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Motivated by the linear operator studied by the author in [2, 3], the author introduce and study a new general integral operator defined on the class of normalized analytic function in the unit disc. This operator is motivated by many researchers. With this operator univalence conditions for the normalized analytic function in the open unit disc are obtained. Indeed, the author present a few conditions of univalency for our integral operator. . Having the integral operator, there are interesting properties of normalized function in the unit disc for univalent conditions for an integral operator. In addition, the author also find some interesting corollaries on the class of normalized analytic of functions in the open unit disc.

وحدة التكافؤ للمؤثر التكاملي الجديد

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ملخص

باستخدام المؤثر الخطي الذي درسه المؤلف في [2، 3]، عرض المؤلف ودرس مؤثر تكاملي جديد معرف على مجموعة الدوال التحليلية في دائرة الوحدة ، هذا المؤثر يكون تعميماً للعديد من الباحثين. مع هذا المؤثر شروط وحدة التكافؤ للدوال التحليلية في دائرة الوحدة المفتوحة تم الحصول عليها.

عرض المؤلف في الواقع بعض الخواص مثيرة للاهتمام في وحدة التكافؤ للمؤثر التكاملي ، بالإضافة إلى ذلك ، أوجد أيضا العديد من النتائج الأخرى على مجموعة الدوال التحليلية في دائرة الوحدة. .

1 Introduction

Let A denote the class of functions f normalized with $f(0)=f'(0)-1=0,$ in the open unit disc in the complex plane

$$\mathrm{U}=\{z\in\mathrm{C}:|z|<1\},$$

given by the normalized power series

$$f(z)=z+\sum_{k=1}^{\infty}a_k z^{k+1}\quad (z\in U)\quad ,$$

Let S be the subclass of A consisting of all univalent functions f in U. For two functions $f\in A$ and $g(z)=z+\sum_{k=2}^{\infty}b_k z^k,$ $(z\in\mathrm{U}),$ where a_k, b_k are a complex number.

The Hadamard product of two functions (or convolution) is defined by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k .$$

Many authors studied the problem of integral operators, acting on functions in S, to belong to the class S. In this sense, the following result due to Ozaki and Nunokawa [1] is useful to study the univalence of integral operator for certain subclass of S.

Theorem 1.1 *Let $f\in A$ satisfy the following inequality:*

$$\left|\frac{z^2 f'(z)}{f^2(z)}-1\right|\le 1\quad \textit{for all } (z\in\mathrm{U}), \tag{1}$$

then the function f is univalent in U.

Let the function $\varphi(a,c;z)$ be given by

$$\varphi(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_k}{(c)_k}z^{k+1},\quad (z\in\mathrm{U}, c\ne 0,-1,-2,-3,\ldots),$$

where $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial).

Corresponding to the function $\varphi(a,c;z)$, Carlson and Shaffer [9] introduced a linear operator $L(a,c)$ by

$$L(a,c)f(z) := \varphi(a,c;z) * f(z) = \sum_{n=0}^{\infty} \frac{(a)_k}{(c)_k} a_k z^{k+1}.$$

The author [2, 3] has recently introduced a new linear operator $D_l^{m,\lambda}(a,b)f(z)$ as the following:

Definition 1.2 *Let*

$$\phi_l^{m,\lambda}(a,b;z) = \sum_{k=0}^{\infty} \left(\frac{1+\lambda k + l}{1+l} \right)^m \frac{(a)_k}{(b)_k} z^{k+1},$$

where $(z \in U, b \neq 0,-1,-2,-3,...), \lambda \geq 0, m \in \mathrm{Z}, l \geq 0,$ and $(x)_k$ is the Pochhammer symbol.

