The existence and uniqueness of periodic solutions for nonlinear neutral dynamic equation with infinite delay on a Time Scale.

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Abstract.

Let $\mathbb{T}$ be a periodic time scale. We study the following nonlinear neutral dynamic equation with infinite delay

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)^{\Delta} + \int_{-\infty}^{t} \big(D(t,s)f(x(s)) + h(s)\big)\Delta s, t \in \mathbb{T}$$

by using a fixed point theorem due to Krasnoselskii, we show that the nonlinear neutral dynamic equation with an infinite delay has a periodic solution. In addition, through utilizing the contraction mapping principle, we have shown that this periodic solution is unique.

Key words: Fixed point, infinite delay, time scales, periodic solution.

الملخص

تختص هذه الورقة بدراسة المعادلة الديناميكية المحايدة غير خطية ذات تأخير لانهائي على فضاء يعرف ب **(Time Scale)**

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)^{\Delta} + \int_{-\infty}^{t} \big(D(t,s)f(x(s)) + h(s)\big)\Delta s, t \in \mathbb{T}$$

باستخدام نظرية كراسنوليسكى تم إيجاد حل دوري للمعادلة الديناميكية المحايدة غير خطية ذات تأخير لا نهائي ويتم ذلك عن طريق إنشاء مشغلين مناسبين، أحدهما مضغوط والآخر له خاصية الانكماش، كذلك تم استخدام نظرية بناخ لنقطة الثابتة لضمان وجود حل وحيد.

1. Introduction

In recent decades, scientific researchers have been able to explore other avenues that help to show the existence of solution of nonlinear neutral dynamic equation with infinite delay such as fixed-point theory, Picard's successive approximation and the non-expansive operators technique. Fixed points has been revealed as a very effective and useful method for the study of nonlinear neutral dynamic equation. Moreover, the fixed-point theory establish conditions to which maps have solutions and applied in variety of fields and mathematics. Recently, **(Makhzoum, H. A and Elmansouri, R. A, 2018)**, studied the existence of solutions for the nonlinear neutral dynamic equation with an infinite delay,

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right) + \int_{-\infty}^{t}(D(t,s)f(x(s)) + h(s))ds. \qquad (1.1).$$

By the use of the Krasnoselskii's fixed-point theorem and proved the existence of periodic solutions of Eq (1.1). Then they used the contraction mapping principle to show the existence of a unique periodic solution of Eq (1.1).

In the present paper, we show the following the nonlinear neutral dynamic equation with an infinite delay, for $t \in \mathbb{T}$

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)^{\Delta} + \int_{-\infty}^{t}(D(t,s)f(x(s)) + h(s))\Delta s \qquad (1.2),$$

by assuming $a(t)$ is a continuous real-valued function.Taking into consideration $Q: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $D: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, $x: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$ are continuous function, and to ensure periodicity the following assumption has been made
$a(t)$, $g(t)$, $D(t,x)$, $Q(t,x)$are periodic functions.

We are interested to study the existence of periodic solutions of Eq (1.2) on the space called Time scale. Time scale is a relatively new subject it has been presented by the following definition a time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. The main point of this space is unifying the theory of difference equations with that of differential equations. Let $0 \in \mathbb{T}$ and g: $\mathbb{T} \to \mathbb{R}$, $id - g : \mathbb{T} \to \mathbb{T}$ is strictly increasing this leads that $x(t - g(t))$ is well-defined over $\mathbb{T}$. The work is inspired and motivated by the works done by (**Ardjouni, A and Djoudi, A, 2016)**, for a wealth of reference material on the subject, we refer to [3, 4, 5, 6, 7, 8] and [11], and the references in them.

To achieve the intended result we have to follow the requirements of Krasnoselskii's fixed point where the theory asks for $z = Az + Bz$ yields $z \in M$ where M is a convex set and Az is continuous and compact, Bz is a contraction. The methodology used in this paper is transformed Eq (1.2) into an integral equation that allows us to create two mappings and it is the condition of the fixed-point theorem of Krasnoselskii and this is done in lemma 3.2. Afterward, we proved that Az is continuous and compact, Bz is a contraction. It helped us to implement Krasnoselskii's theorem and to grant us to prove the existence of periodic solutions. In the end, we show the uniqueness of the periodic solution by the use of the contraction mapping principle.

This paper structured as follows. In Section 2, we present outlines some preliminary background material to be used in the upcoming sections. In addition, some facts will provide about the exponential function on a time scale well. The main result has been presented in Section 3.

