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The Hyperreals enlargement sets and its application on compact topology

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ABSTRACT

This paper presents a study of hyperreal numbers and their relations to real numbers. The hyperreals are a number system extension of the real number system. With this system, simpler and more intuitively natural definition (topological notions of special points and compactness), proofs (Heine-Borel theorem), and new concepts (limited, unlimited numbers, enlargement sets, and halos) of mathematical interest are offered.

Keywords: Hyperreal numbers, infinitesimal, ultrafilter, enlargement set, nonstandard elements, halo, compact topology.

1- Introduction

In 1960, Abraham Robinson (1918–1974) successfully addressed the longstanding issue of providing a rigorous foundation for calculus based on the concept of infinitesimals.

This breakthrough by Robinson stands as one of the most significant mathematical advancements of the twentieth century.

The construction of the hyperreal number system relied on algebraic techniques, which involved selecting an arbitrary parameter like an ultrafilter.

The process of constructing these hyperreal numbers shares similarities with Cantor's construction of real numbers using Cauchy sequences of rational numbers.

A bigger set $\mathbb{R}^{\mathbb{N}}$ (the set of all infinite sequences of real numbers) is needed to be created. A typical member of $\mathbb{R}^{\mathbb{N}}$ has the form $A = \langle a_1, a_2, a_3, \dots, a_n, \dots \rangle$, where $a_1, a_2, a_3, \dots \in \mathbb{R}$.

The operations addition and multiplication were defined on it, as

$$A + B = \langle \dots, a_i, \dots \rangle + \langle \dots, b_i, \dots \rangle = \langle \dots, a_i + b_i, \dots \rangle$$
$$A.B = \langle \dots, a_i . b_i, \dots \rangle$$

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1.1.Theorem

This structure $(\mathbb{R}^{\mathbb{N}}, +, \cdot)$ forms a commutative ring.

Theorem. the relation $\cong$ on $\mathbb{R}^{\mathbb{N}}$ defined by

if and only if $\{n \in \mathbb{N}: a_n = b_n\} \in \mathcal{F}$ $A \cong B$ is an equivalence relation. where $\mathcal{F}$ is a non-principal ultrafilter on $\mathbb{N}$. Which means that the two sequences: agree at almost all n.

Definition. The set of hyperreal numbers $^*\mathbb{R}$ will be the quotient set $\mathbb{R}^{\mathbb{N}} /\cong$.

$$^*\mathbb{R} = \{[A]: A \in \mathbb{R}^{\mathbb{N}}\} = \{\mathcal{A}: A \in \mathbb{R}^{\mathbb{N}}\} = \mathbb{R}^{\mathbb{N}} /\cong$$

Theorem. This structure $\langle {}^*\mathbb{R}, +, \cdot, \leq \rangle$ is an ordered field. Where

$$\mathcal{A} + \mathcal{B} = [A + B] = [\langle \dots, a_n + b_n, \dots \rangle]$$
$$\mathcal{A}.\mathcal{B} = [A.B] = [\langle \dots, a_n . b_n, \dots \rangle].$$

if and only if $[\![a_n \leq b_n]\!] \in \mathcal{F}$ if and only if $\mathcal{A} \leq \mathcal{B}$

$$\{n \in \mathbb{N}: a_n \leq b_n\} \in \mathcal{F}$$

Remark

Any constant sequence $r = [\langle r, r, r, \dots \rangle]$ identify a real number.

The number $\varepsilon = \left[\left\langle 1, \frac{1}{2}, \frac{1}{3}, \dots \right\rangle\right]$, $\varepsilon \neq 0 = \langle 0,0,0, \dots \rangle$ is called an infinitesimal since $\varepsilon < x, \forall x \in \mathbb{R}$. Hence $\varepsilon \notin \mathbb{R}$.

