Subclass of Starlike functions of complex order defined by a generalized Srivastava– Attiya operator

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الفئة الفرعية من الدوال نجمي للنظام المعقدة يحددها المشغل المعمم سريفاستافا – عطية

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ملخص

المؤلف [1] و[2] عرف عائلة جديدة لمشغلي تكاملي معمم حيث تمّ تعريفه عن طريق الدالة العامة هورويتز يرش زيتا باستخدام نهج مماثل قدمه سريفاستافا و عطية (2007) المعرفة على فئة من الدوال التحليلية طبيعية في القرص الوحدة المفتوح .

وجعل الدراسة ممتعة لهذا المشغل من قبل العديد من الباحثين وهم Srivastava , Owa Alexander وغيرهم من الباحثين . مع هذا المشغل قمنا بتعريف فئة فرعية جديدة من الدوال نجمي لنظام المعقد مع معاملات السالبة المحددة في القرص الوحدة المفتوح والذي يرمز له بالرمز $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$ والتحقيق من خصائصها المختلفة ، وعلاوة على ذلك نحصل على معامل عدم المساواة ، والنقاط المتطرفة ، والنمو والنظريات التشويه وأنصاف أقطار التحدب لدوال التي تنتمي إلى هذه الفئة فرعية .

Abstract

The main object of this paper is to study some properties of certain subclass of starlike functions of complex order with negative coefficients denoted by $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$ defined by a general integral operator in the open unit disc and investigate its various characteristics. Further, we obtain coefficient inequalities, extreme points, growth and distortion property and radii of close−to−convexity, starlikeness and convexity for functions belonging to the subclass $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$.

1 Introduction

Let A denote the class of all analytic functions in the open unit disc

$$\mathrm{U}=\{z\in\mathrm{C}:|z|<1\},$$

given by the normalized power series

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k,\quad (z\in\mathrm{U})\quad, \tag{1.1}$$

where a_k is a complex number.

For functions $f\in A$ given by (1.1) and $g\in A$ given by $g(z)=z+\sum_{k=2}^{\infty}b_k z^k$, we define the Hadamard product (or convolution) of f and g by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k.$$

The authors [1] and [2] introduce a general integral operator $\Im_{s,b}^{\alpha}f(z)$ which is defined by means of a general Hurwitz Lerch Zeta function defined on the class of normalized analytic functions in the open unit disc by using the similar approach Srivastava and Attiya operator [7]. This operator is motivated by many researchers namely Owa and Srivastava Alexander and many others. as we will show in the following:

Definition 1.1 *(Srivastava and Choi [4]) A general Hurwitz–Lerch Zeta function* $\Phi(z,s,b)$ *defined by*

$$\Phi(z,s,b)=\sum_{k=0}^{\infty}\frac{z^k}{(k+b)^s},$$

where $(s\in\mathrm{C},\ b\in\mathrm{C}-\mathrm{Z}_0^-)$ *when* $(|z|<1)$, and $(\Re(b)>1)$ *when* $(|z|=1)$.

Note that:

$$\Phi^{*}(z,s,b)=(b^{s}z\,\Phi(z,s,b))*f(z)=z+\sum_{k=2}^{\infty}\frac{b^{s}}{(k+b-1)^{s}}a_{k}z^{k}.$$

Owa and Srivastava [5] introduced the operator $\Omega^{\alpha}:\mathrm{A}\to\mathrm{A}$, which is known as an extension of fractional derivative and fractional integral as follows:

$$\Omega^{\alpha}f(z)=\Gamma(2-\alpha)z^{\alpha}D_{z}^{\alpha}f(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}a_{k}z^{k}\qquad(\alpha\neq 2,3,4,\ldots),$$

where $D_{z}^{\alpha}f(z)$ the fractional derivative of f of order α (see [6]).

For $s\in\mathrm{C}$, $b\in\mathrm{C}-\mathrm{Z}_{0}^{-}$, and $0\le\alpha<1$, the generalized integral operator $(\Im_{s,b}^{\alpha}f):\mathrm{A}\to\mathrm{A}$ is defined by

$$\Im_{s,b}^{\alpha}f(z)=\Gamma(2-\alpha)z^{\alpha}D_{z}^{\alpha}\Phi^{*}(z,s,b),\quad(\alpha\neq 2,3,4,\ldots)$$

$$=z+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left(\frac{b}{k-1+b}\right)^{s}a_{k}z^{k},\quad(z\in\mathrm{U}).$$

Note that : $\Im_{0,b}^{0}f(z)=f(z)$.