2. Preliminaries

This section focus to provide the significant notations related to concepts concerning the calculus on time scales for dynamic equations mostly all definitions, lemmas, and theorems can be found in **(Bohner, M, and Peterson, A, 2001 and 2003)**. A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as

$$\sigma(t) = inf\{s \in \mathbb{T} : s > t\} \; and \; \rho(t) = sup\{s \in \mathbb{T} : s < t\}.$$

These operators allow elements in the time scale to be classified as follows. We say t is

i. right scattered if $\sigma(t) > t$,

ii. right dense if $\sigma(t) = t$,

iii. left scattered if $\rho(t) < t$,

iv. left dense if $\rho(t) = t$.

The graininess function μ: $\mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t)\text{-}t$ and gives the distance between an element and its successor. We set inf $\emptyset$ = sup $\mathbb{T}$ and sup $\emptyset$ = inf $\mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f: \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted by $f^{\Delta}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\Delta}(t) = f'$ (t) is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}$ ($\mathbb{T}$, $\mathbb{R}$), if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point t ∈ T. function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all t ∈ $\mathbb{T}^k$.

We are now able to state some properties of the delta-derivative of f. Note that f^{σ}(t) = $f(\sigma(t))$.

Theorem 2.1. [7]. Assume that $f, g: \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.

i. $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$.
ii. $(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$.
iii. $(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f^{\sigma}(t) g^{\Delta}(t)$.
iv. $(fg)^{\Delta}(t) = f(t) g^{\Delta}(t) + f^{\Delta}(t) g^{\sigma}(t)$. (The product rules)
v. If $g(t) g^{\sigma}(t) \neq 0$ then

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g^{\sigma}(t)}.$$

The first two theorems deal with the composition of two functions. The first theorem is the chain rule on time scales **(Bohner, M, and Peterson, A, 2001: Theorem 1.93)**.

Theorem 2.2 (Chain Rule). Assume, $\boldsymbol{v}$: $\mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale.

Let w: $\widetilde{\mathbb{T}} \to$ R. If $v^{\Delta}(t)$ and $w^{\tilde{\Delta}}(v(t))$ exist for t ∈ $\mathbb{T}^k$, then $(w \circ v)^{\Delta} = (w^{\tilde{\Delta}} \circ v) v^{\Delta}$.

In the sequel, we will need to differentiate and integrate functions of the form $f(t-g(t)) = f(v(t))$, where $v(t) := t - g(t)$. The second theorem is the substitution rule **(Bohner, M, and Peterson, A, 2001: Theorem 1.98)**.

Theorem 2.3 (Substitution). Assume $v: \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. If $f: \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and v is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,

$$\int_a^b f(t)v^{\Delta}(t)\Delta t = \int_{v(a)}^{v(b)} (f \circ v^{-1})(s)\,\tilde{\Delta} s$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $\in \mathbb{T}^k$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by

$$\mathcal{R}^+ = \{f \in \mathcal{R}: 1 + \mu(t)f(t) > 0 \quad for\ all\ t \in \mathbb{T}\}.$$

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \frac{1}{\mu(z)} log\big(1 + \mu(z)p(z)\big)\,\Delta z\right). \tag{2.1}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in T$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\Delta} = p(t)y, y(s) = 1$. Other properties of the exponential function are given in the following lemma, **(Bohner, M, and Peterson, A, 2001: Theorem 2.36)**.

Lemma 2.4. Let $p, q \in \mathcal{R}$. Then

i. $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$,

ii. $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$,

iii. $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s),$ *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,

iv. $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,

v. $e_p(t,s)e_p(s,r) = e_p(t,r)$,

vi. $(\frac{1}{e_p(.,s)})^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(.,s)}$.

The notion of periodic time scales and the next two definitions are quoted from **(Atici, F.M *et al*, 1999)** and **(Kaufmann, E.R and Raffoul, Y.N, 2006)**.

Definition 2.5. We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$, such that, if $\in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p with this property called the period of the time scale.

Example 2.6. The following time scales are periodic.

1. $\mathbb{T} = \cup_{i=-\infty}^{\infty}[2(i-1)h, 2ih], h > 0$ *has period* $p = 2h$.
2. $\mathbb{T} = hZ$ *has period* $p = h$.
3. $\mathbb{T} = \mathbb{R}$
4. $\mathbb{T} = \{t = k - q^m : k \in z, m \in N_0\},$ *where* $0 < q < 1$ *has period* $p = 1$.

