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Comparison between Collocation Method and Chebyshev polynomial Method for Solving Abel Integral Equation

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Abstract

In this paper will be compared between Collocation method and Chebyshev polynomial for solving Abel integral equation . Numerical examples are presented to illustrate the efficiency and accuracy of the proposed methods.

Key words: Volterra integral equation; Abel integral equation; Shifted Legendre Polynomial; Collocation Method; Chebyshev polynomial.

1.Introduction

In this paper, we consider the following Volterra integral equations of the first and second kinds, respectively:

$$\lambda \int_0^x \frac{y(t)}{(x-t)^\alpha} dt = f(x), \quad 0 \le t \le x \le 1, \tag{1}$$

$$y(x) + \lambda \int_0^x \frac{y(t)}{(x-t)^\alpha} dt = f(x), \quad 0 \le t \le x \le 1, \tag{2}$$

where *f(x)* is in $L^2(R)$ on the interval $0 \le x \le 1$ and $0< \alpha < 1$. Here λ, α and the function *f(x)* are given, and *y(x)* is the solution to be determined for $0< \alpha < 1$,

the integral equations (1) and (2) called Abel's integral equations of the first and Second kinds, respectively.

Abel's integral equations frequently appearing many physical and engineering problems, e.g., semi–conductors scattering theory seismology, heat conductors, metallurgy, fluid flow, chemical reactions and population dynamics [1, 13]

Also the author of [14] developed a numerical Technique based on Legendre wavelet approximations for solving (1) and (2) .The numerical treatment is more difficult for first kind than for second kind Abel integral equations, which have been widely studied [4,12,15,16].

In this paper, we apply the shifted Legendre collocation method and Chebyshev polynomial for solving Abel's integral equations.

The outline in this paper is: In Section 2 we present some definitions of fractional Integral, derivative and some properties. In section 3 we apply the collocation method. In section 4 we apply Chebyshev polynomial method. In section 5 some examples are solved to illustrate the accuracy of the proposed methods.

2. Fractional integral and derivative

Definition 2.1. A real function *f(x), x > 0,* is said to be in the space $C_\mu, \mu \in R$ if there exist a real number (p $>\mu$), *such that* $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty]$.

Definition 2.2. [5] D^q (q is real) denotes the fractional differential operator of order q in the sense of Riemann-Liouville , defined by :

$$D^q f(x) = \begin{cases} \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(x)}{(t-x)^{q-n+1}} dx, 0 \le n-1 < q \le n, \\ \frac{d^n f(t)}{dt^n}\ , \quad q = n \in N \end{cases} \qquad (3)$$

Definition 2.3.[5], I^qdenotes the fractional integral operator of order q in the sense of Riemann –Liouville, denoted by:

$$I^q f(x) = D^{-q} f(x) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t \frac{f(x)}{(t-x)^{1-q}} dx\ , \qquad q > 0, \\ f(t), \quad q = 0. \end{cases} \qquad (4)$$

Definition. 2.4. [5] Let $f \in C_{-1}^n,\ n \in N$ Then the Caputo fractional derivative of *f(x)*, defined by:

$$D_*^q f(x) = \begin{cases} \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}}{(t-x)^{q-n+1}} dx, 0 \le n-1 < q \le n, \\ \frac{d^n f(t)}{dt^n}\ , \qquad q = n \in N \end{cases} \qquad (5)$$

Now we will be introduced some basic properties of fractional :

For $f \in C_\alpha, \alpha \ge -1, \mu \ge 1, \eta \ge 0, \beta > -1, \delta \ge 0$

i. $I^\mu \in C_0$.
ii. $I^\eta I^\delta f(x) = I^{\eta+\delta} f(x) = I^\delta I^\eta f(x)$.
iii. $D^\eta D^\delta f(x) = D^{\eta+\delta} f(x)$.
iv. $D^\delta I^\delta f(x) = f(x)$
v. $I^\delta D_*^\delta f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}$,$0 \le n-1 < \delta \le n \in N$.
vi. $I^\delta x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\delta)} x^{\beta+\delta}; x > 0$.
vii. $D^\delta x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\delta)} x^{\beta-\delta}; x > 0$.

3. The Collocation Method

3.1. The shifted of Legendre polynomials

We know Legendre polynomials are denoted on the interval $z \in [-1,1]$ and can be determined with the help of the following formulae [3]:

$$L_0(z) = 1\,, \quad L_1(z) = z\,.$$

$$L_{i+1}(z) = \frac{2i+1}{i+1} z L_i(z) - \frac{i}{i+1} L_{i-1}(z), \qquad i = 1,2,\ldots.$$

In order to use these polynomials on the interval $x \in [0, a]$ called shifted Legendre polynomials by introducing the change of variable: $Z = \frac{2}{a}x - 1$.