Special cases of this operator includes:

- $\Im_{0,b}^{\alpha}f(z)\equiv\Omega^{\alpha}f(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}a_{k}z^{k}$ is Owa and Srivastava operator [5].
- $\Im_{s,b+1}^{0}f(z)\equiv J_{s,b}f(z)=z+\sum_{k=2}^{\infty}\left(\frac{b+1}{k+b}\right)^{s}a_{k}z^{k}$ is Srivastava and Attiya integral operator [7].
- $\Im_{1,1}^{0}f(z)\equiv A(f)(z)=\int_{0}^{z}\frac{f(t)}{t}dt=z+\sum_{k=2}^{\infty}\frac{1}{k}a_{k}z^{k}$, is Alexander integral operators [8].
- $\Im_{1,2}^{0}f(z)\equiv L(f)(z)=\frac{2}{z}\int_{0}^{z}f(t)dt=z+\sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)a_{k}z^{k}$, is Libera integral operators [9].
- $\Im_{1,a+1}^{0}f(z)\equiv L_{a}(f)(z)=\frac{a+1}{z^{a}}\int_{0}^{z}t^{a-1}f(t)dt=z+\sum_{k=2}^{\infty}\left(\frac{a+1}{k+a}\right)a_{k}z^{k}$, $a>-1$, is Bernardi integral operator [10].

- $\Im_{\sigma,2}^{0} f(z) \equiv I^{\sigma} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1} \right)^{\sigma} a_k z^k$, is Jung– Kim– Srivastava integral operator [11].

It is easily verified from the above definition of the operator $\Im_{s,b}^{\alpha} f(z)$ that:

$$z(\Im_{s+1,b}^{\alpha} f(z))' = (1-b)\Im_{s+1,b}^{\alpha} f(z) + b\Im_{s,b}^{\alpha} f(z).$$

Making use of our operator, we introduce a new subclass of analytic functions with negative coefficients, and discuss some properties of geometric function theory in relation to this subclass.

For $(-1 \le \delta < 1)$, $(\beta > 0)$ and $(\gamma \in \mathrm{C} - 0)$ we let $S_{s,b}^{\alpha}(\delta, \beta, \gamma)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re\left\{1 + \frac{1}{\gamma} \frac{z(\Im_{s,b}^{\alpha} f(z))'}{\Im_{s,b}^{\alpha} f(z)} - \delta\right\} \ge \beta \left| 1 + \frac{1}{\gamma} \frac{z\,\Im_{s,b}^{\alpha} f'(z)}{\Im_{s,b}^{\alpha} f(z)} - 1 \right| \quad (z \in \mathrm{U}), \tag{1.2}$$

We further let

$$TS_{s,b}^{\alpha}(\delta, \beta, \gamma) = S_{s,b}^{\alpha}(\delta, \beta, \gamma) \cap T,$$

where

$$T := \left\{ f \in A : f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \text{where } a_k \ge 0, \text{ for all k} \ge 2 \right\}, \qquad (1.3),$$

is a subclass of A introduced and studied by Silverman [12]. By given some specific values to $\alpha, \beta, \gamma, \delta$ and s in the class $TS_{s,b}^{\alpha}(\delta, \beta, \gamma)$ we obtain the following important subclasses studied by various researchers in earlier works.

1− For $s=\mu, \quad \alpha=0$ we obtain the class of functions f satisfying the condition

$$\Re\left\{1+\frac{1}{\gamma}\frac{z\,(J_b^{\mu}f\,(z))'}{J_b^{\mu}f\,(z)}-\delta\right\}\geq\beta\left|1+\frac{1}{\gamma}\frac{zJ_b^{\mu}f\,'(z)}{J_b^{\mu}f\,(z)}-1\right|$$

studied by G. MURUGUSUNDARAMOORTHY and K. UMA see [3]

2− For $s=\alpha=\beta=0, \quad \delta=1$ and $\gamma=b$ we obtain the class of starlike functions of order b satisfying the condition

$$\Re\left\{1+\frac{1}{b}\frac{z\,(f\,(z))'}{f\,(z)}-1\right\}\geq 0$$

studied by Nasr and Aouf see[2]

3− For $s=\alpha=\beta=0, \quad \delta=1$ and $\gamma=1$ we obtain the class of starlike satisfying the condition

$$\Re\left\{\frac{z\,(f\,(z))'}{f\,(z)}\right\}\geq 0$$

studied by Alexander see[3].