Remark 2.7 ([11]). All periodic time scales are unbounded above and below.

Definition 2.8. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f: \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$and T is the smallest number such that $f(t \pm T) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.9 ([11]). If $\mathbb{T}$ is a periodic time scale with period p, then

$$\sigma(t \pm np) = \sigma(t) \pm np.$$

Consequently, the graininess function μ $satisfies$

$$\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t),$$

and so, is a periodic function with period p.

3. Existence of periodic solutions

In this section, we will present the main result. The following conditions should be assumed, Let $C(\mathbb{T},\mathbb{R})$ be the space of all real-valued continuous functions on $\mathbb{T}$.

Define

$$\mathcal{H}_T = \{\varphi \epsilon C(\mathbb{T},\mathbb{R}): \varphi(t+T) = \varphi(t)\}, where\ T > 0;\ T \in \mathbb{T},$$

then $\mathcal{H}_T$ is a Banach space with the supremum norm

$$\|x\| = sup|x(t)|, \quad t \in [0,T].$$

If $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation [a, b] we mean

$$[a,b] = \{t \in \mathbb{T}: a \leq t \leq b\},$$

unless otherwise specified. The intervals $[a,b), (a,b], and\ (a,b)$are defined similarly. For all $\in \mathbb{T}$, let $a(t) > 0$ and $a \in \mathcal{R}^+$, where $a(t)$ is a continuous, and

$$a(t+T) = a(t), g(t+T) = g(t), D(t+T,u+T) = D(t,u) \qquad (3.1)$$

where id is the identity function on $\mathbb{T}$. We also assume that $Q(t,x)$ and $f\ (x)$ are continuous and periodic in t and Lipschitz continuous in *x*. That is,

$$Q(t+T,x) = Q(t,x) \qquad (3.2)$$

and there are positive constants E_1 , E_2, $E_3 and\ E_4$ such that

$$\sum_{i=1}^{p}|Q_i(t,x) - Q_i(t,y)| \leq E_1\|x-y\| \ , \qquad (3.3)$$

$$|f(x) - f(y)| \leq E_2\|x-y\| \ , \qquad (3.4)$$

and,

$$\int_{-\infty}^{t}|D(t,u)|\,\Delta u \leq E_3\ , \quad h(s) \leq E_4, \qquad (3.5)$$

Lemma 3.1. [11]. Let $x \in \mathcal{H}_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$.

The following lemma allow us converting Eq (1.2) to an equivalent integral equation,

Lemma 3.2. Suppose (3.1), (3.2) hold, $if\ x\ \in \mathcal{H}_T$ then x is a solution of Eq (1.2) if and only if

$$x(t) = \sum_{i=1}^{p} Q_i(t, x(t-g(t)) + (1 - e_{\ominus a}(t, t-T))^{-1}$$
$$\times [\int_{t-T}^{t} -a(u) \sum_{i=1}^{p} Q_i^\sigma \left(u, x(u-g(u))\right) e_{\ominus a}(t,u)\Delta u$$
$$+ \int_{t-T}^{t} \int_{-\infty}^{u} \left(D(u,s) f(x(s)) + h(s)\right) \Delta s\, e_{\ominus a}(t,u)\Delta u].$$

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Proof. Let $x\ (t)\ \in \mathcal{H}_T$ be a solution of Eq (1.2). By writing Eq (1.2) as

$$[x(t) - \sum_{i=1}^{p} Q_i\left(t, x(t-g(t))\right)]^\Delta = -a(t)x(t) + \int_{-\infty}^{t} [D(t,s) f(x(s)) + h(s)]\Delta s$$

Adding $a\ (t) \sum_{i=1}^{p} Q^\sigma\ ((t, x(t-g(t)))$ to both sides of the last equation, we obtain

$$\left[x(t) - \sum_{i=1}^{p} Q_i\left(t, x(t-g(t))\right)\right]^\Delta =$$
$$-a(t)\left[x^\sigma(t) - \sum_{i=1}^{p} Q_i^\sigma\ ((t, x(t-g(t)))\right]$$
$$- a(t) \sum_{i=1}^{p} Q_i^\sigma\ ((t, x(t-g(t)))$$
$$+ \int_{-\infty}^{t} [D(t,s) f(x(s)) + h(s)]\Delta s \tag{3.6}$$