Let the shifted Legendre polynomials $L_i\left(\frac{2}{a}x - 1\right)$, be denoted $p_i(x)$ can be obtained as follows:

$$p_0(x) = 1, \quad p_1(x) = \frac{2}{a}x - 1,$$

$$p_{i+1}(x) = \frac{(2i+1)\left(\frac{2}{a}x - 1\right)}{(i+1)} p_i(x) - \frac{i}{i+1} p_{i-1}(x), \; i = 1,2,\ldots. \quad (6)$$

The analytical form of the shifted Legendre polynomial $p_i(x)$ of degree i given by:

$$p_i(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!\left(\frac{x}{a}\right)^k}{(i-k)!\,(k!)^2} \qquad (7)$$

3.2 The solution of singular Volterra Integral Equation

In this section we solve the singular Volterra integral (1) and (2) by using the shifted Legendre collocation method. First of all we approximate *y(x)* as

$$y_n(x) = A_0 x^{\alpha} + \sum_{i=0}^{n} c_i\, p_i(x) \quad , \tag{8}$$

where A$_0$ and the coefficients $c_i(i = 0, \dots, n)$ are unknown , substituting (8) in (1) we have

$$\lambda A_0 \int_0^x \frac{t^{\alpha}}{(x-t)^{\alpha}} dt + \lambda \sum_{i=0}^{n} c_i \int_0^x \frac{p_i(t)}{(x-t)^{\alpha}} dt = f(x). \tag{9}$$

Now, we know that

$$\int_0^x \frac{t^n}{(x-t)^{\alpha}} dt = \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n+2-\alpha)} x^{n+1-\alpha} \;. \tag{10}$$

and

$$\int_0^x \frac{t^{\alpha}}{(x-t)^{\alpha}} dt = \frac{\pi\alpha}{\sin(\pi\alpha)} x \;. \tag{11}$$

Employing (7) and (10) we obtain

$$\int_0^x \frac{p_i(t)}{(x-t)^{\alpha}} dt = \sum_{k=0}^{i} b_{ik}^{(\alpha)} \left(\frac{x}{a}\right)^{k+1-\alpha} \;. \tag{12}$$

$$b_{ik}^{(\alpha)} = (-1)^{i+k} \frac{(i+k)!\,\Gamma(k+1)\Gamma(1-\alpha)}{(k!)^2 (i-k)!\,\Gamma(k+2-\alpha)}$$

(13)

$$= (-1)^{i+k} \frac{(i+k)!\,\Gamma(1-\alpha)}{k!\,(i-k)!\,\Gamma(k+2-\alpha)}$$

By using (11) and (12) hence (9) can be written as

$$\lambda A_0 \frac{\pi\alpha}{\sin(\pi\alpha)} x + \lambda \sum_{i=0}^{n} \sum_{k=0}^{i} c_i b_{ik}^{(\alpha)} \left(\frac{x}{a}\right)^{k+1-\alpha} = f(x). \qquad (14)$$

Similarly by substituting (8) into (2) and by using (11) and (12) we obtain

$$A_0 \left(x^\alpha + \lambda \frac{\pi\alpha}{\sin(\pi\alpha)} x\right) + \sum_{i=0}^{n} c_i p_i(x) + \lambda \sum_{i=0}^{n} \sum_{k=0}^{i} c_i b_{ik}^{(\alpha)} \left(\frac{x}{a}\right)^{k+1-\alpha} = f(x). \qquad (15)$$

To find the solution of the first kind Abel integral equation (1) or the second kind Abel integral equation (2) we collocate (14) or (15) at (n+2) points, respectively.

4. Chebyshev polynomial

In this method, we use the Chebyshev polynomial through the fractional calculus to approximate the solution of Abel's integral equations.

Definition 4. 1.

If $x = \cos(\theta)\ (0 < \theta < \pi)$, the function

$$T_n(x) = \cos(n\theta) = \cos(n \arccos x), \qquad (16)$$

is the polynomial of degree n $(n = 0,1,2,\dots,)$. T_n is called Chebyshev polynomial of degree n .When θ increase from 0 to π x decrease from 1 to -1. Then the interval $[-1,1]$ is a domain of $T_n(x)$. Also the roots of Chebyshev polynomial of degree $n+1$ can be obtained by the following formula

$$x_k = \cos\left(\frac{(2k-1)\pi}{2(n+1)}\right), \qquad k = 1,2,\dots,n+1\,. \qquad (17)$$

In addition, the successive Chebyshev polynomial can be obtain by the following recursive relation

$$T_0(x) = 1,$$
$$T_1(x) = x,$$
$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad n = 2,3,4,\dots \qquad (18)$$

recursively generates all the polynomials $\{T_n(x)\}$ very efficiently.