The number $\mathcal{N} = [\langle 1,2,3, \dots \rangle]$, is called an unlimited since $\mathcal{N} > x, \forall x \in \mathbb{R}$. Hence $\mathcal{N} \notin \mathbb{R}$.

The properties observed of ε and $\mathcal{N}$ shows that $^*\mathbb{R}$ is a proper extension of $\mathbb{R}$, and hence a new structure.

Our discussion of ε and $\mathcal{N}$ shows in fact that if A is any real-valued sequence converging to zero, then A is an infinitesimal in $^*\mathbb{R}$, while if A diverges to $\pm\infty$ then A is unlimited in $^*\mathbb{R}$. The elements of $^*\mathbb{R} - \mathbb{R}$ are called nonstandard elements.

The relation between $\mathbb{R}$ and $^*\mathbb{R}$ is demonstrated by transfer principle.

1.2.Theorem (Łoś's theorem)

If $\varphi(x_1, x_2, \dots)$ is a first-order formula in the language of the ordered field $\mathbb{R}$ and $r_1, \dots r_n \in \mathbb{R}$, then

$$\mathbb{R} \vDash \varphi[r_1, \dots, r_n] \iff {}^*\mathbb{R} \vDash \varphi[r_1, \dots, r_n]$$

2- Enlarging Sets

2.1.Definition

For each $A \in \mathbb{R}^{\mathbb{N}}$, the enlargement set $^*X$ is

$$[A] \in {}^*X \text{ if and only if } \{n \in \mathbb{N} : a_n \in X\} \in \mathcal{F}.$$

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Thus declaring, by the almost-all criterion, that $[A]$ is in $^*X$ if and only if a_n is in X for almost all n.

2.1.Examples

1- If $A = \mathbb{N}$, then $\mathcal{N} \in {}^*\mathbb{N}$ since $\mathcal{N} = \langle 1,2,3, \ . \ . \ . \rangle$ and the elements of this sequence are all natural numbers belongs to $\mathbb{N}$. Hence $\mathcal{N} \in {}^*\mathbb{N} - \mathbb{N}$. Then $^*\mathbb{N}$ consists of $\mathbb{N}$ together with all positive unlimited elements.

2- If $A = \mathbb{Q}$, then ε and $\mathcal{N}$ belongs to $^*\mathbb{Q} - \mathbb{Q}$, since the elements of these sequences are all rational. But we note that $^*\mathbb{Q} \neq {}^*\mathbb{R}$ since $\pi = \langle \pi, \ \pi, \ \pi, \ \ \dots \rangle \notin {}^*\mathbb{Q}$.

3- If $A = \mathbb{Q}^{`}$, then $\varepsilon + \pi$ and $\mathcal{N} - 2e$ belongs to $^*\mathbb{Q}^{`} - \mathbb{Q}^{`}$, since the elements of these sequences are all irrational. Also we note that $^*\mathbb{Q}^{`} \neq {}^*\mathbb{R}$ since $2 = \langle 2,2,2, \dots \rangle \notin {}^*\mathbb{Q}^{`}$.

4- If $A = \mathbb{R}$, then for example $\varepsilon, r + \ \varepsilon$ and $\mathcal{N}$ belongs to $^*\mathbb{R} - \mathbb{R}$. Then these elements are "nonstandard real numbers". Then $^*\mathbb{R}$ consists of $\mathbb{R}$ together with all non-standard elements. And $^*\mathbb{R}$ is the enlargement of $\mathbb{R}$. By transfer principle $^*\mathbb{R} = {}^*\mathbb{Q} \cup \ {}^*\mathbb{Q}^{`}$.

5- If $A = (0, 2]$, then $\varepsilon \in {}^*A$, since $\varepsilon = \langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle$ and every element of this sequence belongs to A. Also $1 + \varepsilon \in {}^*A$ since $1 + \varepsilon = \langle 2, \frac{3}{2}, \frac{4}{3}, \dots \rangle$ and every element in this sequence belongs to A. Then $^*A$ consists of A together with all non-standard elements $\{x \in {}^*\mathbb{R}: 0 < x \leq 2\}$. For example $\varepsilon, 1 + \varepsilon \ \in {}^*A - A$, and $^*A = {}^*(0, 2]$.

