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PERIODIC SOLUTIONS FOR IMPULSIVE NEUTRAL DYNAMIC EQUATIONS WITH INFINITE DELAY ON TIME SCALES SPACE

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Abstract.

This study addresses the problem of determining the existence and uniqueness of periodic solutions to a class of impulsive neutral dynamic equations that incorporate infinite delay, defined over a periodic time scale denoted by .

$$x^{\Delta}(t) = -\prod_{i=1}^{p} B_i(t)x^{\sigma}(t) + \sum_{i=1}^{p} W_i^{\Delta}(t, x(t - f(t)))$$

$$+\int_{-\infty}^{t} \left[\prod_{i=1}^{p} E_i(t,u)h(x(u)) + g(u)\right] \Delta u + D(t, x(t), x(t - \Theta(t))), t \neq t_j, t \in \mathbb{T},$$

$$x(t_j^+) = x(t_j^-) + I_j\left(x(t_j)\right), j \in \mathbb{Z}^+,$$

The focal point of this work is a complex dynamic system that integrates multiple mathematical features: neutral terms, impulsive discontinuities at discrete instances, and an integral representation of the system's historical behavior extending indefinitely into the past.

The dynamic model under consideration involves a delta derivative, multiplicative operator terms, and delayed functional components, and is governed by impulsive effects at specified time points. The analysis is grounded in a general framework that accommodates both discrete and continuous behavior through the unifying language of time scale calculus.

To establish the existence of periodic solutions, we utilize Krasnoselskii's fixed point theorem—an essential tool in nonlinear operator theory known for its effectiveness in handling non-compact and non-linear mappings in Banach spaces. In contrast, the uniqueness of the solution is ensured by applying the Banach contraction principle, which demands more restrictive structural conditions on the system's parameters but provides strong guarantees of solution distinctiveness.

The theoretical contributions presented herein not only address the inherent analytical challenges posed by the neutral and impulsive dynamics but also offer valuable insights into systems exhibiting long-term memory. Such systems are prevalent in various scientific domains,

including automatic control mechanisms with feedback delays, macroeconomic models driven by historical trends, and biological oscillators subject to abrupt environmental perturbations.

By integrating advanced methods from time scale calculus, infinite-dimensional functional analysis, and fixed point theory, this work offers a comprehensive approach that enhances both the theoretical understanding and practical applicability of periodic solutions in delay-dominated dynamic systems.

.**Key words and phrases**. Periodic , dynamic equations, impulses, Krasnoselskii fixed point, time scales.

حلول دورية لمعادلات ديناميكية محايدة نبضية مع تأخير محدود على مقاييس زمنية

الملخص

في هذا البحث، درسنا وجود وتفرد الحلول الدورية للمعادلات الديناميكية المحايدة النبضية مع تأخير لانهائي على مقاييس زمنية. أثبتنا أن هذه المعادلات تمتلك حلولاً دورية تحت بعض الشروط باستخدام مبرهنة نقطة التثبيت لـ **Krasnoselskii** ومفهوم التقلص. كما أظهرنا أن الحل الدوري فريد تحت شروط أكثر صرامة، خاصة تلك التي تنطوي على استمرارية ليبشيتز وتحديدية الدوال المرتبطة. تقدم طريقتنا أساساً قوياً لدراسة المعادلات الديناميكية على مقاييس زمنية من خلال دمج الطرق التحليلية مع نظرية نقاط التثبيت، مما يمكن توسيعه ليشمل العديد من أنواع الأنظمة الديناميكية التي تتشارك في بنية مماثلة. تتيح النتائج فهماً أعمق للأنظمة التي تتميز بتأثيرات الذاكرة، مثل تلك التي تدفعها معادلات التأخير التفاضلية، وتكشف عن السلوك طويل الأجل لهذه الأنظمة في مجالات مثل نظرية التحكم، الاقتصاد، والنمذجة البيولوجية.

1. INTRODUCTION

Stephan Hilger's introduction of time scales (or measure chains) in 1988 provided a unifying framework for discrete and continuous calculus, as detailed in references [7, 8, 9]. This groundbreaking work initiated a thriving area of research, leading to significant advancements in the theory of dynamic equations on time scales. The scope of this theory has expanded considerably beyond its initial formulation, now encompassing a wide array of mathematical problems, as evidenced by [2, 3, 4, 18, 19] and their respective bibliographies. Furthermore, the study of impulsive initial and boundary value problems within this context has become an active area of investigation. For a comprehensive understanding of the theoretical foundations and established results in this field, the monographs [6, 16, 20] are highly recommended. In a recent contribution, A. Ben Fayed, M. Illafe, H. A. Makhzoum, and R.A. Elmansouri in [2] employed

Krasnoselskii's fixed point theorem to demonstrate the existence and uniqueness of solutions for a specific class of nonlinear neutral dynamic equations with infinite delay.

$$x^{\Delta}(t) = -\prod_{i=1}^{p} B_i(t)x^{\sigma}(t) + \sum_{i=1}^{p} W_i^{\Delta}\left(t, x\left(t - f(t)\right)\right) + \int_{-\infty}^{t} \left[\prod_{i=1}^{p} E_i(t,u)h\left(x(u)\right) + g(u)\right]\Delta u$$
$$+D\left(t, x(t), x\left(t - \Theta(t)\right)\right), t \neq t_j, t \in \mathbb{T}$$

In this paper, our central focus lies on delving into the qualitative properties of periodic solutions concerning impulsive neutral dynamic equations. Building upon the foundation laid by previous research in this area, particularly the insights and methodologies presented in the works [1-5, 10-15, 20-22] and their respective bibliographies, we aim to contribute to a deeper understanding of the behavior of solutions within the framework of the specific system under consideration. Our investigation will explore conditions for the existence, uniqueness, and stability of these periodic solutions, while also examining the impact of impulsive effects and neutral terms on the overall dynamics.

$$x^{\Delta}(t) = -\prod_{i=1}^{p} B_i(t)x^{\sigma}(t) + \sum_{i=1}^{p} W_i^{\Delta}\left(t, x\left(t - f(t)\right)\right) + \int_{-\infty}^{t} \left[\prod_{i=1}^{p} E_i(t,u)h\left(x(u)\right) + g(u)\right]\Delta u$$
$$+D\left(t, x(t), x\left(t - \Theta(t)\right)\right), t \neq t_j, t \in \mathbb{T}$$

$$x\left(t_j^{+}\right) = x\left(t_j^{-}\right) + I_j\left(x(t_j)\right), j \in \mathbb{Z}^{+}, \tag{1.1}$$