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bound, extreme points, radii of close to convexity, starlikeness and convexity .

2 Coefficient Inequalities

In the following section, we obtain coefficient bounds for the class $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$ that shall be used in the next theorem.

Theorem 2.1 *Let the function* f *be defined by (1.3). Then* $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$ *if and only if*

$$\sum_{k=2}^{\infty}[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right||a_k|\leq(1-\delta)+|\gamma|(1-\beta). \qquad (2.1)$$

The result is sharp for the function

$$f(z)=z-\frac{(1-\delta)+|\gamma|(1-\beta)}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|(\frac{b}{k-1+b})^{s}\right|}z^{k}. \qquad (2.2)$$

Proof: Assume that the inequality (2.1) holds and let $|z|<1$. Then by hypothesis, we have

$$\Re\left\{1+\frac{1}{\gamma}\frac{z(\Im_{s,b}^{\alpha}f(z))'}{\Im_{s,b}^{\alpha}f(z)}-\delta\right\}-\beta\left|1+\frac{1}{\gamma}\frac{z\Im_{s,b}^{\alpha}f'(z)}{\Im_{s,b}^{\alpha}f(z)}-1\right|.$$

We note that

$$1+\frac{1}{|\gamma|}\frac{(1-\delta)-\sum_{k=2}^{\infty}(k-\delta)\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right||a_k||z^{k-1}|}{1-\sum_{k=2}^{\infty}\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right||a_k||z^{k-1}|}-$$

$$\beta\left(1-\frac{1}{|\gamma|}\frac{\sum_{k=2}^{\infty}(k-1)\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right||a_k||z^{k-1}|}{1-\sum_{k=2}^{\infty}\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right||a_k||z^{k-1}|}\right),$$

which implies

$$1+\frac{1}{|\gamma|}\frac{(1-\delta)-\sum_{k=2}^{\infty}(k-\delta)\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right||a_k|}{1-\sum_{k=2}^{\infty}\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right||a_k|}-$$

$$\beta\left(1-\frac{1}{|\gamma|}\frac{\sum_{k=2}^{\infty}(k-1)\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|\,|a_k|}{1-\sum_{k=2}^{\infty}\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|\,|a_k|}\right)\geq 0\,,$$

that is,

$$\sum_{k=2}^{\infty}[(k+|\gamma|\,)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|\,|a_k|\;\leq(1-\delta)+|\gamma|(1-\beta)\,.$$

Hence, $f\in TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$.

In order to prove the sufficiency, assume that $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$.

$$\Re\left\{1+\frac{1}{\gamma}\frac{z\,(\Im_{s,b}^{\alpha}f\,(z))'}{\Im_{s,b}^{\alpha}f\,(z)}-\delta\right\}\geq\beta\left|1+\frac{1}{\gamma}\frac{z\,\Im_{s,b}^{\alpha}f\,'(z)}{\Im_{s,b}^{\alpha}f\,(z)}-1\right|$$

$$\Re\left\{1+\frac{1}{\gamma}\frac{z\,(1-\delta)-\sum_{k=2}^{\infty}(k-\delta)\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left(\frac{b}{k-1+b}\right)^{s}a_k z^k}{z-\sum_{k=2}^{\infty}\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left(\frac{b}{k-1+b}\right)^{s}a_k z^k}\right\}\geq$$

$$\beta\left|1-\frac{1}{\gamma}\frac{\sum_{k=2}^{\infty}(k-1)\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left(\frac{b}{k-1+b}\right)^{s}a_k z^k}{z-\sum_{k=2}^{\infty}\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left(\frac{b}{k-1+b}\right)^{s}a_k z^k}\right|$$

If we let $z\to 1^{-}$ along the real axis, we have

$$\left\{1+\frac{1}{|\gamma|}\frac{(1-\delta)-\sum_{k=2}^{\infty}(k-\delta)\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|\,|a_k|\,|z^{k-1}|}{1-\sum_{k=2}^{\infty}\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|\,|a_k|\,|z^{k-1}|}\right\}\geq$$

$$\beta\left|1-\frac{1}{|\gamma|}\frac{\sum_{k=2}^{\infty}(k-1)\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)|\left(\frac{b}{k-1+b}\right)^{s}|\ |a_k|}{1-\sum_{k=2}^{\infty}\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)|\left(\frac{b}{k-1+b}\right)^{s}|\ |a_k|}\right| .$$

The simple computational leads the desired inequality

$$\sum_{k=2}^{\infty}[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|\ |a_k|\leq(1-\delta)+|\gamma|(1-\beta) .$$

Remark In the special case $\alpha=0$ and $b=b+1$, Theorem 2.1 yields a result given earlier by [12].

