The Existence and Uniqueness of Periodic Solutions for Nonlinear Neutral first order differential equation with Functional Delay

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ABSTRACT

This paper is devoted to employ the fixed point theorem of Krasnoselskii, to show the existence and uniqueness of periodic solutions of the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -\prod_{i=1}^{p} a_i(t)\,x(t) + \frac{d}{dt}\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right) + \int_{-\infty}^{t}\left[\prod_{i=1}^{p} D_i(t,s)\,f\big(x(s)\big) + h(s)\right] ds + G\left(t, x(t), x\big(t-\tau(t)\big)\right)$$

By modifying the given neutral differential equation into an equivalent integral equation using lemma (2.1). This is done by creating a suitable operators, one is a compact and the other is contraction , which allow us to prove the existence of periodic solutions. Also, we used the Banach fixed point theorem to guarantee a unique periodic solution.

Key Words: Fixed Krasnoselskii's fixed point Theorem, Nonlinear Neutral equation, Functional Delay

الملخص

هذه الورقة مخصصة لتوظيف نظرية النقطة الثابتة لكراسنوسيلسكي ، لإظهار وجود وتفرد الحلول الدورية للمعادلة التفاضلية المحايدة غير الخطية

$$\frac{d}{dt}x(t) = -\prod_{i=1}^{p} a_i(t)\,x(t) + \frac{d}{dt}\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right) + \int_{-\infty}^{t}\left[\prod_{i=1}^{p} D_i(t,s)\,f\big(x(s)\big) + h(s)\right] ds + G\left(t, x(t), x\big(t-\tau(t)\big)\right)$$

عن طريق تعديل المعادلة التفاضلية المحايدة إلى معادلة تكاملية مكافئة باستخدام التمهيدية (2.1). يتم ذلك عن طريق إنشاء عوامل تشغيل مناسبة، أحدها مدمج والآخر هو الانكماش، مما يسمح لنا بإثبات وجود حلول دورية. أيضا، استخدمنا نظرية بناخ الثابتة لضمان وجود حل فريد.

1. INTRODUTION

In the recent decades,the fixed point theorem was a power full tool to show the existence and uniqueness of solutions in a widerange of mathematical issues. Theorem of Krasnoselskii is one of the most interesting results which is introduced by [8], the main idea of this theory is the blending between the Banach contraction principle [2], Schauder's fixed point theorem."which is produced by the famous scholer [12], it has a big effect on the fixed point theory". Krasnoselskii theorem has attracted many scientists and experts in this field.For a wealth of reference material on the subject, we refer to [1, 3 ,4, 6,7, 8,10,11,12] and the references in them.

This study is mainly inspired by the work of [9] in which they obtained adequate conditions for the existence of periodic solutions for the equation

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}\sum_{i=1}^{p} Q_i(t, x(t-g(t))) + \int_{-\infty}^{t}[D(t,s)f(x(s)) + h(s)]\,ds,$$

by assuming a(t) is a continuous real-valued function.Taking into consideration $Q: \mathbb{R}\times\mathbb{R}\to\mathbb{R}, D: \mathbb{R}\times\mathbb{R}\to\mathbb{R}\,, f: \mathbb{R}\to\mathbb{R}, x: \mathbb{R}\to\mathbb{R}\ \ and\ h: \mathbb{R}\to\mathbb{R}$ are continuous function, and to ensure periodicity the following assumption has been made $a(t), g(t), D(t,x)$ and $Q(t,x)$are periodic functions.with supposing C_T be the set of all continuous scalar functions $x(t)$, periodic in t of the period T.This paper discusses the existence and uniqueness of periodic solutions of the form

$$\frac{d}{dt}x(t) = -\prod_{i=1}^{p} a_i(t)\,x(t) + \frac{d}{dt}\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right) + \int_{-\infty}^{t}\left[\prod_{i=1}^{p} D_i(t,s)\,f\big(x(s)\big) + h(s)\right] ds + G\left(t, x(t), x\big(t-\tau(t)\big)\right) \qquad (1.1)$$

