High Accurate Investigation of the Nuclear Hydrodynamical Moment of Inertia

استنتاج عالي الدقة لعزوم القصور الذاتي النووي الهيدروديناميكي

د. عوض حمدي الشريف: أستاذ الرياضيات المشارك. كلية العلوم جامعة بنغازي

د. مجدي متولي أمين. أستاذ الرياضيات المشارك. كلية العلوم. العريش. جامعة العريش. مصر.

Dr. Awad. H. Elsharif: Department of Mathematics, Faculty of Science, Benghazi University, Libya
Email: ahelsharif @ yahoo.com
Dr. Magdy M. Amin: Department of Mathematics, Faculty of Science at El-Arish, El-Arish University, Egypt
Email: drmagdyamin@ yahoo.com

الملخص: طريقة تستنتج عزوم القصور الذاتي غير المحورية داخل إطار النموذج الهيدروديناميكي تم تطويرها لإعطاء صيغة عالية الدقة لعزوم القصور الذاتي الهيدروديناميكي، في هذه الاستنتاجات معامل التشوه من الدرجة الثالثة أخذت في الاعتبار.

الكلمات المكفتاحية: عزوم القصور الذاتي النووي، النموذج الهيدروديناميكي، عزوم القصور الذاتي غير المحورية.

abstract: A method for investigating the non-axial moments of inertia within the framework of the hydrodynamical model is developed to give a high accurate formula of the hydrodynamical moments of inertia. In these investigations the third order of the deformation parameter is taken into account.
Keywords: nuclear moments of inertia, hydrodynamic model, non-axial moments of inertia.

1. INTRODUCTION

The nuclear collective motion [1] is a topic of the nuclear structure theory some fifty years old which has grown steadily both in the sophistication of its theory and in the range of data to which it relates.

 Lying at the basis of the so-called generalized model of the nucleus was proposed by Bohr (1952) and Bohr and Mottelson (1953) [3-4]. The detailed theory of the collective excited states of the nuclei is described in the monograph by Davydov (1967) [5]. The generalized model is a synthesis of the shell and liquid drop models, which is the assumption that the individual nucleons move independently in a slowly varying self-consistent field. In this model as in the shell model, degrees of freedom associated with the motion of one nucleon or several weakly coupled nucleons in the self consistent field are taken into account. In the generalized model, as in the liquid drop model, collective degrees of freedom associated with change of the shape of the nucleus and of its orientation in space are taken into account. In the generalized model; it is usually assumed that the internal and collective motions are separable; they are therefore treated independently. The collective motion of the nucleons in nuclei reduces principally to deformations of the shape of the nuclei, with no change in their volume. In the case of spherical nuclei, the collective excitations correspond to vibrations of the surface of the nucleus of the nucleus about the equilibrium shape. In non-spherical nuclei, the collective excitations can be associated with the vibrations of the surface of the nucleus and with rotation of the nucleus in space.

Until now the description of the nucleus as a drop of an ideal incompressible liquid, we can show that, in the case of irrotational flow of this liquid, the hydrodynamical (irrotational) moments of inertia coincide with the effective one,

$$\mathfrak{I}_i^{hyd} = \mathfrak{I}_i^{eff} = \frac{3}{2\pi} Am\, \beta^2\, R_{sph}^2 sin^2 \left(\gamma - \frac{2\pi(i+1)}{3} \right), i \in \mathbb{Z}_3 \qquad (1.1)$$

Where β and γ are the axially, non-axially deformation parameters and R_{sph} is the radius of the nucleus if it were spherical having the same volume, respectively, $\mathbb{Z}_3 = \{0,1,2\}$. The groups of integers modulo 3, addition modulo 3.

In the present paper we have reinvestigated the hydrodynamical moments of inertia by considering the nucleus as a liquid in a non-axially ellipsoidal rotating container. To a certain extent, the formula resulting from these investigations may introduce high accurate calculation of the hydrodynamical moments of inertia that takes into account the deviation from the effective moment of inertia.

2. THE HYDRODYNAMICAL MOMENTS OF INERTIA

The moments of inertia of the nucleus $\mathfrak{I}_i$ is defined as the ratio between the rotational angular momentum $\hbar \mathfrak{R}_i$ and the angular frequency ω_i^f

$$\mathfrak{I}_i = \frac{\hbar \mathfrak{R}_i}{\omega_i^f}, \qquad i \epsilon \mathbb{Z}_3 \tag{2.1}$$

To determine the moments of inertia $\mathfrak{I}_i$, it is necessary to make use of specific ideas on the internal structure of the nucleus. As limiting cases of the rotating model of the nucleus, we can consider the rotation of a solid and the potential motion of an ideal liquid in rotating shell. The difference between these limiting cases is very clearly manifested in the example of the rotation spherical system.