From Theorem 2.1, we obtain the following corollary.

Corollary 2.2 *Let the function* f *be defined by (1.3) and* $f\in TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$, *then*

$$|a_k|\leq\frac{(1-\delta)+|\gamma|(1-\beta)}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}|\left(\frac{b}{k-1+b}\right)^{s}|},\quad(k\geq 2),$$

with equality only for functions of the form (2.2).

3 Extreme points

We state and prove the following theorem.

Theorem 3.1 *Let* $f_1(z)=z$ *and*

$$f_k(z)=z-\frac{(1-\delta)+|\gamma|(1-\beta)}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}z^{k},\quad(k\geq 2).$$

Then f is in the class $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\omega_k f_k(z), \tag{3.1}$$

where $\omega_k \geq 0$ and $\sum_{k=1}^{\infty}\omega_k = 1.$

Proof: Suppose f can be written as in (3.1). Then

$$f(z)=\sum_{k=1}^{\infty}\omega_k f_k(z)$$

$$=\omega_1 f_1(z)+\sum_{k=2}^{\infty}\omega_k f_k(z)$$

$$=\omega_1 f_1(z)+\sum_{k=2}^{\infty}\omega_k\left(z-\frac{(1-\delta)+|\gamma|(1-\beta)}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}z^k\right)$$

$$=\omega_1 z+\sum_{k=2}^{\infty}\omega_k z-\left(\sum_{k=2}^{\infty}\omega_k\frac{(1-\delta)+|\gamma|(1-\beta)}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}z^k\right)$$

$$=\left(\sum_{k=1}^{\infty}\omega_k\right)z-\left(\sum_{k=2}^{\infty}\omega_k\frac{(1-\delta)+|\gamma|(1-\beta)}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}z^k\right)$$

$$=z-\left(\sum_{k=2}^{\infty}\omega_k\frac{(1-\delta)+|\gamma|(1-\beta)}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}z^k\right).$$

Now,

$$f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k=z-\sum_{k=2}^{\infty}\omega_k\frac{(1-\delta)+|\gamma|(1-\beta)}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}z^k,$$

so that

$$|a_k|=\omega_k\frac{(1-\delta)+|\gamma|(1-\beta)}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|} \qquad (3.2).$$

Since

$$\sum_{k=2}^{\infty}\omega_k=1-\omega_1\le 1,$$

therefore

$$\sum_{k=2}^{\infty}\omega_k=\sum_{k=2}^{\infty}\frac{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{(1-\delta)+|\gamma|(1-\beta)}\omega_k$$

$$\times\frac{(1-\delta)+|\gamma|(1-\beta)}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|},$$

$$=\sum_{k=2}^{\infty}\frac{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{(1-\delta)+|\gamma|(1-\beta)}|a_k|\le 1.$$

That is

$$\sum_{k=2}^{\infty}[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right||a_k|\le(1-\delta)+|\gamma|(1-\beta).$$

Then (2.1) holds. Thus $f\in TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$, by Theorem 2.1 (Sufficiency).

Conversely, assume that f defined by (1.3) belongs to the class

$TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$. Then by using (3.2), we set

$$\omega_k = \frac{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{(1-\delta)+|\gamma|(1-\beta)}|a_k|, \qquad (k\geq 2),$$

and $\omega_1 = 1-\sum_{k=2}^{\infty}\omega_k$.

Then

$$\begin{aligned}
f(z) &= z-\sum_{k=2}^{\infty}|a_k|z^k \\
&= z-\sum_{k=2}^{\infty}\frac{\omega_k[(1-\delta)+|\gamma|(1-\beta)]}{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}z^k \\
&= z-\sum_{k=2}^{\infty}\omega_k[z-f_k(z)] \\
&= z-\sum_{k=1}^{\infty}\omega_k z+\sum_{k=2}^{\infty}\omega_k f_k(z) \\
&= \left(1-\sum_{k=1}^{\infty}\omega_k\right)z+\sum_{k=2}^{\infty}\omega_k f_k(z) \\
&= \omega_1 z+\sum_{k=2}^{\infty}\omega_k f_k(z) \\
&= \omega_1 f_1(z)+\sum_{k=2}^{\infty}\omega_k f_k(z)=\sum_{k=1}^{\infty}\omega_k f_k(z).
\end{aligned}$$

Thus the proof is complete.