BY assuming $a\ (t)$ is a continuous real-valued function. Taking into consideration $Q: \mathbb{R}\times\mathbb{R}\to\mathbb{R},\ D: \mathbb{R}\times\mathbb{R}\to\mathbb{R}, G: \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, f: \mathbb{R}\to\mathbb{R}\ and\ h: \mathbb{R}\to\mathbb{R}$ are continuous functions. The neutral term $\frac{d}{dt}\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)$ in Eq (1.1) produces non-linearity in the derivative term which is more general compared to the neutral term provided in [9]. Also, Eq(1.1) contains a non-constant function $g(t)$ as the delay term unlike other studies,where they are dealing with constant delay. So we provide a new conditions to construct the mappings to employ fixed point theorems.

The technique used in this paper is convert Eq(1.1) into an integral equation which allow us to create two mappings and it is the requirement of the fixed point theorem of Krasnoselskiiand this done in lemma(2.1).Thereafter, as shown in lemma

(3.2) and lemma (3.3) we proved that Az is continuous and compact. Bz is acontraction.It allowed us to apply the theorem of Krasnoselskii and grant us to proo f the existence of periodic solutions. In the end, we show the uniqueness of the periodic solution by using the contraction mapping principle.

The rest of the paper is organized as follows: section 2 provides the preliminaries that will be used in the further sections, also it introduce lemma 2 which transforms Eq (1.1) to an integral equation and section 3 the main results have been presented.

2. Preliminaries

This section introduces some significant notations. We start by supposing that for $T > 0$ define C_T be the set of all continuous scalar functions $x\ (t)$,periodic in t of the period T . Afterwards $(C_T\ ; \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = sup_{t\in[0,T]}|x|$$

It is appropriate to assume the following conditions

$$a(t+T) = a(t) \quad , g(t+T) = g(t)$$

$$D(t+T, x) = D(t, x)$$

$$G(t+T, x, y) = G(t, x, y) \qquad (2.1)$$

With $g\ (t)$ being scalar, continuous, and $g\ (t) > 0$. Also, we assume that

$$\int_0^T \prod_{i=1}^{p} a_i(s)ds > 0 \quad (2.2)$$

We also a assume that the function $Q\ (t,x)$ is periodic in t of period T,

$$Q(t,x) = Q(t+T,x) \quad (2.3)$$

As long as we are looking for periodic solutions, it is necessary to assume $Q(t,x)$and $f(x)$ globally Lipschitz functions. So for E_1 and E_2 are positive constants such that,

$$\sum_{i=1}^{p}|Q_i(t,x) - Q_i(t,y)| \le E_1\|x-y\| \quad (2.4)$$

$$|f(x) - f(y)| \le E_2\|x-y\| \quad (2.5)$$

and,

$$|G(t,x,y) - G(t,w,z)| \le E_3\|x-w\| + E_4\|y-z\| \quad (2.6)$$

Also, there is E_3 , E_4Such that,

$$\int_{-\infty}^{t} \prod_{i=1}^{p}|D_i(t,s)|\,ds \le E_3 < \infty,\ h(s) \le E_4 \quad (2.7)$$

Now, the following lemma helps to convert Eq (1.1) to an equivalent integral equation.

Lemma 2.1. Let $Q(t,x),\ D(t,s),\ a(t),\ f(t),\ x(t),\ g(t)\ and\ h(t)$ are defined as above, then $x(t)$ is a solution of Eq (1.1) if and only if

$$x(t) = \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)$$
$$+\left(1-e^{-\int_0^T \prod_{i=1}^{p} a_i(k)dk}\right)^{-1} \times \left[\int_{t-T}^{t} -\prod_{i=1}^{p} a_i(t) \sum_{i=1}^{p} Q_i\left(u, x\big(u-g(u)\big)\right)\right] e^{-\int_u^t \prod_{i=1}^{p} a_i(k)dk}du$$
$$+\int_{t-T}^{t} \int_{-\infty}^{u} \left[\prod_{i=1}^{p} D_i(u,s) f\big(x(s)\big) + h(s)\right] ds\ e^{-\int_u^t \prod_{i=1}^{p} a_i(k)dk}du$$
$$+\int_{t-T}^{t} \left[G\left(u, x(u), x\big(u-\ \tau(u)\big)\right)\right] e^{-\int_u^t \prod_{i=1}^{p} a_i(k)dk}du$$

