EXISTENCE OF SOLUTION FOR FRACTIONAL IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS

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الملخص

في هذا البحث، قمت بإثبات الوجود والوحدانية لحل معادلة تفاضلية تكاملية ذات رتبة كسرية شبه خطية باستخدام الحساب الكسري .

هذا العمل يبادر لسبل جديدة لاستنتاج حلول عددية لمعادلات التفاضلية التكاملية ذات رتبة كسرية.

هذا البحث يحتوي:

في البند الأول: مقدمة،

في البند الثاني: مراجعة لتعريفات التفاضل ذات الرتبة الكسرية لكابيتو (the Caputo fractional derivative) و تكامل ريمان ليوفيل (Riemann-Liouville integral)

في البند الثالث: درست الوجود والوحدانية لحل معادلة تفاضلية تكاملية ذات رتبة كسرية شبه خطية.

Abstract

In this paper, provided the existence and uniqueness of solution to impulsive semilinear fractional integro-differential equations by using the fractional calculus.

This work initiates new avenues for obtaining numerical solutions of impulsive fractional integro-differential equation.

The paper is organnized as follows.

The first section gives the Introduction and recalled the definition of the Caputo fractional derivative and Riemann-Liouville integral in section 2.

And in section 3 studded the existence of solution to Semilinear Evolution problem.

1. Introduction

Fractional differential equations have recently been addressed by several researchers for a variety of problems. The interest in the study of differential equations of fractional order lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. With this advantage, the fractional-order models become more realistic and practical than the classical integer-order models, in which such effects are not taken into account. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [11, 17, 18, 19]. For some recent development on the topic, see [3, 12, 13] and the references therein.

In [8, 10] the authors have proved the existence of solutions of abstract differential equations by using semigroup theory and fixed point theorem. Many partial fractional differential equations can be expressed as fractional differential equations in some Banach spaces.[9]

Subsequently several authors [6, 15] have discussed the problem for different types of nonlinear differential and integro-differential equations including functional differential equations in Banach spaces. Numerical experiments for fractional models on population dynamics are discussed in [7, 16]. On the other hand, the theory of impulsive differential equations has undergone rapid development over the years and played a very important role in modern applied mathematical models of real processes arising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics: see for instance the monograph by Lakshmikantham et al. [14]. Benchohra and Seba [5] studied the existence of fractional impulsive differential equations in Banach spaces. Belmekki et al. [2] proved the existence of periodic solutions of nonlinear fractional differential equations whereas (Ahmad and Nieto [1]) discussed the existence results for nonlinear boundary value problem of fractional integro-differential equations with integral boundary conditions. Recently, the study of impulsive differential equations has attracted a great deal of attention in fractional dynamics and its theory has been treated in several works.[14 ،4]

In this paper, motivated by [4], we prove the existence and uniqueness of solution to impulsive semilinear fractional integro–differential equations by using the fractional calculus and fixed point theorems. This work initiates new avenues for obtaining numerical solutions of impulsive fractional integro–differential equations the paper is organnized as follows.

Recalled in section 2 the definition of the Caputo fractional derivative and Riemann–Liouville integral, and in section 3 studied the existence of solution to Semilinear Evolution problem.

2. Preliminaries

In this section, we shall introduce some basic definitions, notations and lemmas which are used throughout this paper.

Definition 2.1. Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}^+$and $\alpha \geq 0$. Then the expression

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} ds \qquad t > 0$$

is called the Riemman–Liouville integral of order α.

Definition 2.2. Let $f: \mathbb{R}^+ \to \mathbb{R}$. The Caputo fractional derivative of order α of f is defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds \quad t > 0,$$

Where $\alpha \in (n-1, n), n \in \mathbb{N}$.

Let $B(x)$ denote the algebra of all bounded linear operator acting on a Banach space X. Now let us consider the set of functions

$$PC(I,X) = \{u: I \to X: u \in c\big((t_{k-1}, t_k], X\big), k = 1,2,\cdots, m$$

and there exist $u(t_k^+), k = 1,2,\cdots, m$ with $u(t_k^-) = u(t_k)\}$.

Endowed with the norm

$$\| u \|_{PC} = \sup_{t\in I} \| u(t) \|$$

$(PC(I,X), \|\cdot\|_{PC})$ is a Banach space, see [14].

3. Semilinear Evolution problem

Consider the linear fractional impulsive evolution equation

$$D_t^\alpha u(t) = A(t)u(t) + f\big(t, u(t)\big) \qquad t \neq t_k \quad 0 < \alpha \leq 1$$

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)) \qquad k = 1,\cdots,m.$$
(3.1)

$$u(0) = u_0 \in X$$

Where $A(t)$ is a bounded linear operator on a Banach space X, $I_k: X \to X$, $k = 1,\cdots,m$. And $u_0 \in X$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$.