4 Growth and distortion theorems

A growth and distortion property for function f to be in the class $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$ will be given as follows.

Theorem 4.1 *Let the function f defined by (1.3) be in the class $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$. Then for $|z|=r<1$, we have*

$$r-\frac{(1-\delta)+|\gamma|(1-\beta)}{[(1+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^{s}\right|}r^{2}\le|f(z)|$$

$$\le r+\frac{(1-\delta)+|\gamma|(1-\beta)}{[(1+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^{s}\right|}r^{2}, \qquad (4.1),$$

and

$$1-\frac{2[(1-\delta)+|\gamma|(1-\beta)]}{[(1+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^{s}\right|}r\le|f'(z)|$$

$$\le 1+\frac{2[(1-\delta)+|\gamma|(1-\beta)]}{[(1+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^{s}\right|}r \qquad (4.2).$$

Proof: Since $f\in TS_{s,b}^{\alpha}(\delta,\beta,\gamma),$ and in view of inequality (2.1) of Theorem 2.1, we have

$$[(1+|\gamma|)(1-\beta)-(\beta-\delta)]\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left|\left(\frac{b}{1+b}\right)^{s}\right|\sum_{k=2}^{\infty}|a_k|$$

$$\le\sum_{k=2}^{\infty}[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left|\left(\frac{b}{k-1+b}\right)^{s}\right||a_k|$$

$$\le(1-\delta)+|\gamma|(1-\beta), \quad (k\ge 2).$$

Then

$$\sum_{k=2}^{\infty}a_k\le\frac{(1-\delta)+|\gamma|(1-\beta)}{[(1+|\gamma|)(1-\beta)-(\beta-\delta)]\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left|\left(\frac{b}{1+b}\right)^{s}\right|}. \qquad (4.3).$$

After the inequality obtained by $(1,3)$ and (4.3), assume that $|z|=r,$ in order to get the next inequality. Since

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k,$$

we have that

$$|f(z)| = \left| z - \sum_{k=2}^{\infty} a_k z^k \right| \le$$

$$|f(z)| \le |z| + |z|^2 \sum_{k=2}^{\infty} |a_k| |z^{k-2}| \le$$

$$r + r^2 \sum_{k=2}^{\infty} |a_k| \le$$

$$r + \frac{(1-\delta)+|\gamma|(1-\beta)}{[(1+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|} r^2.$$

So

$$|f(z)| \ge r - r^2 \sum_{k=2}^{\infty} |a_k| \ge$$

$$r - \frac{(1-\delta)+|\gamma|(1-\beta)}{[(1+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|} r^2.$$

Further

$$|f'(z)| \le 1 + \sum_{k=2}^{\infty} k |a_k| |z^{k-1}| \le 1 + 2r \sum_{k=2}^{\infty} |a_k| \le$$

$$1 + \frac{2(1-\delta)+|\gamma|(1-\beta)}{[(1+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|} r,$$

and

$$|f'(z)| \ge 1 - \sum_{k=2}^{\infty} k |a_k| |z^{k-1}| \ge 1 - 2r \sum_{k=2}^{\infty} |a_k| \ge$$

$$1-\frac{2(1-\delta)+|\gamma|(1-\beta)}{[(1+|\gamma|)(1-\beta)-(\beta-\delta)]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|}r.$$

This completes the proof.

5 Radius of starlikeness and convexity

In the next theorems, we will find the radius of starlikeness, convexity and close–to–convexity for the class $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$.

Theorem 5.1 *Let the function* f *defined by (1.3) belong to the class* $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$. *Then* f *is close–to–convex of order* λ, $(0\le\lambda<1)$ *in the disc* $|z|<r$, *where*

$$r:=\inf_{k\ge2}\left(\frac{(1-\lambda)[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left|\left(\frac{b}{k-1+b}\right)^s\right|}{k[(1-\delta)+|\gamma|(1-\beta)]}\right)^{\frac{1}{k-1}}.$$

The result is sharp, with extremal function f given by (2.2).