4.1. Solving Abel's integral equations by Chebyshev polynomials

- **Solving Abel's integral equation of a first kind**

According to (1) and (4), Abel's integral equation of the first kind can be rewritten as follows:

$$f(x) = \Gamma(1-\alpha) I^{1-\alpha} u(x), \tag{19}$$

we will use Chebyshev polynomials for approximating $u(x)$. We assume $u(x)$ on the interval [-1, 1] can be written as an infinite series of Chebyshev basis

$$u(x) = \sum_{i=0}^{\infty} a_i \, G_i(x), \tag{20}$$

where $G_i(x) = T_i(x)$, is Chebyshev polynomial . We express $u(x)$ as a truncated Chebyshev series as follows:

$$u_n(x) = \sum_{i=0}^{n} a_i \, G_i(x), \tag{21}$$

Such that $u_n(x)$ will be approximated solution of Abel's integral equation. Now, we can write (19) in the form:

$$f(x) = \Gamma(1-\alpha) \sum_{i=0}^{n} a_i \, I^{1-\alpha} G_i(x), \tag{22}$$

Note that, we applied the linear combination property of fractional integral according to equation (4). So it is sufficient to obtain $I^{1-\alpha} G_i(x)$.

Assume that

$$G_i(x) = \sum_{k=0}^{i} b_k \, x^k \,, \tag{23}$$

where b_k are the coefficients of a Chebyshev polynomial of degree i that are defined by (18) . Now, by taking $I^{1-\alpha}$ to both sides of the equation (23) then we have

$$I^{1-\alpha} G_i(x) = \sum_{k=0}^{i} b_k \, I^{1-\alpha} x^k \,. \tag{24}$$

From properties of fractional confirms the validity of (23) and utilizes computation of $I^{1-\alpha} T_i$. So substitution from (24) into (22) gives the following form

$$f(x) = \Gamma(1-\alpha) \sum_{i=0}^{n} a_i \sum_{k=0}^{i} b_k \, I^{1-\alpha} x^k \,. \tag{25}$$

Now, we collocate the roots of a Chebyshev polynomial of degree $n+1$ in (25). It leads to a system of linear equations. By solution this system of equations we have the approximate solution of Abel's integral equation as (21). For more efficiency of this method, we suggest reordering Chebychev series as follows

$$\sum_{i=0}^{n} a_i\, G_i(x) = \sum_{i=0}^{n} c_i\, x^i \ , \tag{26}$$

where c_i is linear combination of a_i. Then (22) is reformed

$$f(x) = \Gamma(1-\alpha)\sum_{i=0}^{n} c_i\, I^{1-\alpha} x^i \,. \tag{27}$$

This reformation leads to reduce computation the term $I^{1-\alpha}x^i$.We remind using directly { $1, x, \dots, x^i$} as basis instead of Chebyshev.

- Solving Abel's integral equation of the second kind

  We can rewrite (2) with consideration (4) in the form

$$u(x) = f(x) + \Gamma(1-\alpha) I^{1-\alpha}\, u(x). \tag{28}$$

Similarly, substituting from (21) into (28). So we have

$$\sum_{i=0}^{n} a_i\, G_i(x) = f(x) + \Gamma(1-\alpha)\sum_{i=0}^{n} a_i\, I^{1-\alpha} G_i(x). \tag{29}$$

or equivalently by using (26) we obtain

$$\sum_{i=0}^{n} a_i\, G_i(x) = f(x) + \Gamma(1-\alpha)\sum_{i=0}^{n} c_i\, I^{1-\alpha} x^i. \tag{30}$$

after computing $I^{1-\alpha}x^i$and substitute the collocation points [6] we have a system of linear equations. Solution of the system leads to the approximated solution of Abel's integral equation.

5. Numerical Examples

This section is devoted to computational results. We apply the presented methods in this paper and solve several examples.

Example 5. 1.

Consider the first Kind Abel integral equation [2, 14]

$$\int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt = \frac{32}{35}x^{7/2} + \frac{4}{3}x^{3/2} - 2x^{1/2}$$

which has the exact solution $y(x) = x^3 + x - 1$.