Multiply both sides of (3.6) by $e_a(t,0)$and then integrate from $t\ -\ T\ to\ t$ to get

$$[x(t)-\sum_{i=1}^{p} Q_i\left(t,x\left(t-g(t)\right)\right)]e_a(t,0)-[x(t-T)$$
$$-\sum_{i=1}^{p} Q_i(t-T,x(t-T-g(t-T)))]e_a(t-T,0)$$
$$=\int_{t-T}^{t}[-a(u)\sum_{i=1}^{p} Q_i^{\sigma}\left(u,x\left(u-g(u)\right)\right)e_a(u,0)\Delta u$$
$$+\int_{-\infty}^{u}(D(u,s)f(x(s))+h(s))\Delta s]e_a(u,0)\Delta u$$

By dividing both sides of the above equation by $e_a(t,0)$, and due to the fact that $x(t)$ is a periodic function of period T and using equations (3.1), (3.2), we arrive at

$$x(t)=\sum_{i=1}^{p} Q_i(t,x\left(t-g(t)\right)+(1-e_{\ominus a}(t,t-T))^{-1}.$$
$$\times[\int_{t-T}^{t}-a(u)\sum_{i=1}^{p} Q_i^{\sigma}\left(u,x\left(u-g(u)\right)\right)e_{\ominus a}(t,u)\Delta$$
$$+\int_{t-T}^{t}\int_{-\infty}^{u}\left(D(u,s)f\left(x(s)\right)+h(s)\right)\Delta s\, e_{\ominus a}(t,u)\Delta u]$$

We will introduce the state of Krasnoselskii's fixed-point theorem and apply this theorem to prove the existence of a periodic solution

Theorem 3.3 (Krasnoselskii). Let $\mathbb{M}$be a closed convex nonempty subset of a

Banach space $(B,\|.\|$ Suppose that A and B are two mappings from $\mathbb{M}$ into $\mathbb{B}$ such that

(i) $x,y\ \in\mathbb{M}, implies\ Ax+By\ \in\mathbb{M}$

(ii) A is compact and continuous,

(iii) B is a contraction mapping.

Then there exists $z\in\mathbb{M}\ with\ z=Az+Bz$.

For its proof, we refer the reader to **(Smart, D.R, 1980).** As the structure hypothesis of theorem 3.3 states, there are two mappings, one is a contraction and the other is compact. Therefore, we will define an operator as following، let $P: \mathcal{H}_T \to \mathcal{H}_T$ such that,

$$(P\varphi)(x) = \sum_{i=1}^{p} Q_i(t, \varphi\,(t - g(t)))$$
$$+\left(1 - e_{\ominus a}(t, t-T)\right)^{-1} [\int_{t-T}^{t} -a(u) \sum_{i=1}^{p} Q_i^{\sigma}\left(u, \varphi\left(u - g(u)\right)\right) e_{\ominus a}(t,s)\Delta u$$
$$+\int_{t-T}^{t} \int_{-\infty}^{u} \left(D(u,s) f\left(\varphi(s)\right) + h(s)\right) \Delta s \;\; e_{\ominus a}(t,s)\Delta u].$$

By using the same steps in **(Althubiti, S *et al*, 2013)**, we can prove that, $(P\varphi)(x)$ is periodic in t of period T.

Now by expressing equation (3.7) as

$$(P\varphi)(t) = (B\varphi)(t) + (A\varphi)(t);$$

where A and B are given by

$$(B\varphi)(t) = \sum_{i=1}^{p} Q_i(t, \varphi\,(t - g(t))), \tag{3.8}$$

and,

$$(A\varphi)(t) = \left(1 - e_{\ominus a}(t, t-T)\right)^{-1} [\int_{t-T}^{t} -a(u) \sum_{i=1}^{p} Q_i^{\sigma}\left(u, \varphi\left(u - g(u)\right)\right) e_{\ominus a}(t,s)\Delta u$$
$$+ \int_{t-T}^{t} \int_{-\infty}^{u} \left(D(u,s) f\left(\varphi(s)\right) + h(s)\right) \Delta s \;\; e_{\ominus a}(t,s)\Delta u] \qquad (3.9).$$

We are trying to achieve that $(B\varphi)\,(t)$ is contraction and $(A\varphi)\,(t)$ is compact this can be done by providing these two lemmas. Before introducing the lemmas we define the following constants

$$\tau := max_{t\in[0,T]} \left|\left(1 - e_{\theta a}(t, t_T)\right)^{-1}\right|,$$
$$\nu := \left| max_{u\in[t-T,t]}\, e_{\theta a}(t,u)\right|, \tag{3.10}$$
$$\rho := max_{t\in[0,T]} |a(t)|.$$