where $\mathbb{T}$ represents an ω-periodic time scale, $0 \in \mathbb{T}$ and $x^{\sigma} = x \circ \sigma$. For all interval U of $\mathbb{R}$, we denote by $U_{\mathbb{T}} = U \cap \mathbb{T}$, $x\left(t_j^{+}\right)$and $x\left(t_j^{-}\right)$represent the right and the left limit of $x\left(t_j\right)$ in the sense of time scales, in addition, if t_j is left-scattered, then $x\left(t_j^{-}\right) = x\left(t_j\right)$, $B(t) = \operatorname{diag}\left(b_i(t)_{n\times n}(b_i \in \mathcal{R}^{+})\right)$, $W: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $E: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $x: \mathbb{R} \to \mathbb{R}$, $h: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$, D: $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, are Continuous functions. Also, to ensure periodicity, we assumed that, $\mathcal{R}^{+} = \{b \in C(\mathbb{T}, \mathbb{R}): 1 + \mu(t)b(t) > 0\}$ where $\mu(t) = \sigma(t) - t$, $g \in C(\mathbb{T}, \mathbb{T})$, $f = (f_1, f_2, \dots, f_n) \in C(T \times \mathbb{R}, \mathbb{R})$, $h = (h_1, h_2, \dots, h_n) \in C(\mathbb{R}^n, \mathbb{R}^n)$, $I_j = \left(I_j^{(1)}, I_j^{(2)}, \dots, I_j^{(n)}\right) \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $B(t)$, $g(t), W(t, x(t - g(t)))$ are all ω-periodic functions with respect to $t, E(t + \omega, u + \omega) = E(t, u), \omega > 0$ without loss of generality, we also assume that $[0, \omega)_{\mathbb{T}} \cap \{t_j, j \in \mathbb{Z}^{+}\} = \{t_1, t_2, \dots, t_p\}$

Having converted the original system (1.1) into an equivalent integral system, the subsequent objective is to establish the existence of periodic solutions. To achieve this, we leverage Krasnoselskii's fixed point theorem, a powerful tool detailed in [21]. A crucial step involves demonstrating that the resulting integral equation can be expressed as the product of a compact operator and a contraction operator, setting the stage for applying the theorem. Furthermore, we aim to prove the uniqueness of the periodic solution. This will be accomplished by transforming system (1.1) into an integral equation and then employing the contraction mapping principle. The structure of the paper is as follows: Section 2 lays the groundwork by introducing the essential principles, notations, and necessary preliminary results. Section 3 then presents the core contributions of the paper, where we prove our key findings regarding the existence and uniqueness of periodic solutions, utilizing both the contraction mapping concept and Krasnoselskii's fixed point theorem.

.2. Preliminaries:

For notational convenience, we adopt the convention that for a time scale in$\mathbb{T}$, the interval $[a_1, a_n]_{\mathbb{T}}$ represents the set of all points t in $\mathbb{T}$ such that $a_1 \leq t \leq a_n$, where a_1 and a_n are points in T and $a_1 < a_n$. Furthermore, if a_n is equal to infinity, we denote the half-line $\mathbb{T}_a^+ = [a_1, \infty[_{\mathbb{T}}$, representing all points in T greater than or equal to a_1. We refer the reader to the comprehensive work by Bohner and Peterson (references (2) and (3)) for a detailed exposition of fundamental concepts and results in the theory of time scales. The subsequent analysis and results presented in this paper are based on the following assumptions, which will hold throughout.

(H_1)Let $f = (f_1, f_2, \dots, f_n)$ be a vector-valued function that adheres to a Lipschitz continuity condition with respect to the variable x. That is, for each component $j \in \{1,2,\dots,n\}$, there exists a non-negative constant L_j such that

$$\left|W_j(t,x) - W_j(t,y)\right| \leq L_j \|x - y\|$$

for all $t \in T$ and $x, y \in \mathbb{R}^n$. This condition ensures that the variation in each component of the function is bounded linearly by the normed distance between inputs, a critical property for ensuring uniqueness and stability of solutions.

(H_2 **)**Consider the function $h = (h_1, h_2, \dots, h_n)$, which similarly satisfies a Lipschitztype criterion with respect to the spatial variable x. Specifically, for each $\in \{1,2,\dots,n\}$, there exists a constant $M_{1j} > 0$ such that

$$\left|h_j(x) - h_j(y)\right| \leq M_{1j} \|x - y\|,$$

for all $x, y \in \mathbb{R}^n$. This condition guarantees controlled sensitivity of the function h to perturbations in its arguments.

(H_3)Let $D = (D_1, D_2, \dots, D_n)$ be a function that is also Lipschitz continuous in its arguments $x_1, x_2 \in \mathbb{R}^n$. For each index $j \in \{1,2,\dots,n\}$, there exist constants $M_{2j}, M_{3j} > 0$ such that

$$\left|D_j(t, x_1, x_2) - D_j(t, y_1, y_2)\right| \leq M_{2j} \|x_1 - y_1\| + M_{3j} \|x_2 - y_2\|$$

for all $t \in T$, and $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$. This dual Lipschitz condition permits multi-variable interactions to be estimated in a norm-controlled manner.

(H_4 **)**For each integer $k \in \mathbb{Z}$, let the mapping $I_k^{(j)}: \mathbb{R}^n \to \mathbb{R}$ fulfill a Lipschitz continuity condition. For every $j \in \{1,2,\dots,n\}$, there exists a constant $P_k^{(j)} > 0$ such that

$$\left|I_k^{(j)}(x) - I_k^{(j)}(y)\right| \leq P_k^{(j)} \|x - y\|$$

for all $x, y \in \mathbb{R}^n$. This ensures that impulsive or discrete modifications modeled via $I_k^{(j)}$ remain stable under small perturbations in state.

(H_5 **)**There exists a positive constant N_j such that the following integral inequality is satisfied:

$$\int_{-\infty}^{t} \left| \prod_{i=1}^{p} E_{ij}(t,u) \right| \Delta u \leq N_j$$

This condition bounds the accumulated effect of the matrix product E_{ij}, ensuring the integral remains finite and tractable, which is particularly vital in the context of dynamic equations on time scales.

To investigate the solvability of system (1.1) under the framework of Theorem 2.2, we first define the functional space:

$$PC(T) = \{x: T \to \mathbb{R}^n | x|_{(t_k,t_{k+1})_T} \in C(t_k, t_{k+1})_T, \exists \text{ limits from the left and right at } t_k\}.$$

This denotes the space of piecewise continuous functions on T, where continuity is preserved on subintervals between impulsive moments.

We further define the Banach space

$$X = \{x \in PC(T): x(t+\omega) = x(t)\},$$

equipped with the supremum norm

$$\|x\| = \max_{t\in[0,\omega]_T} |x(t)|_0, \text{ where } |x(t)|_0 = \max_{1\leq j\leq n} |x_j(t)|.$$

This space accommodates periodic solutions and supports the application of fixed-point theorems due to its completeness.

The following fixed-point result, commonly attributed to Krasnoselskii, provides a robust method for proving the existence of solutions in Banach spaces:

Theorem 2.1. Let M be a nonempty, closed, and convex subset of a Banach space (B, $\|\cdot\|$). Suppose there exist two operators R and Z mapping M into B such that:

- For all $x, y \in M$, the combination $Rx + Zy \in M$;
- R is compact;
- Z is a contraction, i.e., there exists $0 < \lambda < 1$ such that $\|Zx - Zy\| \leq \lambda\|x - y\|$.