In the case of the solid, on going over to spherically symmetric case the moment of inertia tends to finite value, namely, the moment of inertia of the solid sphere.

In the case of the hydrodynamic model, in the rotation of the spherical shell the velocity of each point on the surface q_s on the surface is directed along a tangent, and the normal component of the velocity is equal to zero. Therefore, only a liquid at rest will satisfy the equations of motion of ideal hydrodynamics in such a vessel (container), and consequently, the moment of inertia of the system will be equal to zero.

If the shell has non-spherical shape, the normal component of the velocity at the surface will be non zero ($q_n \neq 0, q_n = \vec{q} \cdot \vec{n} = \vec{q}_s \cdot \vec{n}$) and the liquid will be dragged along the shell rotates.

The energy of rotation for a given angular velocity will be greater the more the shape of the shell differs from that of the sphere.

We shall represent the nucleus in the form of an ellipsoid of revolution, with semi-axis of lengths [1-2]:

$$R_i = R_{sph} \; exp(\sqrt{\frac{5}{4\pi}} \beta \cos\left(\gamma - \frac{2\pi}{3}(i+1)\right)), \quad i \in \mathbb{Z}_3 \tag{2.2}$$

If the nucleus regarded as a solid, whose components of moment of inertia are calculated according to the well-known formula for mechanics:

$$\mathfrak{I}_i^{rig} = \frac{Am}{5}(R_{i+1}^2 + R_{i+2}^2), i \in \mathbb{Z}_3 \tag{2.3}$$

Consequently, Am is the mass of the ellipsoidal nucleus whose equation is given by

$$(\sum_{i=0}^{2} \frac{x_i^2}{R_i^2} = 1)$$

.

We now determine the moment of inertia for the potential motion of an ideal liquid in an ellipsoidal vessel having the shape of the nucleus. The simplest assumption about the flow pattern of the fluid we can make is that

incompressible, it means that the density inside the nucleus is constant ($\rho^{\bullet} = 0$), and we get from the equation of continuity ($\nabla \bullet \vec{q} = 0$). The next assumption is that of irrotationality or vortex-free flow ($\nabla \wedge \vec{q} = 0$), but for the moment let us take the velocity $\vec{q}$ of the liquid is given by the gradient of the potential Φ. The potential Φ satisfying Laplace's equation

$$\nabla^2 \Phi = 0 \tag{2.4}$$

In the case of an ideal liquid, the boundary conditions reduce to the requirement that the normal component of the liquid at the surface ($\vec{q} \bullet \vec{n}$) coincide with the normal component of the velocity of the vessel wall($\vec{q}_s \bullet \vec{n}$) [5]. The velocity of a point with position vector $\vec{r} \equiv (X_0, X_1, X_2)$ at the surface of the rotating vessel is equal tois the angular $\vec{\omega} \equiv (\omega_0, \omega_1, \omega_2)$, where $\vec{q}_s = \vec{\omega} \wedge \vec{r}$ velocity of the vessel.

The unit normal to the surface of the ellipsoid is characterized by the components

$$(n_i = \frac{X_i}{cR_i^2}, i \in \mathbb{Z}_3) \text{ where } c = \sqrt{\sum_{i=0}^2 (\frac{X_i}{R_i^2})^2} .$$

We write the condition that the normal components of the velocities $\vec{q}$ and $\vec{\omega} \wedge \vec{r}$ be equal at the surface in the form

$$\sum_{i=0}^2 \frac{X_i}{R_{i+1}^2} \left\{ \frac{\partial \Phi}{\partial X_i} - X_{i+2}\omega_{i+1} + X_{i+1}\omega_{i+2} \right\} = 0, \quad i \in \mathbb{Z}_3 \tag{2,5}$$

The last condition must be satisfied over the entire surface. We can satisfy this relation by choosing as a solution of the Laplace's equation the potential function in the form

$$\Phi = \sum_{i=0}^2 \Phi_i = \sum_{i=0}^2 A_i X_{i+1} X_{i+2}, \quad i \in \mathbb{Z}_3 \tag{2.6}$$

Where $A_i, i \in \mathbb{Z}_3$ are constants. Substituting this function in its boundary conditions (2.5) we have

$$\sum_{i=0}^2 \left\{ \frac{X_i}{R_i^2} \left(\frac{\partial \Phi_i}{\partial X_i} \right) + \frac{X_{i+1}}{R_{i+1}^2} \left(\frac{\partial \Phi_i}{\partial X_{i+1}} + X_{i+2}\omega_i \right) + \frac{X_{i+1}}{R_{i+1}^2} \left(\frac{\partial \Phi_i}{\partial X_{i+2}} - X_{i+1}\omega_i \right) \right\} = 0, i \in \mathbb{Z}_3 \tag{2.7}$$