Proof:

Let $x(t) \in B_T$ be a solution of Eq (1.1). By writing Eq (1.1) as

$$\frac{d}{dt}\left[x(t)-\sum_{i=1}^{p}Q_i\left(t,x\big(t-g(t)\big)\right)\right] = -\prod_{i=1}^{p}a_i(t)\,x(t)+\int_{-\infty}^{t}\left[\prod_{i=1}^{p}D_i(t,s)\,f\big(x(s)\big)+h(s)\right]ds$$

$$+\,G\left(t,x(t),x\big(t-\tau(t)\big)\right)$$

Adding $\prod_{i=1}^{p}a_i(t)\sum_{i=1}^{p}Q_i\left(t,x\big(t-g(t)\big)\right)$ to both sides of the last equation ,we find :

$$\frac{d}{dt}\left[x(t)-\sum_{i=1}^{p}Q_i\left(t,x\big(t-g(t)\big)\right)\right]=-\prod_{i=1}^{p}a_i(t)\left[x(t)-\sum_{i=1}^{p}Q_i\left(t,x\big(t-g(t)\big)\right)\right]$$

$$-\prod_{i=1}^{p}a_i(t)\sum_{i=1}^{p}Q_i\left(t,x\big(t-g(t)\big)\right)+\int_{-\infty}^{t}\left[\prod_{i=1}^{p}D_i(t,s)\,f\big(x(s)\big)+h(s)\right]ds$$

$$+\;G\left(t,x(t),x\big(t-\tau(t)\big)\right) \qquad (2.8)$$

Now , multiply both sides of (2.8) by $e^{\int_0^t\prod_{i=1}^{p}a_i(k)dk}$, then integrate from $t-T$ $to\ t$,we have

$$\left[x(t)-\sum_{i=1}^{p}Q_i\left(t,x\big(t-g(t)\big)\right)\right]e^{\int_0^t\prod_{i=1}^{p}a_i(k)dk}$$

$$-\left[x(t-T)-\sum_{i=1}^{p}Q_i\left(t-T,x\big(t-T-g(t-T)\big)\right)\right]e^{\int_0^t\prod_{i=1}^{p}a_i(k)dk}$$

$$=\int_{t-T}^{t}\left[-\prod_{i=1}^{p}a_i(t)\sum_{i=1}^{p}Q_i\left(u,x\big(u-g(u)\big)\right)+\int_{-\infty}^{t}\left[\prod_{i=1}^{p}D_i(u,s)\,f\big(x(s)\big)+h(s)\right]ds\right.$$

$$\left.+\,G\left(u,x(u),x\big(u-\tau(u)\big)\right)\right]e^{\int_0^t\prod_{i=1}^{p}a_i(k)dk}du$$

Now ,by dividing both sides of the above equation by $e^{\int_0^t\prod_{i=1}^{p}a_i(k)dk}$, and due to the fact that $x(t)$ is a periodic function of period T and using Eq(2.1) and Eq(2.3), we get :

$$x(t)=\sum_{i=1}^{p}Q_i\left(t,x\big(t-g(t)\big)\right)+\left(1-e^{-\int_0^T\prod_{i=1}^{p}a_i(k)dk}\right)^{-1}$$