$\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-), u(t_k^+) = \lim_{h\to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h\to 0^-} u(t_k + h)$ represent the right and left limits $u(t)$ at $t = t_k$.

Definition 3.1. We mean an abstract function u by a solution of Eq. (3.1) such that the following condition are satisfied

(i) $u \in PC(I,X)\ and\ u \in D(A(t))$ for all $t \in I' \subseteq I$;

(ii) $\frac{\partial^\alpha u}{\partial t^\alpha}$ exists on I' where $0 < \alpha < 1$;

(iii) Satisfy Eq. (3.1) on I' and satisfy the conditions

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)) \qquad k = 1,\cdots,m.$$
(3.1)

$$u(0) = u_0 \in X$$

Now, we assume the following conditions to prove the existence of a solution of the Eq. (3.1).

(H1) $A(t)$ is a bounded linear operator on X for each $t \in I$ and the function $t \to A(t)$ is continuous in the uniform operator topology and there exists a constant M such that

$$M = \max_{t\in I} \| A(t) \|$$

(H2) The functions $I_k: X \to X$ are continuous and there exists a constant $L_1 > 0$ and $p_1 > 0$ such that

$\| I_k(u) - I_k(v) \| \leq L_1 \| u - v \|$ and $\| I_k(u) \| < p_1 \| u \|$ for each $u, v \in X$ and $k = 1,2,\cdots,m$, and $f: I \times X \to X$ is continuous and there exists a constant L_2 such that

$$\| f(t,u) - f(t,v) \| \leq L_2 \| u - v \| \quad \text{for all} \quad u, v \in X.$$

For brevity let us take $\frac{T^\alpha}{\Gamma(\alpha+1)} = \gamma$.

Definition 3.2. Let a function $u \in PC(I,X)$ be solution of the fractional integral

equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(A(s)u(s) + f(s,u(s)))ds$$

will be called a mild solution of Eq.(3.1).

Lemma 3.3. Let $f: I \times X \to X$ be a continuous function and A be a bounded linear operator. If $u \in C(I,X)$ is a mild solution of Eq. (3.1) in the sense of Def. (3.2), then for any $t \in (t_{k-1}, t_k]$, $k = 1, \cdots, m$, then Eq. (3.1) is to the equivalent integral equation

$$u(x) = u_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j - s)^{\alpha-1}(A(s)u(s) + f(s,u(s)))ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}(A(s)u(s) + f(s,u(s)))ds + \sum_{0<t_k<t} I_k(u(t_k^-)) \quad (3.2)$$

If $t \in (t_k, t_{k+1}]$

Proof. Using Def. (3.2), we have

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[A(s)u(s) + f(s,u(s))]ds \quad t \in (0, t_1],$$

leads to

$$u(t_1) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1 - s)^{\alpha-1}[A(s)u(s) + f(s,u(s))]ds$$

Since $u(t_1^+) = u(t_1^-) + I_1(u(t_1^-)) = u(t_1) + I_1(u(t_1^-))$, then

$$u(t_1^+) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1 - s)^{\alpha-1}[A(s)u(s) + f(s,u(s))]ds + I_1(u(t_1^-)).$$

Moreover, for $t \in (t_1, t_2]$,

$$u(t) = u(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}[A(s)u(s) + f(s,u(s))]ds$$

$$= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1 - s)^{\alpha-1}[A(s)u(s) + f(s,u(s))]ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}[A(s)u(s)+f(s,u(s))]ds+I_1\big(u(t_1^-)\big)$$

For $t=t_2$ and $t=t_2^+$,

$$u(t_2)=u_0+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}[A(s)u(s)+f(s,u(s))]ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}[A(s)u(s)+f(s,u(s))]ds+I_1\big(u(t_1^-)\big)$$

Since $u(t_2^+)=u(t_2^-)+I_2\big(u(t_2^-)\big)=u(t_2)+I_2\big(u(t_2^-)\big)$

Then

$$u(t_2^+)=u_0+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}[A(s)u(s)+f(s,u(s))]ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}[A(s)u(s)+f(s,u(s))]ds+I_1\big(u(t_1^-)\big)+I_2\big(u(t_2^-)\big)$$

Hence for any $t\in(t_2,t_3]$

$$u(t)=u(t_2^+)+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t-s)^{\alpha-1}[A(s)u(s)+f(s,u(s))]ds$$

$$=T(t)u_0+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}[A(s)u(s)+f(s,u(s))]ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}[A(s)u(s)+f(s,u(s))]ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t-s)^{\alpha-1}[A(s)u(s)+f(s,u(s))]ds+I_1\big(u(t_1^-)\big)+I_2\big(u(t_2^-)\big)$$

By repeating the same procedure, we can easily deduceThat

$$u(t)=T(t)u_0+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}[A(s)u(s)$$

$$+f(s,u(s))]ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}[A(s)u(s)+f(s,u(s))]ds$$

$$+\sum_{0<t_k<t}I_k\big(u(t_k^-)\big)$$

For any $t\in(t_{k-1},t_k]$, $k=1,\cdots,m$.