he method of splitting the original problem into three simpler ones, is known as the method of superposition rule, it can be used here since the differential equation and the boundary conditions involved in the problem are all linear, so we can split our problem into three similar problem of an axially symmetric nuclei in the form of an ellipsoid of revolution are rotating about an axis perpendicular to the symmetry axis like as was proposed by Moszkowski (1957) [7-8], we have three of Laplace's equation

$$\nabla^2 \Phi_i = 0, \quad i \in \mathbb{Z}_3 \tag{2.8}$$

And the corresponding solutions as follows

$$\Phi_i = A_i X_{i+1} X_{i+2}, \quad i \in \mathbb{Z}_3 \tag{2.9}$$

Must satisfy the boundary conditions

$$\frac{X_i}{R_i^2} \left(\frac{\partial \Phi_i}{\partial X_i} \right) + \frac{X_{i+1}}{R_{i+1}^2} \left(\frac{\partial \Phi_i}{\partial X_{i+1}} + X_{i+2}\omega_i \right) + \frac{X_{i+1}}{R_{i+1}^2} \left(\frac{\partial \Phi_i}{\partial X_{i+2}} - X_{i+1}\omega_i \right) = 0, \quad i \in \mathbb{Z}_3 \tag{2.10}$$

Substituting from (2.9) in (2.10) we find the values of A_i

$$A_i = \left(\frac{R_{i+1}^2 - R_{i+2}^2}{R_{i+1}^2 + R_{i+2}^2}\right)\omega_i, \qquad i \in \mathbb{Z}_3 \tag{2.11}$$

We shall calculate the kinetic energy of the liquid in the rotating shell, this energy are equal to $\sum_{i=0}^{2} T_i$, where

$$T_i = \frac{\rho}{2}\int_\tau (q_i)^2 \, d\tau = \frac{\rho}{2}\int_\tau (\nabla\Phi_i)^2 \, d\tau = \frac{A_i^2}{2}\int_\tau \rho(X_{i+1}^2 + X_{i+2}^2) \, d\tau = \frac{A_i^2}{2}\mathfrak{I}_i^{rig}, \qquad i \in \mathbb{Z}_3$$

This can be rewritten as

$$T_i = \frac{1}{2}\left[\frac{Am\left(R_{i+1}^2 - R_{i+2}^2\right)^2}{5\left(R_{i+1}^2 + R_{i+2}^2\right)}\right]\omega_i^2 = \frac{1}{2}\mathfrak{I}_i^{hyd}\omega_i^2, \qquad i \in \mathbb{Z}_3 \tag{2.12}$$

It is apparent that the coefficient of ${\omega_i^2}/{2}$ is the hydrodynamical moments of inertia

$$\mathfrak{I}_i^{hyd} = \frac{Am\left(R_{i+1}^2 - R_{i+2}^2\right)^2}{5\left(R_{i+1}^2 + R_{i+2}^2\right)}, \qquad i \in \mathbb{Z}_3 \tag{2.13}$$

After a tedious but straightforward derivation, one can simplify equation (2.13) by substituting from (2.2), we deduce high accurate of the hydrodynamical moments of inertia

$$\mathfrak{I}_i^{hyd} = \frac{3}{2\pi}Am\,p^2 R_{sph}^2 \sin^2\left(\gamma - \frac{2\pi(i+1)}{3}\right) \times \left(1 + \beta\sqrt{\frac{5}{4\pi}}\cos\left(\gamma - \frac{2\pi(i+1)}{3}\right)\right), i \in \mathbb{Z}_3 \tag{2.14}$$

The difference between equations (1.1) and (2.14) is that the latter is pointed out the third order of the deformation parameter.

3. THE EFFECTIVE AND RIGID BODY MOMENTS OF INERTIA

In the unified model one assumes the nucleus as consists of an even-even core plus one or more nucleons moving in the shell model (distorted shell model, that is the shell model potential is assumed non-spherical) orbits of the potential produced by the core, and interacting with it (or them). The coupling of the external nucleons with the core may be weak, strong, or intermediate.

As we move away from closed shells we may expect nuclei to show firstly some vibrational features (associated with the change of the shapes of nuclei) characteristic of the motion of soft undeformed nuclei and then rotational features characteristic of the motion of permanently deformed nuclei. The kinetic energy for collective oscillations is given in terms of β and γ by

$$T = T_{vib} + T_{rot} = \frac{B_2}{2}\left(\left(\dot{\beta}\right)^2\gamma^2 + (\dot{\gamma})^2\beta^2\right) + \sum_i \frac{\hbar^2}{2\mathfrak{I}_i}\mathfrak{R}_i^2, \tag{3.1}$$