Then there exists at least one point $p \in M$ such that

$$p = Rp + Zp$$

This theorem plays a pivotal role in demonstrating the existence of periodic or bounded solutions to complex dynamic systems, especially in the presence of nonlinearity and discontinuities.

Lemma 2.2. A function x qualifies as an ω-periodic solution to Equation (1.2) if and only if it also satisfies the alternative equation as an ω-periodic solution.

$$x(t)=\sum_{i=1}^{p} W_i(t,x(t-f(t)))+\int_t^{t+\omega} G(t,\tau)\left[\int_{-\infty}^{\tau}\left(\prod_{i=1}^{p} E_i(s,u)h(x(u))+g(u)\right)\Delta u\right.$$
$$\left.+\mathrm{D}(s,x(s),x(s-\Theta(s)))-\sum_{j=1}^{p} B(s)W_j^{\sigma}(s,x(s-f(s)))\right]\Delta s+\sum_{j,t_j\in[t,t+\omega]} G(t,t_j)I_j\left(x(t_j)\right)$$

where

$$G(t,s)=\operatorname{diag}\left(G_i(t,s)\right)_{n\times n}, G_i(t,s)=\left(1-e_{\ominus}\prod_{i=1}^{p} b_i(\omega,0)\right)^{-1} e_{\ominus\prod_{i=1}^{p} b_i}(t+\omega,s)$$
$$B(t)=\operatorname{diag}\left(\prod_{i=1}^{p} b_i(t)\right)_{n\times n}, e_{\ominus\prod_{i=1}^{p} b_i}(t,s)=\frac{1}{e_{\prod_{i=1}^{p} b_i}(t,s)},$$
$$\ominus\prod_{i=1}^{p} b_i(t)=-\frac{\prod_{i=1}^{p} b_i(t)}{1+\mu(t)\prod_{i=1}^{p} b_i(t)},\sum_{i=1}^{p} W_i^{\sigma}(t,x(t-f(t)))=\sum_{i=1}^{p} W_i(\sigma(t)),x^{\sigma}(t-f(t)).$$

Proof. Suppose that x is an ω-periodic solution of (1.1). Then, for every $t\in T$, there exists an integer $j\in\mathbb{Z}$ such that t_j denotes the first impulsive moment occurring after t. Under this construction, and for each index $j=1,2,\dots,n$, the corresponding segment x_j inherits the ω-periodicity and satisfies the associated equation

$$x_j^{\Delta}(t)+\prod_{i=1}^{p} b_{ij}(t)x_j^{\sigma}(t)=\sum_{i=1}^{p} W_{ij}^{\Delta}\left(t,x_j(t-f(t))\right)+\int_{-\infty}^{t}\left(\prod_{i=1}^{p} E_{ij}(t,u)h_j(x(u))+g(u)\right)\Delta u$$
$$+\mathrm{D}_j\left(t,x_j(t),x_j(t-\Theta(t))\right) \quad (2.1)$$

To proceed with the derivation, we multiply both sides of Equation (1.2) by the exponential function $e_{b_{ij}}(t,0)$, and subsequently perform integration over the interval $[t,s]_T$, where $s\in[t,t_k]_T$. We obtain

$$\int_t^s\left[e_{\prod_{i=1}^{p} b_{ij}}(\tau,0)x_j(\tau)\right]^{\Delta}\Delta\tau$$
$$=\int_t^s e_{\prod_{i=1}^{p} b_{ij}}(\tau,0)\left[\left[\sum_{i=1}^{p} W_{ij}^{\Delta}\left(\tau,x_j(\tau-f(\tau))\right)+\int_{-\infty}^{\tau}\left(\prod_{i=1}^{p} E_{ij}(\tau,u)h_j\left(x_j(u)\right)+g(u)\right)\Delta u+G_j\left(\tau,x_j(\tau)\right.\right.\right.$$
$$+\mathrm{D}_j\left(\tau,x_j(\tau),x_j(\tau-\Theta(\tau))\right)]\Delta\tau, t\in\mathbb{T}$$

or

$$e_{\prod_{i=1}^{p} b_{ij}}(s,0)x_j(s)=e_{\prod_{i=1}^{p} b_{ij}}(t,0)x_j(t)+\int_t^s e_{\prod_{i=1}^{p} b_{ij}}(\tau,0)\left[\sum_{i=1}^{p} W_{ij}^{\Delta}\left(\tau,x_j(\tau-f(\tau))\right)\right.$$
$$\left.+\int_{-\infty}^{\tau}\left(\prod_{i=1}^{p} E_{ij}(\tau,u)h_j\left(x_j(u)\right)+g(u)\right)\Delta u+\mathrm{D}_j\left(\tau,x_j(\tau),x_j(t-\Theta(\tau))\right)\right]\Delta\tau$$

then

$$x_j(s) = e_{\ominus}\prod_{i=1}^{p} b_{ij}(s,t)x_j(t) + \int_t^s e_{\ominus\prod_{i=1}^{p} b_{ij}}(s,\tau)\left[\sum_{i=1}^{p} W_{ij}^{\Delta}(\tau, x_j(\tau - f(\tau)))\right.$$
$$\left. + \int_{-\infty}^{\tau}\left(\prod_{i=1}^{p} E_{ij}(\tau,u)h_j(x_j(u)) + g(u)\right)\Delta u + D_j(\tau, x_j(\tau), x_j(\tau-\Theta(\tau)))\right]\Delta\tau, j = 1,2,\dots,n,$$

hence

$$x_j(t_k) = e_{\ominus\prod_{i=1}^{p} b_{ij}}(t_k,t)x_j(t) + \int_t^{t_k} e_{\ominus\prod_{i=1}^{p} b_{ij}}(t_k,\tau)\left[\sum_{i=1}^{p} W_{ij}^{\Delta}(\tau, x_j(\tau - f(\tau)))\right.$$
$$\left. + \int_{-\infty}^{\tau}\left(\prod_{i=1}^{p} E_{ij}(\tau,u)h_j(x_j(u)) + g(u)\right)\Delta u + D_j(\tau, x_j(\tau), x_j(\tau-\Theta(\tau)))\right]\Delta\tau, j = 1,2,\dots,n, \quad (2.2)$$