6- If $A = \{1, 2, 3,4\}$, then any sequence of a non-standard element intersect with A is a finite set, Hence there are no nonstandard elements inside it, and $^*A = A$.

2.1.Theorem (Gordon, Kusraer, & Kutatelodzo)

The enlargement $^*X$ of any infinite subset X of $\mathbb{R}$ has nonstandard elements.

2.2.Theorem

If A is finite, then $^*A = A$ and hence A has no nonstandard members.

2.1.Proposition

. $X \subseteq Y$ if and only if $^*X \subseteq {}^*Y$

Corollary 2.1

if and only if $^*X = {}^*Y \ X = Y$

2.2.Proposition

1- $^*(X \cup Y) = {}^*X \cup {}^*Y$
2- $^*(X \cap Y) = {}^*X \cap {}^*Y$
3- $^*\emptyset = \emptyset$

2.3.Proposition

If $A \subseteq \mathbb{R}$, then $^*A \cap \mathbb{R} = A$.

3.Halos

3.1.Definition

A hyperreal a is infinitely close to b, dented by $a \simeq b$, if $a - b$ is infinitesimal.

This defines an equivalence relation on $^*\mathbb{R}$, (Rayo, 2015).

3.2.Definition

The halo of a is the $\simeq$-equivalence class

hal$(a) = \{b \in {}^*\mathbb{R}: a \simeq b\}$

So, a is infinitesimal if and only if $a \simeq 0$.

Thus hal$(0) = \mathbb{I}$, the set of infinitesimal.

3.1.Proposition

$$\text{hal}(a) = \{a + \varepsilon: \varepsilon \in \text{hal}(0)\}$$

4.Topological Spaces

In the hyperreal context of usual topology on $\mathbb{R}$ we can make the idea of nearness quite explicit by taking near to mean infinitely close . As we shall see, this leads to a very natural formulation and treatment of many topological ideas.

4.1.**Definition**

If $A \subseteq \mathbb{R}$, then

1. A is open if and only if for all $r \in A$, if x is infinitely close to r, then $x \in {}^*A$;
2. A is closed if and only if for all real r, if r is infinitely close to some $x \in {}^*A$, then $r \in A$.

4.1.Theorem (Goldblat, 1998)

If $A \subseteq \mathbb{R}$ and $r \in \mathbb{R}$,

(1) r is interior to A if and only if $r \simeq x$ implies $x \in {}^*A$, if and only if hal$(r) \subseteq {}^*A$.

(2) r is a limit point of A if and only if there is an $x \neq r$ such that $r \simeq x \in {}^*A$, if and only if hal$(r) \cap {}^*A$ contains a point other than r.

(3) r is a closure point of A if and only if r is infinitely close to some $x \in {}^*A$, if and only if hal$(r) \cap {}^*A$ is nonempty.

Compactness

4.2.Definition

A set $B \subseteq \mathbb{R}$ is compact if every open cover of B has a finite subcover, if whenever $B \subseteq \bigcup_{i \in I} A_i$ and each A_i is open in $\mathbb{R}$, then there is a finite $J \subseteq I$ such that $B \subseteq \bigcup_{i \in I} A_i$.

This concept did not simply arise out of thin air. It originated from research conducted in the nineteenth century on bounded and closed intervals in the real line, which eventually led to the proof that such intervals are compact according to the definition provided by the Heine-Borel theorem. As this definition solely pertains to open sets, it becomes the suitable one to employ in an abstract topological space where there is no concept of numerical distance to determine boundedness.

4.2.Theorem (Robinson's compactness criterion).

B is compact subset of $\mathbb{R}$ if and only if

1- Every $x \in {}^*B, x \simeq r$, for some $r \in B$.