Lemma 3.4. *If A is defined by (3.9), then A is continuous and the image of A is contained in a compact set.*

Proof. We will start by proving A is continuous we define A as (3.9). Let $\varphi, \psi \in \mathcal{H}_T$, for a given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{\mathrm{N}}$ with $N = \tau\nu T[\rho E_1 + E_2 E_3]$, now for $\|\varphi - \psi\| < \delta$, and by using (3.3) into (3.5), we get

$$\left\|A_\varphi - A_\psi\right\| \leq \tau\nu T[\rho E_1 + E_2 E_3]\|\varphi - \psi\| \leq \mathrm{N}\|\varphi - \psi\| \leq \mathrm{N}\delta \leq \varepsilon.$$

This shows that A is continuous. The second step is showing A is a compact set using Ascoli-Arzela's theorem **(DiBenedetto, E, and Debenedetto, E, 2002)** which states that for $A \subset X$, A is compact if and only if A is bounded, and equicontinuous.

Let $\Omega = \{\varphi \in \mathcal{H}_T : \|\varphi\| \leq \Upsilon\}$, where Υ is any fixed positive constant, from (3.3) and (3.4) we have,

$$\sum_{i=1}^{p} |Q_i(t,x)| = \sum_{i=1}^{p} |Q_i(t,x) - Q_i(t,0) + Q_i(t,0)|$$
$$\leq \sum_{i=1}^{p} [|Q_i(t,x) - Q_i(t,0)| + |Q_i(t,0)|]$$
$$\leq E_1\|x\| + \alpha,$$

where $\alpha = sup_{t\in[0,T]} \sum_{i=1}^{p} |Q_i(t,0)|$.

In the same way,

$$|f(x)| = |f(x) - f(0)| \leq E_2\|x\|.$$

Taking into consideration, $f(0) = 0$. Let $\varphi_n \in \Omega$ where n is a positive integer with $L = \tau\nu T[\rho(E_1\Upsilon + \alpha) + \Upsilon E_2 E_3 + E_4]$ where $L > 0$, Therefore,

$$\|A_{\varphi_n}\| = \left|\left(1 - e_{\ominus a}(t, t-T)\right)^{-1} [\int_{t-T}^{t} -a(u) \sum_{i=1}^{p} Q_i^\sigma \left(u, \varphi_n(u - g(u))\right) e_{\ominus a}(t,s)\Delta u \right.$$
$$\left. + \int_{t-T}^{t} \int_{-\infty}^{u} \left(D(u,s) f(\varphi_n(s)) + h(s)\right) \Delta s \; e_{\ominus a}(t,s)\Delta u \,]\right|$$
$$\leq max_{t\in[0,T]} \left|\left(1 - e_{\ominus a}(t, t-T)\right)^{-1} [\int_{t-T}^{t} -a(u) \sum_{i=1}^{p} Q_i^\sigma \left(u, \varphi_n(u - g(u))\right) e_{\ominus a}(t,s)\Delta u \right.$$
$$\left. + \int_{t-T}^{t} \int_{-\infty}^{u} \left(D(u,s) f(\varphi_n(s)) + h(s)\right) \Delta s \; e_{\ominus a}(t,s)\Delta u \,]\right|$$

$\leq \tau v \int_{t-T}^{t} [\rho(E_1\|\varphi_n\| + \alpha) + \int_{-\infty}^{u} |D(u,s) f(\varphi_n(s)) + h(s)| \Delta s] \, \Delta u$

$\leq \tau v \int_{t-T}^{t} [\rho(E_1\|\varphi_n\| + \alpha) + E_2 E_3 \|\varphi_n\| + E_4] \, du$

$\leq \tau v T [\rho(E_1\|\varphi_n\| + \alpha) + E_2 E_3 \|\varphi_n\| + E_4]$

$\leq \tau v T [\rho(E_1 \Upsilon + \alpha) + \Upsilon E_2 E_3 + E_4] \leq L.$

This is showing that A is bounded. To prove A is equicontinuous we need to find $(A\varphi_n)^\Delta(t)$ and prove that it is uniformly bounded. Therefore, after derivative (3.9) with using (3.3) - (3.5) we get,

$(A\varphi_n)^\Delta(t)=$

$$-a(t) A(\varphi_n)^\sigma(t) - a(t) \sum_{i=1}^{p} Q_i^\sigma \left(t, \varphi_n(t - g(t))\right) + \int_{-\infty}^{t} (D(t,s) f(\varphi_n(s)) + h(s)) \Delta s.$$

The above expression yields $\|(A\varphi_n)^\Delta\| \leq Z$ where Z is some positive constant. Hence, by Ascoli-Arzela's theorem $A\varphi$ is compact.