$$\times\left[\int_{t-T}^{t}-\prod_{i=1}^{p}a_i(t)\sum_{i=1}^{p}Q_i\left(u,x\big(u-g(u)\big)\right)\right]e^{-\int_u^t\prod_{i=1}^{p}a_i(k)dk}du$$

$$+\int_{t-T}^{t}\int_{-\infty}^{u}\left[\prod_{i=1}^{p}D_i(u,s)\,f\big(x(s)\big)+h(s)\right]ds\;\;e^{-\int_u^t\prod_{i=1}^{p}a_i(k)dk}du$$

$$+\int_{t-T}^{t}\left[G\left(u,x(u),x\big(u-\tau(u)\big)\right)\right]\;e^{-\int_u^t\prod_{i=1}^{p}a_i(k)dk}du$$

the proof is complete.

Now define a mapping $P\varphi(t)$ by

$$(P\varphi)(t)=\sum_{i=1}^{p}Q_i\left(t,\varphi\big(t-g(t)\big)\right)+\left(1-e^{-\int_0^T\prod_{i=1}^{p}a_i(k)dk}\right)^{-1}$$

$$\times\left[\int_{t-T}^{t}-\prod_{i=1}^{p}a_i(t)\sum_{i=1}^{p}Q_i\left(u,\varphi\big(u-g(u)\big)\right)\right]e^{-\int_u^t\prod_{i=1}^{p}a_i(k)dk}du$$

$$+\int_{t-T}^{t}\int_{-\infty}^{u}\left[\prod_{i=1}^{p}D_i(u,s)\,f\big(\varphi(s)\big)+h(s)\right]ds\;\;e^{-\int_u^t\prod_{i=1}^{p}a_i(k)dk}du$$

$$+\int_{t-T}^{t}\left[G\left(u,\varphi(u),\varphi\big(u-\tau(u)\big)\right)\right]\;e^{-\int_u^t\prod_{i=1}^{p}a_i(k)dk}du \tag{2.9}$$

We need to prove $(P\varphi)(t)$ is a periodic function of period T for $\varphi\in B_T$

$$(P\varphi)(t+T)=\sum_{i=1}^{p}Q_i\left(t+T,\varphi\big(t+T-g(t+T)\big)\right)$$

$$+\left(1-\;e^{-\int_0^T\prod_{i=1}^{p}a_i(k)dk}\right)^{-1}\left[\int_{t}^{t+T}-\prod_{i=1}^{p}a_i(u)\sum_{i=1}^{p}Q_i\left(u,\varphi\big(u-g(u)\big)\right)\right]e^{-\int_u^t\prod_{i=1}^{p}a_i(k)dk}du$$

$$+\int_{t}^{t+T}\int_{-\infty}^{u}\left[\prod_{i=1}^{p}D_i(u,s)\,f\big(\varphi(s)\big)+h(s)\right]ds\;\;e^{-\int_u^{t+T}\prod_{i=1}^{p}a_i(k)dk}du$$

$$+\int_{t}^{t+T}\left[G\left(u,\varphi(u),\varphi\big(u-\tau(u)\big)\right)\right]\;e^{-\int_u^{t+T}\prod_{i=1}^{p}a_i(k)dk}du$$

The first term by using Eq (2.1) and (2.3), we obtain :

$$Q\left(t+T,\varphi\big(t+T-g(t+T)\big)\right)=Q\left(t,\varphi\big(t-g(t)\big)\right)$$

We put $v=u-T$ in the second part, we have :

$$\left(1-e^{-\int_0^T\prod_{i=1}^{p}a_i(k)dk}\right)^{-1}\left[\int_{t-T}^{t}-\prod_{i=1}^{p}a_i(v+T)\sum_{i=1}^{p}Q_i\left(v+T,\varphi\big(v+T-g(v+T)\big)\right)\right]e^{-\int_{v+T}^{t+T}\prod_{i=1}^{p}a_i(k)dk}dv$$
$$+\int_{t-T}^{t}\int_{-\infty}^{v+T}\left[\prod_{i=1}^{p}D_i(v+T,s)\,f\big(\varphi(s)\big)+h(s)\right]ds\ e^{-\int_{v+T}^{t}\prod_{i=1}^{p}a_i(k)dk}dv$$
$$+\int_{t-T}^{t}\left[G\left(v+T,\varphi(v+T),\varphi\big(v+T-\tau(v+T)\big)\right)\right]\ e^{-\int_{v+T}^{t+T}\prod_{i=1}^{p}a_i(k)dk}dv$$