Theorem 3.4. If the hypotheses (H1) and (H2) are satisfied, then Eq. (3.1) has

a unique solution on I.

Proof. The proof is based on the application of Picards iteration method. Let

$M = \max_{0\leq t\leq T} \| A(t) \|$ and define a mapping $F: PC([0,T]:X) \to PC([0,T]:X)$

by

$$Fu(t) = u_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left(t_j - s\right)^{\alpha-1}[A(s)u(s) + f(s,u(s))]ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}[A(s)u(s) + f(s,u(s))]ds$$
$$+\sum_{0<t_k<t} I_k\left(u(t_k^-)\right) \text{ if } t \in (t_{k-1}, t_k]. \quad (3.3)$$

Let $u, v \in PC(I,X)$. Then from Eq. (3.3), we have for each $t \in I$

$$\| Fu(t) - Fv(t) \| \leq \frac{T^{\alpha}(m+1)}{\Gamma(\alpha+1)}(M + L_2) \| u - v \| + mL_1 \| u - v \|.$$

Then by induction we have

$$\| F^n u(t) - F^n v(t) \| \leq \frac{(\gamma(m+1)(M+L_2)+mL_1)^n}{n!} \| u - v \|.$$

Since $\frac{(\gamma(m+1)(M+L_2)+mL_1)^n}{n!} < 1$ for large n, then by well known generalization of the Banach contraction principle, F has a unique fixed point $u \in PC([0,T]:X)$.This fixed point is the unique solution of Eq.(3.1).

References.

[1] B.Ahmad, J.J. Nieto, Existence results for nonlinear boundary value problems of fractional integro-differential equations with integral boundary conditions, Bound. Value Probl. Article ID. 708576 (2009), 1-11.

[2] K. Balachandran, S. Kiruthika, J. J. Trujillo, Existence of periodic solutions for a nonlinear fractional differential equation, Bound. Value Probl. Article ID. 324561 (2009), 1-18.

[3] K. Balachandran, J.J. Trujillo, The nonlocal Cauchy problem for nonlinear fractional integro-differential equations in Banach spaces, Nonlinear Anal. 72 (2010), 4587-4593.

[4] M. Benchohra, B.A. Slimani, Existence and uniqueness of solutions to impulsive fractional differential equations, Electron. J. Differential Equations 10 (2009), 1-11.

[5] M. Benchohra, D. Seba, Impulsive fractional differential equations in Banach Spaces, Electron.

J. Qual. Theory Differ. Equ. 8 (2009), 1-14.

[6] L. Byszewski, H. Akca, Existence of solutions of a semilinear functional differential evolution nonlocal problem, Nonlinear Anal. 34 (1998), 65-72.

[7] Y.K. Chang, J.J. Nieto, Existence of solutions for impulsive neutral integro-differential inclusions with nonlocal initial conditions via fractional operators, Numer. Funct. Anal. Optim. 30 (2009), 227-44.

[8] M. M. El-Borai, Semigroups and some nonlinear fractional differential equations, Appl. Math.Comput. 149 (2004) 823–831.

[9] M. A. El-Sayed, Fractional order diffusion wave equation, Internat. J. Theoret. Phys. 35.322–311 ،(1996).

[10] O.K. Jaradat, A. Al-Omari, S. Momani, Existence of the mild solution for fractional semilinear initial calue problems, Nonlinear Anal. 69 (2008), 3153–3159.

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland

Mathematics Studies, vol. 204, Elsevier Science B.V., Amster- dam, 2006.

[12] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. 69 (8) (2008), 2677–2682.

[13] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, 2009.

[14] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations.Singapore, world scientific; 1989.

[15] J.Liang, J.Liu, T.J. Xiao, Nonlocal Cauchy problems governed by compact operator families،

Nonlinear Anal. 57 (2004), 183–9.

[16] NGu´er´ekata G. M., A Cauchy problem for some fractional abstract differential equation with nonlocal conditions, Nonlinear Anal. 70 (2009), 1873–6.

[17] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[18] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), Advances in Fractional Calculus: Theoret- ical Developments and Applications in Physics and Engineering, Springer, Dordrecht, 2007.

[19] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, 1993.