Proof: Given $f\in T$ and f is close–to–convex of order λ in the disc $|z|<r$ if and only if we have

$$|f'(z)-1|<1-\lambda,\quad \text{whenever}\quad |z|<r. \tag{5.1}$$

For the left hand side of (5.1) we have

$$|f'(z)-1|\le\sum_{k=2}^{\infty}k\,|a_k||z|^{k-1}.$$

Then (5.1) is implied by

$$\sum_{k=2}^{\infty}\frac{k}{1-\lambda}|a_k||z|^{k-1}<1.$$

Using the fact that $f\in TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$, if and only if

$$\sum_{k=2}^{\infty}\frac{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\dfrac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left|\left(\dfrac{b}{k-1+b}\right)^{s}\right|}{(1-\delta)+|\gamma|(1-\beta)}|a_k|\leq 1,$$

it follows that (5.1) is true if

$$\frac{k}{1-\lambda}|z|^{k-1}\leq\frac{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\dfrac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left|\left(\dfrac{b}{k-1+b}\right)^{s}\right|}{(1-\delta)+|\gamma|(1-\beta)},$$

whenever $|z|<r$ .We obtain

$$r:=\inf_{k\geq 2}\left(\frac{(1-\lambda)[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\dfrac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left|\left(\dfrac{b}{k-1+b}\right)^{s}\right|}{k[(1-\delta)+|\gamma|(1-\beta)]}\right)^{\frac{1}{k-1}}.$$

This completes the proof.

Theorem 5.2 *Let the function* f *defined by (1.3) belong to the class* $TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$. *Then*

(1) f is starlike of order $\lambda,\ (0\leq\lambda<1)$ in the disc $|z|<r$, that is,

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\lambda,\quad (|z|<r,0\leq\lambda<1),$$

where

$$r:=\inf_{k\geq 2}\left(\frac{(1-\lambda)[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\dfrac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left|\left(\dfrac{b}{k-1+b}\right)^{s}\right|}{[(1-\delta)+|\gamma|(1-\beta)](k-\beta)}\right)^{\frac{1}{k-1}}.$$

f is convex of order $\lambda,\ (0\leq\lambda<1)$ in the disc $|z|<r$, that is,

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\lambda,\quad (|z|<r,0\leq\lambda<1),$$

where

$$r := \inf_{k\geq 2}\left(\frac{(1-\lambda)[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\dfrac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left|\left(\dfrac{b}{k-1+b}\right)^{s}\right|}{[(1-\delta)+|\gamma|(1-\beta)](k-\lambda)}\right)^{\frac{1}{k-1}}.$$

Each of these results is sharp for the extremal function given by (2.2).

Proof: (1) Given $f\in T$ and f is starlike of order $\lambda,$ in the disc $|z|<r$ if and only if

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\lambda, \text{whenever } |z|<r. \qquad (5.2)$$

For the left hand side of (5.2) we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{\sum_{k=2}^{\infty}(k-1)|a_k|\,|z|^{k-1}}{1-\sum_{k=2}^{\infty}|a_k|\,|z|^{k-1}}.$$

Then (5.2) is implied by

$$\sum_{k=2}^{\infty}\frac{k-\lambda}{1-\lambda}|a_k|\,|z|^{k-1}<1.$$

Using the fact that $f\in TS_{s,b}^{\alpha}(\delta,\beta,\gamma)$ if and only if

$$\sum_{k=2}^{\infty}\frac{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\dfrac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left|\left(\dfrac{b}{k-1+b}\right)^{s}\right|}{[(1-\delta)+|\gamma|(1-\beta)]}|a_k|\leq 1.$$

(5,2) is true for every z in the disc $|z|<r$ if

$$\frac{k-\lambda}{1-\lambda}|z|^{k-1}\leq\frac{[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\dfrac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left|\left(\dfrac{b}{k-1+b}\right)^{s}\right|}{[(1-\delta)+|\gamma|(1-\beta)]}.$$

Thus

$$r := \inf_{k\geq 2}\left(\frac{(1-\lambda)[(k+|\gamma|)(1-\beta)-(\beta-\delta)]\dfrac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left|\left(\dfrac{b}{k-1+b}\right)^{s}\right|}{[(1-\delta)+|\gamma|(1-\beta)](k-\lambda)}\right)^{\frac{1}{k-1}}.$$

This completes the proof.

(2) Using the fact that f is convex of order λ if and only if $zf'(z)$ is starlike of order λ, we can prove (2) using similar methods to the proof of (1).

Remark In the special case $\alpha = 0$ and $b = b + 1$, Theorem 2.1 yields a result given earlier by [12].

6 Conclusion

The work presented here is generalization of work done by earlier researchers. Further the research can be done by using fractional calculus operators for this class.

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