Put $k=L+T$ in the last equation ,we get

$$\left(1-e^{-\int_{-T}^{0}a(L+T)dL}\right)^{-1}\left[\int_{t-T}^{t}-\prod_{i=1}^{p}a_i(v+T)\sum_{i=1}^{p}Q_i\left(v+T,\varphi\big(v+T-g(v+T)\big)\right)\right]e^{-\int_{v}^{t}a(L+T)dL}dv$$
$$+\int_{t-T}^{t}\int_{-\infty}^{v+T}\left[\prod_{i=1}^{p}D_i(v+T,s)\,f\big(\varphi(s)\big)+h(s)\right]ds\ e^{-\int_{v}^{t}a(L+T)dL}dv$$
$$+\int_{t-T}^{t}\left[G\left(v+T,\varphi(v+T),\varphi\big(v+T-\tau(v+T)\big)\right)\right]\ e^{-\int_{v}^{t}a(L+T)dL}dv$$

By using Eq(2.1) , Eq (2.3) we have :

$$\left(1-e^{-\int_0^T\prod_{i=1}^{p}a_i(k)dk}\right)^{-1}\left[\int_{t-T}^{t}-\prod_{i=1}^{p}a_i(u)\sum_{i=1}^{p}Q_i\left(u,\varphi\big(u-g(u)\big)\right)\right]e^{-\int_u^t\prod_{i=1}^{p}a_i(k)dk}du$$
$$+\int_{t-T}^{t}\int_{-\infty}^{u}\left[\prod_{i=1}^{p}D_i(u,s)\,f\big(\varphi(s)\big)+h(s)\right]ds\ e^{-\int_u^t\prod_{i=1}^{p}a_i(k)dk}du$$
$$+\int_{t-T}^{t}\left[G\left(u,\varphi(u),\varphi\big(u-\tau(u)\big)\right)\right]\ e^{-\int_u^t\prod_{i=1}^{p}a_i(k)dk}du.$$

3. Existence and Uniquness of Periodic Solutions

This section presents the state of the fixed point theorem of Krasnoselskii a nd uses this theorem to show the the existence of a periodic solution.

Theorem 3.1. (Krasnoselskii). Let $\mathcal{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathcal{B}, \| . \|)$. suppose that A and B map $\mathcal{M}$ into $\mathcal{M}$ such that

(i) $x, y \in \mathcal{M}$, implies $Ax + By \in \mathcal{M}$ ،

(ii) A is continuous and $A\mathcal{M}$ is contained in a compact set subset of $\mathcal{M}$,

(iii) B is a contraction mapping.

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$.

As theorem 3.1 states there are two mappings, one is a contraction and the other is compact. Therefore, we will define the operator $P: C_T \to C_T$ as Eq (2.9) and by rewriting Eq (2.9) as follows