2-${}^*B \subseteq \bigcup_{r \in B} hal(r)$.

3-If $x \in {}^*B$ then $sh(x) \in B$.

4.1.Proposition

The three definitions are equivalent.

Proof

From 1 to 2

let $x \in {}^*B \; \exists \; r \in B$ such that $x \simeq r$, then $x \in hal(r)$, hence $\Rightarrow$

$x \in \bigcup_{r \in B} hal(r)$.

$\Leftarrow$let $x \in {}^*B$, then $x \in \bigcup_{r \in B} hal(r)$, hence $x \in hal(s)$ for some $s \in B$.

From 1 to 3

$\Rightarrow$let $x \in {}^*B$ and let x is not limited, then $\exists$ no $y \in B$ such that $x \simeq y$ contradiction, then x is limited and $sh(x) = y \in B$.

let $x \in {}^*B$, then $sh(x) \in B$, hence $x \simeq y$ and $y \in B$. $\Leftarrow$

This criterion gives an intuitively appealing and useful characterisation of the notion of compactness. Constructions involving open covers are replaced by elementary reasoning about hyperreal points. For instance:

1- The open interval $(1,3) \subseteq \mathbb{R}$ is not compact, because $1 + \varepsilon \in {}^*(1,3)$ as $1 < 1 + \varepsilon < 3$, but $1 + \varepsilon$ is not infinitely close to any member of $(1,3)$ because its shadow is $1 \notin (1,3)$.

2- Any closed interval $[a, b] \subseteq \mathbb{R}$ is compact, because if $x \in {}^*[a, b]$, then $a \leq x \leq b$, so x is limited and its shadow r must also satisfy $a \leq r \leq b$. Thus $x \simeq r \in [a, b]$.

3- Any finite set is compact, because if B is finite, then ${}^*B = B$, so each member of ${}^*B$ is infinitely close to itself in B.

4- If $B \subseteq \mathbb{R}$ is unbounded above, in the sense that

$(\forall x \in \mathbb{R})(\exists y \in B)(x < y)$, Then B cannot, be compact: taking any unlimited $x \in {}^*\mathbb{R}$, by transfer there exists $y > x$ with $y \in {}^*B$. Then y is unlimited, so cannot be infinitely close to any member of B. Similarly, B cannot be compact if it is unbounded below.

Altogether then, a compact set must be bounded above and below.

5- If B is not closed, then B cannot be compact: it must have a closure point r that does not belong to B. As a closure point, r is infinitely close to some $x \in {}^*B$. But then x is not infinitely close to any member of B, since $\mathrm{sh}(x) = r \notin B$. Hence a compact set must be closed.

The Proof of Robinson's Criterion can be found in (Goldblat, R.,1998).

4.3.Theorem (Heine-Borel) (Habil, & Ghneim, 2015)

A set $B \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof

It has already been seen that if B satisfies Robinson's criterion, then it is closed and bounded (above and below).

Conversely, if B is closed and bounded, then there is some real b such that

$$(\forall x \in B)(|x| \leq b).$$

Now, to proof Robinson's criterion, suppose $x \in {}^*B$. Then by transfer, $|x| \leq b \in \mathbb{R}$. Hence x is limited, and so has a shadow $r \in \mathbb{R}$. Then $r \simeq x \in {}^*B$, and so $r \in B$ because B is closed. Thus it has been shown that x is infinitely close to the member r of B, proving that B is compact.

Robinson's criterion can be formulated for an abstract compact topological space, and it provides an elegantly simple proof of Tychonoff's theorem, which states that the product of compact spaces is also compact.

Conclusion

This paper presents the relation between the hyperreal numbers and the real numbers, the enlargement sets which are the extension of the sets in the real number system and the halos ware used to gave a more intuitively natural definitions of special points of topological spaces and compactness and provide a simpler proofs.

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