Similarly, for $s \in (t_k, t_{k+1}]$, we have

$$x_j(s) = e_{\ominus}\prod_{i=1}^{p} b_{ij}(s,t_k)x_j(t_k^+) + \int_{t_k}^s e_{\ominus\prod_{i=1}^{p} b_{ij}}(s,\tau)\left[\sum_{i=1}^{p} W_{ij}^{\Delta}(\tau, x_j(\tau - f(\tau)))\right.$$
$$\left. + \int_{-\infty}^{\tau}\left(\prod_{i=1}^{p} E_{ij}(\tau,u)h_j(x_j(u)) + g(u)\right)\Delta u + D_j(\tau, x_j(\tau), x_j(\tau-\Theta(\tau)))\right]\Delta\tau, i = 1,2,\dots,n,$$
$$= e_{\ominus}\prod_{i=1}^{p} b_{ij}(s,t_k)x_j(t_k^-) + \int_{t_k}^s e_{\ominus\prod_{i=1}^{p} b_{ij}}(s,\tau)\left[\sum_{i=1}^{p} W_{ij}^{\Delta}(\tau, x_j(\tau - h(\tau)))\right.$$
$$\left. + \int_{-\infty}^{\tau}\left(\prod_{i=1}^{p} E_{ij}(\tau,u)h_j(x_j(u)) + g(u)\right)\Delta u + D_j(\tau, x_j(\tau), x_j(\tau-\Theta(\tau)))\right]\Delta\tau$$
$$+ e_{\ominus}\left(\prod_{i=1}^{p} b_{ij}(s,t_k)x_i(t_k) + \int_{t_k}^s e_{\ominus}\prod_{i=1}^{p} b_{ij}(t_k)\right),$$
$$+ \int_{-\infty}^{\tau}\left(\prod_{i=1}^{p} E_{ij}(\tau,u)h_j(x_j(u)) + g(u)\right)W_{ij}^{\Delta}(\tau, x_j(\tau - f(\tau)))$$
$$+ \prod_{i=1}^{p} b_{ij}(s,t_k)I_k^{(i)}\left(x_j(t_k)\right)$$

Repeating the above steps for $s \in [t, t+\omega]_{\mathbb{T}}$, we have

$$x_j(s) = e_{\ominus\prod_{i=1}^{p} b_{ij}}(s,t)x_j(t) + \int_t^s e_{\ominus\prod_{i=1}^{p} b_{ij}}(s,\tau)\left[\sum_{i=1}^{p} W_{ij}^{\Delta}(\tau, x_j(\tau - f(\tau)))\right.$$
$$\left. + \int_{-\infty}^{\tau}\left(\prod_{i=1}^{p} E_{ij}(\tau,u)h_j(x_j(u)) + g(u)\right)\Delta u + D_j(\tau, x_j(\tau), x_{jj}(\tau - \Theta(\tau)))\right]\Delta\tau$$
$$+ \sum_{k, t_k \in [t, t+\omega)} e_{\ominus\prod_{i=1}^{p} b_{ij}}(s,t_k) I_k^{(j)}\left(x_j(t_k)\right),$$

for $i = 1,2,\dots,n$. Let $s = t + \omega$ in the above equality, we have

$$x_j(t+\omega) = e_{\ominus}\prod_{i=1}^{p} b_{ij}(t+\omega,t)x_j(t) + \int_t^{t+\omega} e_{\ominus}\prod_{i=1}^{p} b_{ij}(t+\omega,\tau)\left[\sum_{i=1}^{p} W_{ij}^{\Delta}(\tau, x_j(\tau - f(\tau)))\right.$$
$$\left. + \int_{-\infty}^{\tau}\prod_{i=1}^{p}(E_{ij}(\tau,u)h_j(x_j(u)) + g(u))\Delta u + \mathrm{D}_j(\tau, x_j(\tau), x_j(\tau - \Theta(\tau)))\right]\Delta\tau$$
$$+ \sum_{k, t_k \in [t, t+\omega]} e_{\ominus\prod_{i=1}^{p} b_{ij}}(t+\omega,t_k) I_k^{(j)}\left(x_j(t_k)\right),$$

$j = 1,2,\dots,n$. Noticing that $x_j(t+\omega) = x_j(t)$ and $e_{\ominus}\prod_{i=1}^{p} b_{ij}(t+\omega,t) = e_{\ominus}\prod_{i=1}^{p} b_{ij}(\omega,0)$, we obtain

$$\left(1 - e_{\ominus\prod_{i=1}^{p} b_{ij}}(\omega,0)\right)x_j(t) = \int_t^{t+\omega} e_{\ominus\prod_{i=1}^{p} b_{ij}}(t+\omega,\tau)\left[\sum_{i=1}^{p} W_{ij}^{\Delta}(\tau, x_j(\tau - f(\tau)))\right.$$
$$\left. + \int_{-\infty}^{\tau}\left(\prod_{i=1}^{p}(E_{ij}(\tau,u)h_j(x_j(u)) + g(u))\right)\Delta u + \mathrm{D}_j(\tau, x_j(\tau), x_j(\tau - \Theta(\tau)))\right]\Delta\tau$$
$$+ \sum_{k, t_k \in [t, t+\omega]} e_{\ominus\prod_{i=1}^{p} b_{ij}}(t+\omega,t_k) I_k^{(j)}\left(x_j(t_k)\right) \quad (2.3)$$

for $j = 1,2,\dots,n$. Notice that

(2.4)

$$\int_t^{t+\omega} e_{\ominus \prod_{i=1}^p b_{ij}}(t+\omega,\tau)\left(\sum_{i=1}^p W_{ij}^{\Delta}\left(\tau, x_j(\tau - f(\tau))\right)\right)\Delta\tau$$

$$= e_{\ominus \prod_{i=1}^p b_{ij}}(t+\omega,t+\omega)\left(\sum_{i=1}^p W_{ij}\big(t+\omega, x_j(t+\omega - f(t+\omega))\big)\right)$$

$$-e_{\ominus \prod_{i=1}^p b_{ij}}(t+\omega,t)\sum_{i=1}^p W_{ij}\big(t, x_j(t-g(t))\big)$$

$$= \int_t^{t+\omega} e_{\ominus \prod_{i=1}^p b_{ij}}(t+\omega,\tau)\left(\prod_{i=1}^p b_{ij}(\tau)\right)\left(\sum_{i=1}^p W_{ij}^{\sigma}(\tau, x_i(\tau - f(\tau)))\right)\Delta\tau,$$

$$= \left[1 - e_{\ominus \prod_{i=1}^p b_{ij}}(\omega,0)\right]\sum_{i=1}^p W_{ij}\big(t, x_j(t-f(t))\big)$$

$$-\int_t^{t+\omega} e_{\ominus \prod_{i=1}^p b_{ij}}(t+\omega,\tau)\left(\prod_{i=1}^p b_{ij}(\tau)\right)\left(\sum_{i=1}^p W_{ij}^{\sigma}\big(\tau, x_j(\tau - f(\tau))\big)\right)\Delta\tau, j = 1,2,\dots,n$$