We define a linear operator $D_l^{m,\lambda}(a,b) : A \to A$ by the following Hadamard product:

$$D_l^{m,\lambda}(a,b) f(z) = \phi_l^{m,\lambda}(a,b;z) * f(z) = \sum_{k=0}^{\infty} \left(\frac{1+\lambda k + l}{1+l} \right)^m \frac{(a)_k}{(b)_k} a_k z^{k+1}. \qquad (2)$$

Special cases of this operator includes:

- $D_0^{m,0}(a,b)f(z) = D_l^{0,\lambda}(a,b)f(z) = L(a,b)f(z).$
- the Ruscheweyh derivative operator [11] in the cases: $D_0^{0,0}(\beta+1,1)f(z) = D^{\beta} f(z); \beta \geq -1.$
- the Salagean derivative operator [13]: $D_0^{m,1}(1,1)f(z).$
- the generalized Salagean derivative operator introduced by Al−Oboudi [12]: $D_0^{m,\lambda}(1,1)f(z).$
- the Catas drivative operator [10]: $D_l^{m,\lambda}(1,1)f(z),$ and finally
- The fractional operator introduced by Owa and Srivastava [14]

$$D_0^{0,0}(2,2-\gamma)f(z) = \Omega^{\gamma} f(z) = \Gamma(2-\gamma) z^{\gamma} D_z^{\gamma} f(z);$$

$D_z^{\gamma} f(z)$ is the fractional derivative of f of order $\gamma; \gamma \neq 2,3,4,\cdots.$

Using the operator $D_l^{m,\lambda}(a,b)f(x),$ we now introduce the following integral operator.

Definition 1.3 *For $f_i \in A$ for all $(i = 1,2,...n)$, and the parameters $l \geq 0,\ \ \lambda \geq 0, m \in \mathbb{N}$ and γ is a complex number , we define the integral operator $F_{a,b}^{m,l}(f_1,...,f_n): A^n \to A$ by*

$$F_{a,b}^{m,l}(f_1,...,f_n)(z) = \left((k(\gamma-1)+1)\int_0^z \left(D^m(\lambda,l,a,b)f_1(t)\right)^{\gamma-1} ...\left(D^m(\lambda,l,a,b)f_n(t)\right)^{\gamma-1} dt\right)^{\frac{1}{k(\gamma-1)+1}}. \quad (3)$$

Remark 1.1 *It is interesting to note that the integral operators $F_{a,b}^{m,l}(f_1,...,f_n)$ generalizes many operators which were introduced and studied recently, for example:*

(1) If $m = 0$ and $b = a = 1,$ then the operator $F_{a,b}^{m,l}(f_1,...,f_n)(z)$ reduces to the integral operator

$$\left((k(\gamma-1)+1))\int_0^z (f_1(t))^{\gamma-1} ...(f_n(t))^{\gamma-1} dt\right)^{\frac{1}{k(\gamma-1)+1}},$$

was introduced and studied by Breaz and Breaz [6].

(2) If $l = 1-\lambda$ and $b = a = 1,$ then we obtain the integral operator

$$G_{m,\gamma} = \left((k(\gamma-1)+1))\int_0^z \left(D^m f_1(t)\right)^{\gamma-1} ...\left(D^m f_n(t)\right)^{\gamma-1} dt\right)^{\frac{1}{k(\gamma-1)+1}},$$

was introduced and studied by Bulut [7],where D^m is the Al−Oboudi derivative operator

(3) If $m = 0,\ \ b = a = 1,$ and $k = 1$ then we obtain the integral operator

$$G_\gamma = \left(\gamma\int_0^z f(t)^{\gamma-1} dt\right)^{\frac{1}{\gamma}}.$$

We now state the following results which we need to establish our results in the sequel.

Lemma 1.4 *[5] Let the function f be regular in the disc*

$$U_R = \{z \in \mathbb{C} : |z| < R\},$$

with $|f(z)| < M$ for fixed M. If $f(z)$ has one zero with multiplicity order greater than m for $z = 0,$ then

$$|f(z)| \leq \frac{M}{R^m}|z|^m, (z \in \mathrm{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta}(\frac{M}{R^m})z^m,$$

where θ is a constant.