$$(P\varphi)(t) = (B\varphi)(t) + (A\varphi)(t),$$

Where $A, B : C_T \to C_T$ are given by

$$(B\varphi)(t) = \sum_{i=1}^{p} Q_i(t, \varphi(t - g(t))) \tag{3.1}$$

And,

$$(A\varphi)(t) = \left(1 - e^{-\int_0^T \prod_{i=1}^{p} a_i(k)dk}\right)^{-1} \left[\int_{t-T}^{t} - \prod_{i=1}^{p} a_i(u) \sum_{i=1}^{p} Q_i\left(u, \varphi\big(u - g(u)\big)\right)\right] e^{-\int_u^t \prod_{i=1}^{p} a_i(k)dk} du$$
$$+ \int_{t-T}^{t} \int_{-\infty}^{u} \left[\prod_{i=1}^{p} D_i(u, s)\, f\big(\varphi(s)\big) + h(s)\right] ds\; e^{-\int_u^t \prod_{i=1}^{p} a_i(k)dk} du$$
$$+ \int_{t-T}^{t} \left[G\left(u, \varphi(u), \varphi\big(u - \tau(u)\big)\right)\right] e^{-\int_u^t \prod_{i=1}^{p} a_i(k)dk} du \tag{3.2}$$

The goal here is to show that $(B\varphi)(t)$ is contraction and $(A\varphi)(t)$ is compact. The analysis is introduced in these two lemmas

Lemma 3.2. If B is given by Eq(3.1) with $E_1 < 1$, and (2.4) hold, then B is a contraction.

Proof. Let B be defined by Eq (3.1). Then for $\varphi, \psi \in C_T$ we have

$$\|B\varphi - B\psi\| = sup_{t\epsilon[0,T]}|B\varphi - B\psi| \leq E_1 sup_{t\epsilon[0,T]} \sum_{i=1}^{p} \left|Q_i\left(t, \varphi(t-g(t))\right) - Q_i\left(t, \psi(t-g(t))\right)\right|$$

$$\leq E_1|\varphi - \psi|$$

Hence B defines a contraction . As $E_1 < 1$ therefore B defines a contraction.

Before showing Lemma 3.3. It's appropriate to the following notations:

$$\tau = max_{t\in[0,T]} \left|(1 - e^{-\int_0^T \prod_{i=1}^{p} a_i(k)\mathrm{dk}})^{-1}\right|, \rho = max_{t\in[0,T]} \prod_{i=1}^{p} |a_i(t)|,$$

$$\nu = max_{u\in[t-T,t]} e^{-\int_u^t \prod_{i=1}^{p} a_i(k)dk}$$

Lemma 3.3. If A is defined by Eq(3.2), then A is continuous and the image of A is contained in a compact set.

Proof. We will start by proving A is continuous we define A as Eq(3.2). Let $\varphi, \psi \in C_T$,

for a given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{N}$ with $N = \eta\gamma T[\,\rho E_1 + E_2E_5 + E_3 + E_4]\|\varphi - \psi\|$, now for $\|\varphi - \psi\| < \delta$, and by using (2.4) into Eq(3.2) ,we get

$$\|A_\varphi - A_\psi\| \leq \eta\gamma T[\,\rho E_1 + E_2E_5 + E_3 + E_4]\|\varphi - \psi\| \leq N\|\varphi - \psi\| \leq N\delta \leq \varepsilon.$$

This is show that A is continuous. The second step is showing A is a compact set using Ascoli-Arzela's theorem [5] which states that for $A \subset X$, A is compact if and only if A is bounded, and equicontinuous.

Let $\Omega = \{\varphi \in C_T : \|\varphi\| \leq V\}$, where V is any fixed positive constant, from (2.4), (2.5) we have,

$$\sum_{i=1}^{p} |Q_i(t,x)| = \sum_{i=1}^{p} |Q_i(t,x) - Q_i(t,0) + Q_i(t,0)|$$

$$\leq \sum_{i=1}^{p} [|Q_i(t,x\,) - Q_i(t,0)| + |Q_i(t,0)|]$$

$$\leq E_1\|x\| + \alpha$$

where $\alpha=sup_{t\in[0,T]}\sum_{i=1}^{p}|Q_i(t,0)|$.