It follows from (2.3) and (2.4) that

$$x_j(t) = \sum_{i=1}^p W_{ij}\big(t, x_j(t-f(t))\big) + \int_t^{t+\omega}\left[1 - e_{\ominus \prod_{i=1}^p b_{ij}}(\omega,0)\right]^{-1} e_{\ominus \prod_{i=1}^p b_{ij}}(t+\omega,\tau)$$

$$\times\left[\int_{-\infty}^{\tau}\left(\prod_{i=1}^p E_{ij}(\tau,u)h_j\big(x_j(u)\big) + g(u)\right)\Delta u + \mathrm{D}_j\big(\tau, x_j(\tau), x_j(\tau - \Theta(\tau))\big)\right.$$

$$\left.-\left(\prod_{i=1}^p b_{ij}(\tau)\right)\left(\sum_{i=1}^p W_{ij}^{\sigma}\big(\tau, x_{ij}(\tau - f(\tau))\big)\right)\right]\Delta\tau$$

$$+\sum_{k, t_k \in [t, t+\omega]}\left[1 - e_{\ominus}\prod_{i=1}^p b_{ij}(\omega,0)\right]^{-1} e_{\ominus \prod_{i=1}^p b_{ij}}(t+\omega,t_k) I_k^{(j)}\left(x_j(t_k)\right)$$

$$= \sum_{i=1}^p W_{ij}\big(t, x_j(t-f(t))\big) + \int_t^{t+\omega} G_j(t,\tau)\left[\int_{-\infty}^{\tau}\left(\prod_{i=1}^p E_{ij}(\tau,u)h_j\big(x_j(u)\big) + g(u)\right)\Delta u\right.$$

$$\left.+D_j\big(\tau, x_j(\tau), x_j(\tau-\Theta(\tau))\big) - \left(\prod_{i=1}^p b_{ij}(\tau)\right)\left(\sum_{i=1}^p W_{ij}^{\sigma}\big(\tau, x_j(\tau - f(\tau))\big)\right)\right]\Delta\tau$$

$$+\sum_{k, t_k \in [t, t+\omega]} G_j(t,t_k) I_k^{(j)}\left(x_j(t_k)\right)$$

for $j = 1,2,\dots,n$. Next, we prove the converse. Let

$$x_j(t)=\sum_{i=1}^{p} W_{ij}\big(t,x_j(t-f(t))\big)+\int_t^{t+\omega} G_j(t,\tau)\left[\int_{-\infty}^{\tau}\left(\prod_{i=1}^{p} E_{ij}(s,u)h_j\big(x_j(u)\big)+g(u)\right)\Delta u\right.$$

$$\left.+\mathrm{D}(s,x(s),x(s-\Theta(s)))-\left(\prod_{i=1}^{p} b_i(s)\right)\left(\sum_{i=1}^{p} W_{ij}^{\sigma}(s,x_j(s-f(s)))\right)\right]\Delta s$$

$$+\sum_{k,t_k\in[t,t+\omega]} G_j(t,t_k)I_k^{(j)}\big(x_j(t_k)\big)$$

where

$$G_j(t,s)=\left(1-e_{\ominus}\prod_{i=1}^{p} b_{ij}(\omega,0)\right)^{-1} e_{\ominus\prod_{i=1}^{p} b_{ij}}(t+\omega,s), j=1,2,\dots,n$$

Then if $t\neq t_j, j\in\mathbb{Z}^+$, we have

If $t=t_j, j\in\mathbb{Z}^+$, we obtain

$$x_j^{\Delta}(t)=\sum_{i=1}^{p} W_{ij}^{\Delta}\big(t,x_j(t-f(t))\big)+\int_t^{t+\omega}\Big\{G_j(t,s)[\int_{-\infty}^{s}\left(\prod_{i=1}^{p} E_{ij}(s,u)h_j\big(x_j(u)\big)+g(u)\right)\Delta u$$

$$+\mathrm{D}_j\big(t,x_j(t),x_j(t-\Theta(t))\big)-\Big(\prod_{i=1}^{p} b_{ij}(t)\Big)\sum_{i=1}^{p} W_{ij}^{\Delta}\Big(t,x_j\big(t-f(t)\big)\Big)]\Big\}^{\Delta}\Delta s$$

$$+G_j(t,t+\omega)\left[\int_{-\infty}^{t+\omega}\left(\prod_{i=1}^{p} E_{ij}(t+\omega,u)h_j\big(x_j(u)\big)+g(u)\right)\Delta u\right.$$

$$+\mathrm{D}_j\big(t+\omega,x_j(t+\omega),x_j(t+\omega-\lambda(t+\omega)\big)$$

$$-\left(\prod_{i=1}^{p} b_{ij}(t+\omega)\right)\sum_{i=1}^{p} W_{ij}^{\Delta}\Big(t+\omega,x_j\big(t+\omega-f(t+\omega)\big)\Big)$$

$$-G_j(t,t)[\int_{-\infty}^{t}\left(\prod_{i=1}^{p} E_{ij}(t,u)h_j\big(x_j(u)\big)+g(u)\right)\Delta u$$

$$+\mathrm{D}_j\big(t,x_j(t),x_j(t-\Theta(t))\big)\ -\big(\prod_{i=1}^{p} b_{ij}(t)\big)\sum_{i=1}^{p} W_{ij}^{\Delta}\big(t,x_j(t-f(t))\big)]$$

$$=\sum_{i=1}^{p} W_{ij}^{\Delta}\big(t,x_j(t-f(t))\big)+\int_{-\infty}^{t}\left(\prod_{i=1}^{p} E_{ij}(t,u)h_j\big(x_j(u)\big)+g(u)\right)\Delta u$$

$$+D_j\big(t,x_j(t),x_j(t-\Theta(t))\big)-\left(\prod_{i=1}^{p} b_{ij}(t)\right)\sum_{i=1}^{p} W_{ij}^{\Delta}\big(t,x_j(t-f(t))\big)$$

$$+\int_t^{t+\omega}\Big\{G_j(t,s)[\int_{-\infty}^{s}\left(\prod_{i=1}^{p} E_{ij}(s,u)h_j\big(x_j(u)\big)+g(u)\right)\Delta u$$

$$+\mathrm{D}_j\big(t,x_j(t),x_j(t-\Theta(t))\big)$$

$$-\left(\prod_{i=1}^{p} b_{ij}(t)\right)\sum_{i=1}^{p} W_{ij}^{\Delta}\left(t, x_j(t-f(t))\right)]\Bigg\}^{\Delta} \Delta s$$

$$= -\prod_{i=1}^{p} b_{ij}(t)x_j^{\sigma}(t) + \sum_{i=1}^{p} W_{ij}^{\Delta}(t, x_j(t-g(t)))$$

$$+\int_{-\infty}^{t}\left(\prod_{i=1}^{p} E_{ij}(t,u)h_j\left(x_j(u)\right)+g(u)\right)\Delta u$$

$$+ \mathrm{D}_j\left(t, x_j(t), x_j(t-\Theta(t))\right)$$

$$x_j(t_j^+) - x_j(t_j^-) = \sum_{k,t_k\in[t_k^+,t_k^++\omega]} G_j(t_j,t_k)I_k^{(j)}\left(x_j(t_k)\right) - \sum_{k,t_k\in[t_k^-,t_k^-+\omega]} G_j(t_j,t_k)I_k^{(j)}\left(x_j(t_k)\right)$$
$$= G_j(t_j,t_j+\omega)I_j^{(j)}\left(x_j(t_j+\omega)\right) - G_j(t_j,t_j)I_j^{(j)}\left(x_j(t_j)\right)$$
$$= I_j^{(j)}\left(x_j(t_j)\right), j = 1,2,\ldots,n$$

Iterating the preceding procedure over the interval $s \in [t, t+\omega]_T$, the resulting expression can be generalized accordingly. Consequently, it follows that the function x also satisfies (1.1) as an ω-periodic solution. This conclusion confirms the equivalence of the periodic solutions across the two formulations and thereby finalizes the proof