Lemma 1.5 *[4] Let* $f \in A$ *and* $\beta, c \in \mathrm{C}$ *where* $\Re e\{\beta\} > 0$ *and* $(|c| \leq 1, c \neq -1)$. *If*

$$\left| c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zf''(z)}{\beta f'(z)} \right| \leq 1,$$

for all $z \in \mathrm{U}$, then the function

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1} f'(u)dt\right]^{\frac{1}{\beta}}, \qquad (4)$$

belongs to S.

Lemma 1.6 *[8] Let the function* f *satisfy the inequality (1). Also let*

$$\alpha \in \Re, \quad \left(\alpha \in [1, \frac{3}{2}]\right), \quad c \in \mathrm{C},$$

if

$$|c| \leq \frac{3-2\alpha}{\alpha} \quad (c \neq -1),$$

and

$$|g(z)| \leq 1, \quad (z \in \mathrm{U}).$$

Then the function G_α defined by

$$G_\alpha = \left(\alpha\int_0^z g(t)^{\alpha-1}dt\right)^{\frac{1}{\alpha}}.$$

is analytic and univalent in U.

2 Main results

In this section, we first state the main univalent condition involving the general integral operator given by (3).

Theorem 2.1 *Let* $f_i \in A,\ i = 1,2,..n,\ c \in \mathbb{C},\ \gamma \in \mathbb{R},$ *and* $M \geq 1$*, with*

$$|c| \leq 1 + \frac{1-\gamma}{k(\gamma-1)+1}(2M+1)k, \tag{5}$$

and

$$\gamma \in \left[1, \frac{(2M+1)k}{(2M+1)k-1}\right].$$

If

$$\left|\frac{z^2 (D_l^{m,\lambda}(a,b) f_i(z))'}{(D_l^{m,\lambda}(a,b) f_i(z))^2} - 1\right| \leq 1, \tag{6}$$

where $l, \lambda \geq 0,\ m \in \mathbb{N},$

and

$$|(D_l^{m,\lambda}(a,b) f_i(z)| \leq M, \quad (z \in \mathbb{U},\ i \in \{1,2,...n\}),$$

then the integral operator $F_{a,b}^{m,l}(f_1,...,f_n)$ defined by (3) is analytic and univalent in $\mathbb{U}$.

Proof: Since $i \in 1,2,...,n, f_i \in A,$ we have

$$\frac{F_{a,b}^{m,l}(f_1,...,f_n)(z)}{z} = \frac{z + \sum_{k=2}^{\infty}\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(b)_{k-1}} a_k z^k}{z}$$

$$= 1 + \sum_{k=2}^{\infty}\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(b)_{k-1}} a_k z^{k-1},$$

and

$$\frac{F_{a,b}^{m,l}(f_1,...,f_n)(z)}{z} \neq 0, \text{for all} \quad z \in \mathbb{U}.$$

Let us consider the function defined by

$$F(z)=\int_0^z \prod_{i=1}^{n}\left(\frac{D_l^{m,\lambda}(a,b)f_i(t)}{t}\right)^{\frac{1}{k(\gamma-1)+1}}dt,$$

$$F'(z)=\prod_{i=1}^{n}\left(\frac{D_l^{m,\lambda}(a,b)f_i(z)}{z}\right)^{\frac{1}{k(\gamma-1)+1}}.$$

It is clear that $F(0)=F'(0)-1=0.$

A simple computation yields

$$\frac{zF''(z)}{F'(z)}=(\gamma-1)\sum_{i=1}^{n}\left(\frac{z(D_l^{m,\lambda}(a,b)f_i(z))'}{D_l^{m,\lambda}(a,b)f_i(z)}-1\right),$$

which readily shows that

$$\left|c|z|^{2[k(\gamma-1)+1]}+(1-|z|^{2[k(\gamma-1)+1]})\frac{zF''(z)}{[k(\gamma-1)+1]F'(z)}\right|$$

$$\le |c|+\left(\frac{\gamma-1}{k(\gamma-1)+1}\right)\sum_{i=1}^{n}\left(\left|\frac{z^2(D_l^{m,\lambda}(a,b)f_i(z))'}{(D_l^{m,\lambda}(a,b)f_i(z))^2}\right|\left|\frac{D_l^{m,\lambda}(a,b)f_i(z)}{z}\right|+1\right).$$