In the same way,

$$|f(x)| = |f(x) - f(0)|$$

$$\leq |f(x) - f(0)|$$

$$\leq E_2\|x\|.$$

And,

$$|G(t,x,y)| = |G(t,x,y) - G(t,0,0) + G(t,0,0)|$$

$$\leq |G(t,x,y) - G(t,0,0)| + |G(t,0,0)|$$

$$\leq E_3\|x\| + E_4\|y\|$$

Taking into consideration , $f(0) = 0$ and $G(t,0,0) = 0$, let $\varphi_n \in \Omega$ where n is positive integer with $L = \eta\gamma T[\rho E_1(V+\alpha) + E_2E_5V + E_6 + V(E_3 + E_4)]$, $L \geq 0$ Therefore,

$$\|A\varphi_n\| = \left|\left(1 - e^{-\int_0^T \prod_{i=1}^{p} a_i(k)\mathrm{dk}}\right)^{-1}\left[\int_{t-T}^{t} -\prod_{i=1}^{p} a_i(u)\sum_{i=1}^{p} Q_i\left(u, \varphi_n(u-g(u))\right)\right] e^{-\int_u^t \prod_{i=1}^{p} a_i(k)\mathrm{dk}} du\right.$$
$$+ \int_{t-T}^{t}\int_{-\infty}^{u}\left[\prod_{i=1}^{p} D_i(u,s)\, f(\varphi_n(s)) + h(s)\right] ds\ e^{-\int_u^t \prod_{i=1}^{p} a_i(k)\mathrm{dk}} du$$
$$\left. + \int_{t-T}^{t}\left[G\left(u, \varphi_n(u), \varphi_n(u-\tau(u))\right)\right] e^{-\int_u^t \prod_{i=1}^{p} a_i(k)\mathrm{dk}} du\right|$$

$$\leq \max_{t\epsilon[0,T]}\left|\left(1 - e^{-\int_0^T \prod_{i=1}^{p} a_i(k)\mathrm{dk}}\right)^{-1}\left[\int_{t-T}^{t} -\prod_{i=1}^{p} a_i(u)\sum_{i=1}^{p} Q_i\left(u, \varphi_n(u-g(u))\right)\ e^{-\int_u^t \prod_{i=1}^{p} a_i(k)\mathrm{dk}} du\right.\right.$$
$$+ \int_{t-T}^{t}\int_{-\infty}^{u}\left[\sum_{i=1}^{p} D_i(u,s)\, f(\varphi_n(s)) + h(s)\right] ds\ e^{-\int_u^t \prod_{i=1}^{p} a_i(k)\mathrm{dk}} du$$
$$\left.\left. + \int_{t-T}^{t} G\left(u, \varphi_n(u), \varphi_n(u-\tau(u))\right)\ e^{-\int_u^t \prod_{i=1}^{p} a_i(k)\mathrm{dk}} du\right]\right|$$

$$\leq \eta\gamma\left[-\int_{t-T}^{t}\left[\rho E_1\|\varphi_n\| + \int_{-\infty}^{u}\left[\left|\prod_{i=1}^{p} D_i(u,s)\right|\left|f(\varphi_n(s))\right| + |h(s)|\right]ds + E_3\|\varphi_n\| + E_4\|\varphi_n\|\right]du\right]$$

$$\leq \eta\gamma\left[-\int_{t-T}^{t}[\rho E_1(\|\varphi_n\|+\propto) + E_2E_5\|\varphi_n\| + E_6 + \|\varphi_n\|(E_3+E_4)]du\right]$$

$$\leq \eta\gamma T[\,\rho E_1(\|\varphi_n\|+\propto) + E_2E_5\|\varphi_n\| + E_6 + \|\varphi_n\|(E_3+E_4)]$$

$$\leq \eta\gamma T[\,\rho E_1(V+\alpha) + E_2E_5V + E_6 + V(E_3+E_4)] \leq L$$

This is showing that A is bounded. To prove A is equicontinuous we need to find $(A\varphi_n)'(t)$ and prove that it is uniformly bounded. Therefore, after derivative Eq(3.2) with using (2.2), Eq(2.3) we get,

$$(A\varphi_n)'(t) = -\prod_{i=1}^{p} a_i(t)\,A(\varphi_n(t)) - \prod_{i=1}^{p} a_i(t)\sum_{i=1}^{p} Q_i\left(u,\varphi_n(u-g(u))\right)$$

$$+\int_{-\infty}^{t}\left[\prod_{i=1}^{p} D_i(t,s)\,f(\varphi_n(t)) + h(s)\right]ds + G\left(t,\varphi_n(t),\varphi_n(t-\tau(t))\right).$$

