2), (1,3,2), (3,2,1), (2,1,3)}. There exits some groups that are subsets of Trans(X), but not subsets of the S3. The groups of order 2 are : NG_1={(1,1,3),(3,3,1)},

NG_2={(1,2,1),(2,1,2)},NG_3={(1,2,2),(2,1,1)},NG_4={(1,3,3),(3,1,1)},NG_5={(2,2,3),(3,3,2)},andNG_6={(2,3,2),(3,2,3)}.

Proposition 3-1[5]: Suppose that NG_1 and NG_2 are two NG-groups that are not a subset of S_3, then the union of NG and NG_2 and NG_1 is NG groups, and the intersection of NG_1 and NG_2 is not necessary to be NG-groups.

Definition 3-1: An NG that not a subset of S_3 is (left, right) regular if $\forall f \in NG \exists g \in NG \text{ such that } (f = gf^2, f = f^2 g)$ and is regular if there exists $g \in NG$ such that $f = fgf$.

Definition 3-2: An element f of NG is the anti-reflective of NG if there exists $g \in NG$ such that $fgf = g \text{ and } gfg = f$.

Definition 3-3. An NG-group is an anti-inverse anti-reflective if, for every$f \in NG$, there exists anti-reflective element$g \in NG$. By $\mathfrak{B}$ to denote the class of anti-reflective.

Definition3.3. A NG-groups is called anti quasi-seperative if $\text{for any } f, g \in NG, g^2 = (fg)^2 = (fg)(fg) \Rightarrow ff = f^2 = f$.

Example 3-2: Consider the example 3-1. Take NG= {(1,1,3),(3,3,1)} ;

If f=(1,1,3) and g=(3,3,1), then fgf = (1,1,3) (3,3,1) (1,1,3)= (3,3,1)=g and gfg= (3,3,1) (1,1,3) (3,3,1) = (1,1,3)=f. So, By definition 3-1, then NG is anti-reflective. But, f^2= (1,1,3) (1,1,3)=(1,1,3) and f^2g= (1,1,3)(3,3,1)=(3,3,1)$\neq f$ is not right And $g f^2$=(3,3,1) (1,1,3)=(3,3,1)=g. not left. Moreover, $fgf = g \neq f$ is not regular. However, NG is anti quasi-seperative.

Remark 3.1- All NG groups that are not subsets of S_3 are anti-reflective but necessarily regular but anti-quasi-seperative.

Proposition 3.2: Suppose that $NG \in \mathfrak{B}$, then $g \in$ NG $\Leftrightarrow((\forall g \in NG)(\exists f \in NG)(f^2 = g^2)$

Proof: Suppose that $NG \in \mathfrak{B}$, For all $f \in NG$, there exit anti-reflective element $g \in NG$, $f = \text{from prpostion } 2 - 2.$ $\text{since } f \text{ is anti} - \text{reflective element}, \text{then } (gfg) = f, fgf = g$. Thus, $f^2 = (gfg)f = g(fgf) = gg = g^2$.

Conversely, suppose that $f^2 = g^2$, from proposition $2 - 2, g \in$ NG.

Remark 3.2: Suppose that $NG \in \mathfrak{B}$, then $\exists g \in NG \Leftrightarrow((\forall g \in NG)(\exists f \in NG)(f = g^2)$

Proposition 3.3: Suppose that $NG \in \mathfrak{B}$ and $f \in$ NG, then there exist $g \in$ N $((\forall g \in NG)$

1) $(\exists f \in NG)(gf = f^3 g = fg)$.
2) $(\exists f \in NG)(f = f^5)$.
3) $(\exists f \in NG)(f^3 = f^7)$.

Proof:1) Suppose $NG\in\mathfrak{B}$, then g∈ NG. $gf=(fgf)(gfg)=f(gfg)fg=f(ff)g=f^3g$, from proposition 3.2. Moreover, $f^3g=f(f^2)g=ffg=f^2g=fg$.

2) $f=gfg=(f^3g)g\ from 1, then\ f^3g^2=f^3f^2\ from\ propostion\ 3.2\ then\ f=f^5$.

3) $f^3=f^2f=f^2f^5$from 2, $then=f^7$.

Remark 3-3. Note that $f=f^2=f^3=\cdots$

Proposition 3.3: Suppose that $NG\in\mathfrak{B}$ and $f\in$ NG, $g\in$ N ,then $((\forall\ g\in NG)\ (\exists f\in NG)(f=f^{n})$.

Proof using mathematical induction. For *n=1* , it is true.(Propstion2.2)

We assume it is true for *n=k* $((\forall\ g\in NG)\ (\exists f\in NG)(f=f^{k}))$

Now for *n=k+1*, $f^{k+1}=\left(f^kf\right)=ff^2=ff=f^2=f\ is\ true$.

Then it is true for n∈N. we complete the proof .

Remark 3.2. We denote with $\mathfrak{T}_{m,n}$ the class of NG groups for which holds$(\forall\ g\in NG)(\exists f\in NG)(f^{m}=g^{m})(f^{m}=(fg)^{(m)}(f^{n}=f$, where n=1,2,…).

Proposition 3.4: Suppose that $NG\in\mathfrak{B}$ and $f\in$ NG, then there exist $g\in$ NG, such that $((\forall\ g\in NG)(\exists f\in NG)((fg)^2=f^2)$.

Proof: $(fg)^2=(gf)(gf)=g(fgf)=gg=\ g^2=\ f^2\ from\ propostion\ 3-2$

Corollary 3.1. Suppose that $NG\in\mathfrak{B}$ and $f\in$ NG, then there exist $g\in$ NG, such that $((\forall\ g\in NG)(\exists f\in NG)((fg)^2=f)$.

Conclusion: The NG groups presented as anti-reflective groups. Particularly in this paper, we consider NG not subsets of S_3. Moreover, some properties have been studied by the new concepts.

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