Since

$$|D_l^{m,\lambda}(a,b)f_i(z)|\le M,\quad (z\in \mathrm{U},\quad i\in\{1,2,\ldots n\}),$$

Using inequality (6) and the general Schwarz lemma, we obtain

$$\left|c|z|^{2[k(\gamma-1)+1]}+(1-|z|^{2[k(\gamma-1)+1]})\frac{zF''(z)}{[k(\gamma-1)+1]F'(z)}\right|$$

$$\le |c|+\left(\frac{\gamma-1}{k(\gamma-1)+1}\right)\sum_{i=1}^{n}\left(\left|\frac{z^2(D_l^{m,\lambda}(a,b)f_i(z))'}{D_l^{m,\lambda}(a,b)f_i(z)}\right|M+1\right)$$

$$=|c|+\left(\frac{\gamma-1}{k(\gamma-1)+1}\right)\sum_{i=1}^{n}\left(\left|\frac{z^2(D_l^{m,\lambda}(a,b)f_i(z))'}{D_l^{m,\lambda}(a,b)f_i(z)}-1\right|M+M+1\right)$$

$$=|c|+\left(\frac{\gamma-1}{k(\gamma-1)+1}\right)(2M+1)k,$$

which, by (5), yields

$$\left|c|z|^{2[k(\gamma-1)+1]}+(1-|z|^{2[k(\gamma-1)+1]})\frac{zF''(z)}{[k(\gamma-1)+1]F'(z)}\right|\le 1.$$

Applying Theorem 1.5, we conclude that the function $F_{a,b}^{m,l}(f_1,\ldots,f_n)$ defined by (3) is in the class S.

3 Applications of Theorem 2.1

Putting $M=1$ in Theorem 2.1, we get the following result.

Corollary 3.1 *Let* $f_i \in A$, $i=1,2,..n$, $c \in \mathbb{C}$, $\gamma \in \mathbb{R}$, *and* $M=1$, *with*

$$|c| \le 1+3\left(\frac{1-\gamma}{k(\gamma-1)+1}\right)k. \tag{7}$$

If

$$\gamma \in \left[1, \frac{3k}{3k-1}\right],$$

and

$$|(D_l^{m,\lambda}(a,b)f_i(z)| \le M, \quad (z \in \mathrm{U}, \quad i \in \{1,2,...n\}),$$

then the integral operator $F_{a,b}^{m,l}(f_1,...,f_n)(z)$ defined by (3) is analytic and univalent in U.

If we set $k=1$ in Theorem2.1, we obtain the following result.

Corollary 3.2 *Let* $M \ge 1$,*and suppose* $f \in A$ *and satisfies the inequality*

$$|c| \le 1+\frac{1-\gamma}{\gamma}(2M-1). \tag{8}$$

If $\gamma \in \left[1, \frac{(2M+1)}{(2M+1)-1}\right]$, and

$$|D_l^{m,\lambda}(a,b)f(z)| \le M, \quad (z \in \mathrm{U}, \quad i \in \{1,2,...n\}),$$

then the integral operator $F_{a,b}^{m,l}(f_1,...,f_n)(z)$ defined by (3) is analytic and univalent in U.

Remark 3.1 *If we set* $l=0$ *and* $b=a=1$, *in Theorem 2.1, then we have Theorem 1 [7].*

Remark 3.2 *If we set* $m=0$ *and* $b=a=1$, *in Corollary 3.1,then we have Corollary 1 [6].*

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