The above expression yields $\|(A\varphi_n)'\| \leq Z$ where Z is some positive constant. Hence, by Ascoli-Arzela's theorem $A\varphi$ is compact.

Theorem 3.4. Suppose the hypothesis of Lemma 2.4. Let $\alpha = sup_{t\,[[0,T]}\sum_{i=1}^{p}|Q_i(t,0)|$ and, suppose (2.1)-(2.7) hold. Let J be a positive constant satisfying the inequality

$$\propto + E_1J + \eta\gamma T[\,\rho E_1(J+\alpha) + E_2E_5J + E_6 + J(E_3+E_4)] \leq J$$

Let $\mathcal{M} = \{\varphi \in C_T : \|\varphi\| \leq J\}$. Then Eq(1.1) has a solution in $\mathcal{M}$.

Proof: First of all, we will define $\mathcal{M} = \{\varphi \in C_T : \|\varphi\| \leq J\}$, and by knowing that A is continuous and AM is contained in a compact set. Also, the mapping B is a contraction from lemma (3.2), (3.3) and it is clear that $A, B: C_T \to C_T$. The aim is showing that $\|A_\varphi + B_\psi\| \leq J$. Let $\varphi,\psi \in \mathcal{M}$, Let $\varphi,\psi \in M$ whit $\|\varphi\|, \|\psi\| \leq J$.Then

$$\|A\varphi + B\psi\| \leq \|A\varphi\| + \|B\psi\|$$

We know from Lemma(3.3)that

$$\|A\varphi_n\| \leq \eta\gamma T[\,\rho E_1(\|\varphi_n\|+\propto) + E_2E_5\|\varphi_n\| + E_6 + \|\varphi_n\|(E_3+E_4)]$$

So, we get :

$$\|A\varphi\| + \|B\psi\| \leq \eta\gamma T[\,\rho E_1(\|\varphi\| + \propto) + E_2E_5\|\varphi\| + E_6 + \|\varphi\|(E_3 + E_4)] + \alpha + E_1\|\psi\|$$

$$\leq \propto + E_1J + \eta\gamma T[\,\rho E_1(J + \alpha) + E_2E_5J + E_6 + J(E_3 + E_4)] \leq J$$

This is proving all conditions of Theorem 3.1. Thus, there exists a fixed point z in $\mathcal{M}$. By Lemma 2.1, this fixed point is a solution of Eq(1.1). Hence Eq(1.1) has a T-periodic solution.

Theorem 3.5. Let (2.1)-(2.7) hold if

$$\propto + E_1J + \eta\gamma T[\,\rho E_1(J + \alpha) + E_2E_5J + E_6 + J(E_3 + E_4)] < 1$$

then Eq(1.1) has a unique T-periodic solution..

Proof. We define $(P\varphi)(t)$ as Eq(2.9). Let $\varphi, \psi \epsilon\ C_T$, in view of Eq(2.9) we have,

$$\left\|P_{\varphi -}P_{\psi}\right\| < [E_1 + \tau\nu T(\rho E_1) + TE_3E_2]\|\varphi - \psi\|.$$

This completes the proof of Theorem 3.5.

Conclusion

The goal of this paper is to transform Eq(1.1) into an integral equation and apply Theorem 3.1, which offers the existence of periodic solutions. Obtaining the integral equation allow us to create two mappings, one of them is a contraction and the other is completely continuous. In addition, by using the contraction mapping principle enables us to show the uniqueness of the periodic solution which showing in theorem 3.5.

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