Lemma 2.3 (16). Assume that $x \in X$. It then follows that the norm of the shifted function x^{σ} exists, and moreover, satisfies the identity $\|x^{\sigma}\| = \|x\|$. Importantly, we observe that the Greens function component $G_j(t,s)$ is uniformly bounded above by the constant

$$\left(1 - e_{\ominus b_j}(\omega, 0)\right)^{-1},$$

which we denote as η_j. To streamline the exposition, the following abbreviated notation will be adopted in subsequent developments.

$$\bar{\eta} := \max_{1\le j\le n} \eta_j, \gamma := \max_{1\le j\le n, t\in[0,\omega]_T} |b_j(t)|, L := \max_{1\le j\le n} L_j^p, M := \max_{1\le j\le n} M_j$$
$$N := \max_{1\le j\le n} N_j, P_k := \max_{1\le j\le n} P_k^{(j)}, P := \max_{1\le k\le p} P_k$$

Let the mapping $H: X \to X$ defining as follows

(2.5)

$$H\phi(t) = \sum_{i=1}^{p} W_i(t,\phi(t-f(t))) + \int_t^{t+\omega} G(t,s)\left[\int_{-\infty}^{s}\left(\prod_{i=1}^{p} E_i(s,u)h(\phi(u)) + g(u)\right)\Delta u\right.$$
$$\left. + \mathrm{D}(t,\phi(t),\phi(t-\Theta(t))) - B(s)\sum_{i=1}^{p} W_i^{\sigma}(s,\phi(s-f(s)))\right]\Delta s + \sum_{k,t_k\in[t,t+\omega)} G(t,t_k)I_k(\phi(t_k)),$$

To invoke the conditions of Theorem 2.1, it is necessary to define two separate operators: one that exhibits contractive properties and another that is both continuous and compact. With this objective in mind, we decompose Equation (2.5) into an equivalent operator formulation suitable for applying the theorem.

$$(H\phi)(t) = (R\phi)(t) + (Z\phi)(t)$$

where

$$(R\phi)(t) = \sum_{i=1}^{p} W(t,\phi(t-f(t))), \tag{2.6}$$

and

$$(Z\phi)(t) = \int_t^{t+\omega} G(t,s)\left[\int_{-\infty}^{s}\left(\prod_{i=1}^{p} E_i(s,u)h(\phi(u)) + g(u)\right)\Delta u + \mathrm{D}(t,\phi(t),\phi(t-\Theta(t)))\right.$$
$$\left. -B(t)\sum_{i=1}^{p} W_i^{\sigma}(t,\phi(s-f(s)))\right]\Delta s + \sum_{N,t_{Nj}\in[t_j,t_j+\omega]} G(t,t_j)I_j\left(x_i(t_j)\right) \tag{2.7}$$

Lemma 2.4. Suppose that the condition (H1) is satisfied and that the constant L_1, associated with the Lipschitz behavior of the involved operator, fulfills the inequality $L_1 < 1$. Under these assumptions, the operator $R: X \to X$, as defined in Equation (1.8), satisfies the contraction mapping property.

Proof. Let $R: X \to X$. For $\phi, \psi \in X$, we have

$$\| R(\phi) - R(\psi) \| = \max_{t\in[0,\omega]_{\mathbb{T}}} \max_{1\le j\le n} \sum_{\substack{i=1\\ ij}}^{p} | W_{ij}\left(t,\phi_j(t-h(t))\right) - W_{ij}\left(t,\psi_j(t-h(\mathrm{t}))\right) |$$
$$\le L_1 \| \phi - \Psi \|. \tag{2.8}$$

Lemma 2.5.] Suppose that the assumptions (H1) through (H5) are all satisfied. Then, the operator $Z: X \to X$, as defined in Equation (2.7), is not only continuous but also completely continuous (i.e., compact) on the space X.

Proof. Measuring (2.7) at $t + \omega$ gives

$$(Z\phi)(t+\omega)$$
$$=\int_{t+\omega}^{t+2\omega} G(t+\omega,s)\left[\int_{-\infty}^{s}\left(\prod_{i=1}^{p} E_i(s,u)h(\phi(u))+g(u)\right)\Delta u+\mathrm{D}(s,\phi(s),\phi(s-\Theta(s)))\right.$$
$$\left.-B(s)\sum_{i=1}^{p} W_i^{\sigma}(s,\phi(s-f(s)))\right]\Delta s+\sum_{k,t_k\in[t+\omega,t+2\omega]} G(t+\omega,t_k)I_k(\phi(t_k))$$
$$=\int_{t}^{t+\omega} G(t+\omega,v+\omega)\left[\int_{-\infty}^{v+\omega}\left(\prod_{i=1}^{p} E_i(v+\omega,u)h(\phi(u))+g(u)\right)\Delta u\right.$$
$$\left.+\mathrm{D}(v+\omega,\phi(v+\omega),\phi(v+\omega-\Theta(v+\omega)))-B(v+\omega)\sum_{i=1}^{p} W_i^{\sigma}(v+\omega,\phi(v+\omega-f(v+\omega)))\right]\Delta v$$
$$+\sum_{y,t_y\in[t,t+\omega]} G(t,t_y)I_k\left(\phi(t_y)\right)$$

$$=\int_{t}^{t+\omega} G(t,v)\left[\int_{-\infty}^{v}\left(\prod_{i=1}^{p} E_i(v,u)h_i(\phi(u))+g(u)\right)\Delta u+\mathrm{D}(v,\phi(v),\phi(v-\Theta(v)))\right.$$

$$\left.-B(v)\sum_{i=1}^{p} W_i^{\sigma}(v,\phi(v-f(v)))\right]\Delta v+\sum_{y,t_y\in[t,t+\omega]} G_i(t,t_y)I_k\left(\phi(t_y)\right)$$
$$=(Z\phi)(t)$$

Therefore, $Z: X\to X$

We now proceed to demonstrate the continuity of the operator R. To this end, let $\phi,\psi\in X$ be arbitrary elements, and suppose that $\epsilon>0$ is given. We then choose an appropriate quantitydepending on ϵ and the structure of R to establish that small variations in the input result in correspondingly small changes in the output, thereby confirming the continuity of R.