Where $B_2 = \frac{3}{8\pi}AmR_{sph}^2$ is the mass coefficient. The effective moments of inertia are given by equation (1.1). In the case of an axially-symmetric (x_2 axis is an axis of symmetry) nucleus, $\gamma = 0$ or $\gamma = \pi$ the effective moments of inertia are

$$(\mathfrak{I}_i^{eff})_{sym} = \mathfrak{I}_{sym}^{eff} = 3\beta^2 B_2 = \frac{9}{8\pi}Am\beta^2 R_{sph}^2, \ i = 0,1, \ \ \mathfrak{I}_2^{eff} = 0$$

Equation (1.1) can be rewritten as

$$\mathfrak{I}_i^{eff} = \frac{4}{3}\mathfrak{I}_{sym}^{eff}sin^2\left(\gamma - \frac{2\pi(i+1)}{3}\right), i \in \mathbb{Z}_3 \tag{3.2}$$

The rigid body moments of inertia of an ellipsoid of revolution with the same deformation (2.2) are

$$\frac{\mathfrak{I}_i^{rig}}{\mathfrak{I}_s^{rig}} = \left[1 - \beta\sqrt{\frac{5}{4\pi}}\ cos\left(\gamma - \frac{2\pi(i+1)}{3}\right)\right], i \in \mathbb{Z}_3 \tag{3.3}$$

Where $\mathfrak{I}_s^{rig} = \frac{2}{5}AmR_{sph}^2$ is the moment of inertia of a rigid sphere of radius R_{sph}.

4. RESULTS AND CONCLUSION

Recalling, the two expressions for the hydrodynamical moments of inertia (2.14) and the effective moments of inertia (1.1), we can find the relation between them is given by the expression

$$\frac{\mathfrak{I}_i^{hyd}}{\mathfrak{I}_i^{eff}} = \left[1 + \beta\sqrt{\frac{5}{4\pi}}\ cos\left(\gamma - \frac{2\pi(i+1)}{3}\right)\right], i \in \mathbb{Z}_3 \tag{4.1}$$

However, the hydrodynamical moments of inertia computed in this expression are not equal to the effective moments of inertia of the nucleus.

A more systematic investigation can be carried out by adding the ratio (4.1) to the ratio (3.3), thus one writes

$$\frac{\mathfrak{I}_i^{hyd}}{\mathfrak{I}_i^{eff}} + \frac{\mathfrak{I}_i^{rig}}{\mathfrak{I}_s^{rig}} = 2, i \in \mathbb{Z}_3 \tag{4.2}$$

The relationship connecting of two terms have simple physical interpretation and so the formula gives a useful representation of some overall nuclear properties of different moments of inertia.

We offer a simple demonstration by making a comparison of the investigated expression of the ratio (4.1) with the ratio (3.3), we must assume that the first term in (4.2) is equal to unity (the hydrodynamical moments of inertia coincide with the effective moments of inertia) this leads to the rigid body moments of inertia is coincide with the moment of inertia of a rigid sphere which means that the nucleus not deformed that contradict with the our assumption.

For the axially symmetric shapes ($\gamma = 0$ and $\beta > 0$), the nucleus is a prolate ellipsoid of revolution, with semi-axes of lengths $(R_i)_{\gamma=0}$ or $\gamma = \pi$ and $\beta > 0$, the nucleus is an oblate ellipsoid of revolution, with semi axes of lengths $(R_i)_{\gamma=\pi}$, the relation (4.2) can be rewritten in this case as

$$\frac{(\mathfrak{I}_i^{hyd})_{sym}}{(\mathfrak{I}_i^{eff})_{sym}} + \frac{(\mathfrak{I}_i^{rig})_{sym}}{(\mathfrak{I}_s^{rig})_{sym}} = 2, \qquad i = 0,1,$$

and

$$(\mathfrak{I}_2^{hyd})_{sym} = (\mathfrak{I}_2^{eff})_{sym} = (\mathfrak{I}_2^{rig})_{sym} = 0, \tag{4.3}$$

References
[1] P. Ring and P. Schuck. The Nuclear Many-Body Problem, Springer Verlag, N. Y. (1980).
[2] S. B. Doma and M. M. Amin. International Journal of Applied Mathematics , vol. 25, No. 6 (2000).
[3] A. Bohr. Mat. Fys. Medd. Dan. Vid. Selsk., **26**, No. 14 (1952).
[4] A. Bohr and B. R. Mottelson Mat. Fys. Medd. Dan. Vid. Selsk., **27,** No. 16(1953).
[5] A. S. Davydov. Excited States of Atomic Nuclei, Atomizdat, Moscow (1967).
[6] L. D. Landau and E. M. Lifshitz, Fluid Mechanics, pergamon press, N.Y (1987).
[7] S. A. Moszkowski, Models of nuclear structure, Encyclopadia of Phys. Springer- Verlag, Berlin. Vol. **39**, (1957).
[8] J.M. Eisenberg and W. Greiner. Nuclear Models. North Holand, Amsterdam, (1970).