Lemma 3.5. If B is given by (3.8) with $E_1 < 1$, and (3.3) hold, then B is a contraction.

Proof. Let B be defined by (3.8). Then for $\varphi, \psi \in \mathcal{H}_T$ we have

$$\|(B\varphi)(t)-(B\psi)(t)\| = sup_{t\in[0,T]}|(B\varphi)(t)-(B\psi)(t)|$$
$$= sup_{t\in[0,T]}\sum_{i=1}^{p}|Q_i(t,\varphi(t-g(t))) - Q_i(t,\psi(t-g(t)))|.$$

By using (3.3), then

$$\|(B\varphi)(t)-(B\psi)(t)\| \le E_1 sup_{t\in[0,T]}\|\varphi(t-g(t))-\psi(t-g(t))\|$$

As $E_1 < 1$ therefore, B defines a contraction.

Theorem 3.6. Suppose (3.1)-(3.5) hold. Let α=$sup_{t\in[0,T]}\sum_{i=1}^{p}|Q_i(t,0)|$, and let $\mathcal{K}$ be a positive constant satisfying the inequality

$$\tau\nu T[\rho(E_1\mathcal{K}+\alpha)+E_2E_3\mathcal{K}+E_4]+E_1\mathcal{K}+\alpha \le \mathcal{K}$$

Let $\mathcal{M} = \{\varphi \in \mathcal{H}_T : \|\varphi\| \le \mathcal{K}\}$. Then Eq (1.2) has a solution in $\mathcal{M}$.

Proof. First, we will define $\mathcal{M} = \{\varphi \in \mathcal{H}_T : \|\varphi\| \le \mathcal{K}\}$. By knowing, that A is continuous and AM contained in a compact set from lemma (3.4). In addition, the mapping B is a contraction from lemma (3.5). It is clear that $A, B: \mathcal{H}_T \to \mathcal{H}_T$. The aim is showing that $\|A_\varphi + B_\psi\| \le \mathcal{K}$. Let $\varphi, \psi \in \mathcal{M}$, with $\|\varphi\|, \|\psi\| \le \mathcal{K}$. Then,

$$\|A_\varphi + B_\psi\| \le \|A_\varphi\| + \|B_\psi\|$$

Lemma 3.4 says that,

$$\|A_{\varphi_n}\| \le \tau\nu T[\rho E_1(\|\varphi_n\|+\alpha)+E_2E_3\|\varphi_n\|+E_4]$$

Therefore,

$$\|A\| + \|B\| \le \tau\nu T[\rho E_1(\|\varphi_n\|+\alpha)+E_2E_3\|\varphi_n\|+E_4]+E_1\|\psi\|+\alpha$$
$$\le \tau\nu T[\rho E_1(\mathcal{K}+\alpha)+E_2E_3\mathcal{K}+E_4]+E_1\mathcal{K}+\alpha \le \mathcal{K}.$$

Hence, all conditions of Theorem 3.3 are proven. Thus, there exists a fixed point z in $\mathcal{M}$. By Lemma 3.2, this fixed point is a solution of Eq (1.2). Therefore, Eq (1.2) has a T-periodic solution**.**

Theorem 3.7. Let (3.1)-(3.5) hold if

$$E_1+\tau\nu T(\rho E_1)+TE_3E_2<1$$

Then Eq (1.2) has a unique T-periodic solution.

Proof. Let $\varphi,\psi\epsilon\ \mathcal{H}_T$. We define P as Eq (3.7). We have,

$$\|P_{\varphi}{}_{-}P_{\psi}\|<[E_1+\tau\nu T(\rho E_1)+TE_3E_2]\|\varphi-\psi\|.$$

This completes the proof of Theorem 3.7.

CONCLUSION

The aim of this study to convert Eq (1.2) into an integral equation and employ Theorem 3.3, provide the existence of periodic solutions. The integral equation helps us to create two mappings, one of them is a contraction and the other is completely continuous. Besides, we show the uniqueness of the periodic solution by using the contraction mapping principle that has been mentioned in theorem 3.7.

Acknowledgment

The authors would like to express their sincere thanks to referees for reading the manuscript, correcting errors, and valuable hints.

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