$$\delta=\frac{\varepsilon}{\bar{\eta}[\omega(M_1N+L\gamma)+P]+M_2+M_3}$$

such that for $\parallel \phi-\psi\parallel\le\delta$. By applying the Lipschitz condition, we get

$$\| R\vartheta - R\psi \|$$

$$\leq \max_{t\in[0,\omega]_{\mathbb{T}}} \int_t^{t+\omega} G(t,s)\left[\int_{-\infty}^{s}\left(\prod_{i=1}^{p} E_i(s,u)h(\phi(u)) + g(u)\right)\Delta u - \int_{-\infty}^{s}\left(\prod_{i=1}^{p} E_i(s,u)h(\psi(u)) + g(u)\right)\Delta u\right.$$

$$+ \max_{t\in[0,\omega]_{\mathbb{T}}} |\mathrm{D}(s,\phi(s),\phi(s-\Theta(s))) - \mathrm{D}(s,\psi(s),\psi(s-\Theta(s)))|$$

$$+ \max_{t\in[0,\omega]_{\mathbb{T}}} \left|\int_t^{t+\omega} G(t,s)B(s)\left[\sum_{i=1}^{p} W^{\sigma}(s,\phi(s-f(s))) - \sum_{i=1}^{p} W^{\sigma}(s,\psi(s-F(s)))\right]\Delta s\right|_0$$

$$+ \max_{t\in[0,\omega]_{\mathbb{T}}} \sum_{k,t_k\in[t,t+\omega)} \left|G(t,t_k)\left[I_k\left(\phi(t_k) - I_k(\psi(t_k))\right)\right]\right|_0$$

$$\leq \bar{\eta}\int_{-\infty}^{\omega}\prod_{i=1}^{p} |E_i(s,u)[h(\phi(u)) - h(\psi(u))]|_0 \Delta u\Delta s$$

$$+M_1|\phi-\psi| + M_2|\phi-\psi| + \bar{\eta}\gamma\int_0^{\omega}\left|\sum_{i=1}^{p} W^{\sigma}(s,\vartheta(s-f(s))) - \sum_{i=1}^{p} W^{\sigma}(s,\psi(s-h(s)))\right|_0 \Delta s$$

$$+\bar{\eta}_{1\leq k\leq p}\left|I_k\left(\vartheta(t_k) - I_k(\psi(t_k))\right)\right|_0$$

$$\leq (\bar{\eta}[\omega(M_1N + L\gamma) + P] + M_2 + M_3) \| \vartheta - \psi \| < \varepsilon$$

The preceding argument establishes the continuity of the operator ψ. Moving forward, it remains to demonstrate that ψ is also a compact operator. To this end, consider a sequence of periodic functions $\{\phi_n\} \subset X$ that is uniformly bounded; that is, there exists a constant $\Theta > 0$ such that $\|\phi_n\| \leq \Theta$ for all $n \in \mathbb{N}$. Under the assumptions (H1) - (H3), we are then able to derive the necessary estimates that lead to the relative compactness of $\{\psi(\phi_n)\}$.

where $\alpha_f = \|W^{\sigma}(t,0)\|, \alpha_h = \| h(0) \|, \alpha_D = \| \mathrm{D}(t,0,0) \|, \alpha_g = \| g(0) \|$ and $\alpha_{I_k} = \|I_k(0)\|$ hence

$$\|W^{\sigma}(t,x)\| \leq \|W^{\sigma}(t,x) - W^{\sigma}(t,0)\| + \|W^{\sigma}(t,0)\|$$

$$= \max_{t\in[0,\omega]_{\mathbb{T}} 1\leq j\leq n} \left|W_j^{\sigma}(t,x) - W_j^{\sigma}(t,0)\right| + \alpha_f$$

$$\leq L \| x \| + \alpha_{\varphi} \tag{2.9}$$

$$\| h(x) \| \leq \| h(x) - h(0) \| + \| h(0) \|$$

$$= \max_{1\leq j\leq n} \left|h_j(x) - h_j(0)\right| + \alpha_h$$

$$\| g(x) \| \leq \| g(x) - g(0) \| + \| g(0) \|$$

$$\leq H \| x \| + \alpha_g \tag{2.10}$$

$$\| D(t,x(t),x(t-\tau)) \| \leq \| D(t,x(t),x(t-\Theta(t))) - D(t,0,0) \| + \| D(t,0,0) \|$$

$$\leq \max_{1\leq j\leq n} \left| D\left(t,x_j(t),x_j(t-\Theta(s))\right) - D(t,0,0)\right| + \alpha_D$$

$$\leq (M_2+M_3) \| x \| + \alpha_D \tag{2.11}$$

$$\|I_k(x)\| \leq \|I_k(x) - I_k(0)\| + \|I_k(0)\|$$

$$= \max_{1\leq j\leq n} n \left| I_k^{(j)}(x) - I_k^{(j)}(0)\right| + \alpha_{I_k}$$

$$\leq P_k \| x \| + \alpha_{I_k}, \text{ for } j \in \mathbb{Z}^+ \tag{2.12}$$

$$\| \Psi\phi_n \| \leq \max_{t\in[0,\omega]} \left| \int_t^{t+\omega} G(t,s)\left[\int_{-\infty}^{s} \prod_{i=1}^{p} E_i(s,u)h(\phi_n(u)) + g(\phi_n)\Delta u \right|\right.$$
$$\left. - B(s)\sum_{i=1}^{p} W_i^\sigma\left(s,\phi_n\left(s-f(s)\right)\right]\Delta s\right|_0 + \max_{t\in[0,\omega]} |D(t,\phi_n(t),\phi_n(t-\Theta(t)))|_0$$
$$+ \max_{t\in[0,\omega]_T} \sum_{k,t_k\in[t,t+\omega]} \left|G(t,t_k)I_k\left(\varphi_n(t_k)\right)\right|_0$$
$$\leq \bar{\eta}\int_0^\omega \int_{-\infty}^{s} \left(\prod_{i=1}^{p} |E_i(s,u)h(\phi_n(u))|_0 + |g(\phi_n)|_0\right)\Delta u\Delta s + \bar{\eta}\gamma\int_0^\omega \left|\sum_{i=1}^{p} W_i^\sigma(s,\phi_n(s-f(s)))\right|_0^p \Delta s$$
$$+ |D(t,\phi(t),\phi(t-\Theta(t)))|_0 + \bar{\eta}\sum_{k=1}^{p} \left|I_k\left(\phi_n(t_k)\right)\right|_0$$
$$\leq \bar{\eta}\omega N\left(M_1\|\phi_n\| + \alpha_h + H\|\phi_n\| + \alpha_g\right) + \bar{\eta}\gamma\omega\left(L\|\phi_n\| + \alpha_f\right) + \bar{\eta}\gamma\int_0^\omega \left\|\sum_{i=1}^{p} W_i^\sigma(s,\phi_n(s-f(s)))\right\|_0 \Delta s$$
$$+ (M_2+M_3) \| \phi \| + \alpha_D + \bar{\eta}\sum_{k=1}^{p} \left|I_k\left(\phi_n(t_k)\right)\right|_0$$
$$\leq \bar{\eta}\omega \ominus (M_1N + HN + \gamma L) + \bar{\eta}\left(\omega N\alpha_h + \omega N\alpha_g + \gamma\omega\alpha_f + P\ominus + \alpha\right) + (M_2+M_3)\ominus := D$$

Here, $\alpha = \max_{1\leq k\leq p} \alpha_{I_k}$, where the constant α_{I_k} is associated with the impulse effects at discrete points.

$$(R\phi_n)^{\Delta}(t) = \quad -B(t)(R\phi_n)^{\sigma}(t) + \int_{-\infty}^{t}\left(\prod_{i=1}^{p} E_i(s,u)h(\phi_n(u)) + g(u)\right)\Delta u + D(t,\phi(t),\phi(t-\Theta(t)))$$
$$-B(t)\sum_{i=1}^{p} \phi^{\sigma}(t,\phi_n(t-f(t)))$$

Therefore, it is clear from the Lemma (2.3)and the (2.9)-(2.13) that

$$\left|(R\phi_n)^{\Delta}(t)\right|_0 \leq \| B \| \, \|(R\phi_n)^{\sigma}\| + \left\|\int_{-\infty}^{t}\prod_{i=1}^{p} E_i(s,u)h(\phi_n(u))\Delta u\right\| + \left\|\int_{-\infty}^{t} g(u)\Delta u\right\|$$
$$+\| \mathrm{D}(t,\phi_n(t),\phi_n(t-\Theta(t)) + \left|B(t)\sum_{i=1}^{p} \phi^{\sigma}(t,\phi_n(t-f(t)))\right|_0$$
$$\leq \| B \| \, \|(R\phi_n)\| + N\big(M_1\|\phi_n\| + \alpha_h + H\|\phi_n\| + \alpha_g\big) + (M_2+M_3)\|\phi_n\| + \alpha_D + \| B \| \, \big(L\|\phi_n\| + \alpha_f\big)$$
$$\leq \| B \| \, (D + L \ominus + \alpha_f) + N(NM_1 + \alpha_h) + N(M_2+M_3) + \alpha_D$$

for all n. That is, $\|(R\phi_n)^{\Delta}\| \leq \| B \| \, (D + L \ominus + \alpha_f) + N(NM + \alpha_h) + N(M_2+M_3) + \alpha_D$.

As a consequence, the sequence $\{R\phi_n\}$ is uniformly bounded. With this bound established, it now becomes straightforward to verify that the operator R maps bounded sets into relatively compact sets, or alternatively, to demonstrate equicontinuity of the sequence $\{R\phi_n\}$, depending on the specific properties under consideration.

3. Main Results.

Theorem 3.1.

Suppose that assumptions (H1) through (H5) are satisfied, and that the Lipschitz constant L_1 obeys the condition $L_1 < 1$. Furthermore, assume that there exists a non-negative constant G such that for every solution $x \in X$ of Equation (1.1), the inequality $\|x\| \leq G$ holds uniformly. Under these conditions, the following estimate is satisfied:

$$\frac{\gamma\omega\alpha_f + \omega N a_h + \alpha}{1 \setminus \bar{\eta} - [\omega(\gamma L + M_1 N)] - [L \setminus \bar{\eta}] - P} \leq \mathcal{H}$$

holds. Hence (1.1) has an ω-periodic solution.

Proof. Define the set $M = \{\phi \in X : \|\phi\| \leq H\}$. According to Lemma 2.5, the operator $R: X \to X$ is both compact and continuous. Furthermore, as established in Lemma 2.4, R is a contraction mapping on X. Meanwhile, the operator $Z: X \to X$ is also well-defined. Our goal is to demonstrate that for any pair $\phi, \psi \in M$, the inequality $\|R\phi + Z\psi\| \leq H$ holds.

To proceed, consider arbitrary elements $\phi, \psi \in M$ such that $\|\phi\| \leq H$ and $\|\psi\| \leq H$. Using the estimates derived from Equations (2.9)(2.11), we obtain the necessary bound on the combined operator acting on these elements:

$$\|R\phi + Z\psi\| \leq \|R\phi\| + \|Z\psi\|$$

$$\leq LG + \bar{\eta}\omega G(\gamma L + M_1 N) + \bar{\eta}(\gamma\omega\alpha_f + \omega N\alpha_h + GP + \alpha) \leq G.$$

It follows that $R\phi + Z\psi \in M$, thereby demonstrating that the operator sum preserves the bounded set M. This result, in conjunction with the established properties of the operatorsnamely, that R is a contraction and Z is continuous and compactconfirms that all the necessary conditions for applying Krasnoselskiis Fixed-Point Theorem are fulfilled within the space X.

In light of Lemma 2.3, we then deduce the existence of a fixed point $z \in M$ satisfying the relation

$$z = Rz + Zz.$$

This fixed point corresponds precisely to a solution of the original Equation (1.1), thus proving the existence of at least one solution under the given assumptions

Theorem 3.2

Let $(H_1) - (H_5)$ be hold. If

$$\Gamma := \bar{\eta}[\omega(\gamma + +M_1 N + M_1 + M_2) + P] < 1$$

hence 1.1 has an unique ω-periodic solution

Proof. for $\phi, \psi \in X$, we have

$$\| H\phi - H\psi \| \leq \bar{\eta}\int_0^{\omega}\int_{-\infty}^{s}\left|\prod_{i=1}^{p} E_i(s,u)h(\phi(u)) - \prod_{i=1}^{p} E_i(s,u)h(\psi(u))\right|_0 \Delta u\Delta s$$

$$+\bar{\eta}\omega(M_2 + M_3) \| \phi - \psi \| + \bar{\eta}\gamma\int_0^{\omega}\left|\sum_{i=1}^{p}\phi^{\sigma}(t,\phi(t - f(t))) - \sum_{i=1}^{p}\phi^{\sigma}(t,\psi_n(t - f(t)))\right|_0 \Delta s$$

$$+\bar{\eta}\sum_{j=1}^{p}\left|I_j\left(\phi(t_j)\right) - I_j\left(\psi(t_j)\right)\right|_0$$

$$\leq \bar{\eta}\omega M_1 N \| \phi - \psi \| + \bar{\eta}\gamma\omega L \| \phi - \psi \| + \bar{\eta}P \| \phi - \psi \|$$

$$\leq \bar{\eta}[(\omega\gamma L + M_1 N + M_2 + M_3) + P] \| \phi - \psi \|$$

$$= \gamma \| \phi - \psi \|$$

The proof is complete .

Conclusion

In this work, we investigated the existence and uniqueness of periodic solutions for impulsive neutral dynamic equations with infinite delay on time scales.We proved that under some conditions these equations have periodic solutions by using Krasnoselskii's fixed point theorem and the contraction mapping concept.

We prove that impulsive neutral dynamic equations with infinite delay on periodic time scales admit periodic solutions.This was obtained by applying Krasnoselskii's fixed point theorem and turning the system into an integral equation.

We proved that the periodic solution is unique under more strict conditions—more especially, involving Lipschitz continuity and limitedness of the associated functions.

This was accomplished by applying the contraction mapping concept, therefore guaranteeing that the lone solution under the specified constraints is likewise periodic.

Our method presented a strong foundation for investigating dynamic equations on time scales by combining analytical methods with fixed-point theory. This approach can be expanded to many kinds of dynamic systems having comparable structures.

The findings have important ramifications for systems with memory effects—such as those driven by delay differential equations. Crucially in disciplines including control theory, economics, and biological modeling, the periodic character of the solutions can reveal understanding of the long-term behavior of such systems.

Further research could explore the stability of these periodic solutions and their sensitivity to initial conditions or parameter changes.

Extending the analysis to stochastic dynamic equations on time scales could provide additional insights into real-world systems subject to random fluctuations.

This work contributes to the broader understanding of dynamic equations on time scales, offering both theoretical insights and practical tools for analyzing complex systems with delay and impulsive effects.

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