- **Solution by using collocation method**

For this problem we use equation (14)

$with\ \alpha = {}^{1}/_{2}\ , \lambda = 1\ and\ n = 3.$ We obtain

$$A_0 = 0, c_0 = \frac{-1}{4}\ , c_1 = \frac{19}{20}\ , c_2 = \frac{1}{4}\ , c_3 = \frac{1}{2o}.$$

Therefore using (8) we have

$$y_3(x) = \frac{-1}{4}p_0(x) + \frac{19}{20}p_1(x) + \frac{1}{4}p_2(x) + \frac{1}{20}p_3(x)$$

$$= \frac{-1}{4}(1) + \frac{19}{20}\left(\frac{2}{a}x - 1\right) + \frac{1}{4}\left(\frac{6}{a^2}x^2 - \frac{6}{a}x + 1\right) + \frac{1}{20}\left(\frac{20}{a^3}x^3 - \frac{30}{a^2}x^2 + 12x - 1\right)$$

at a =1, we have

$$y_3(x) = \frac{-1}{4}(1) + \frac{19}{20}(2x - 1) + \frac{1}{4}(6x^2 - 6x + 1) + \frac{1}{20}(20x^3 - 30x^2 + 12 - 1)$$

$$=x^3 + x - 1$$

which is the Exact solution.

- **Solution by using Chebyshev polynomial**

Chebyshev at $n = 3$			
x	**Exact**	**Approximate**	**Abs. Error**
0.1	**-0.899**	**-0.899000**	**0**
0.2	**-0.792**	**-0.792000**	**0**
0.3	**-0.673**	**-0.673000**	**0**
0.4	**-0.536**	**-0.536000**	**0**

0.5	**-0.375**	**-0.3750000001**	$\mathbf{1.10^{-10}}$
0.6	**-0.184**	**-0.1840000001**	$\mathbf{1.10^{-10}}$
0.7	**0.043**	**0.0429999998**	$\mathbf{2.10^{-10}}$
0.8	**0.312**	**0.3119999998**	$\mathbf{2.10^{-10}}$
0.9	**0.629**	**0.6289999997**	$\mathbf{3.10^{-10}}$
1.0	**1.0**	**0.9999999996**	$\mathbf{4.10^{-10}}$

Table 1: Estimate the exact solution, approximate solution and error of Example 5.1.

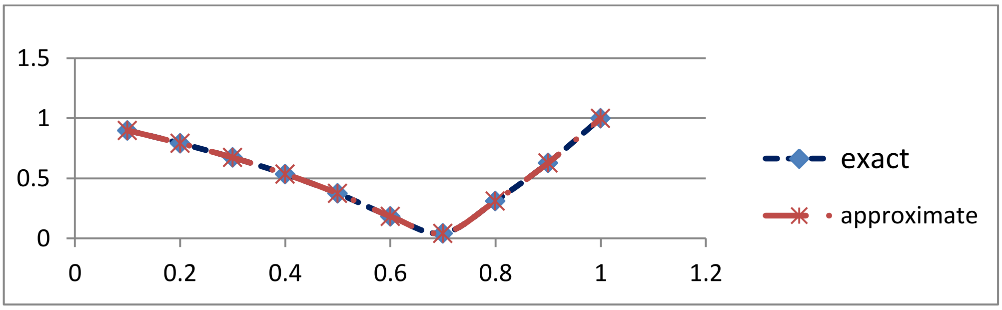

Fig 1: Numerical result for Example 5.1.

Example 5.2.

Consider the first Kind Abel's integral equation:

$$\int_0^x \frac{y(t)}{(x-t)^{1/4}} dt = \frac{\pi}{2\sqrt{2}} x ,$$

which has the exact solution$y(x) = \sqrt[4]{x}$.

- **Solution by using collocation method**

For this problem we use equation (14) with $\alpha = 1/_4$,$\lambda = 1$ and $n = 3$. We obtain

$$A_0 = 1, c_0 = 0 , c_1 = 0 , c_2 = 0, c_3 = 0 .$$

Therefore using (11) we have

$$y_3(x) = \sqrt[4]{x}$$

which is the exact solution.

- **Solution by using Chebyshev polynomial**

Chebyshev at $n = 3$			
x	**Exact**	**Approximate**	**Abs. Error**
0.1	**0.5623413252**	**-0.899000**	**0.0865373343**
0.2	**0.668740305**	**-0.792000**	**0.0166710838**
0.3	**0.7400828045**	**-0.673000**	**0.058867315**
0.4	**0.7952707288**	**-0.536000**	**0.068400085**
0.5	**0.8408964153**	**-0.3750000001**	**0.06552292815**
0.6	**0.8801117368**	**-0.1840000001**	**0.0581640915**
0.7	**0.9146912192**	**0.0429999998**	**0.03709964249**
0.8	**0.945741609**	**0.3119999998**	**0.0349020831**
0.9	**0.9740037464**	**0.6289999997**	**0.2162095076**
1.0	**1.0**	**0.9999999996**	**0.559611511**

Table 2: Estimate the exact solution, approximate solution and error of Example 5.2.

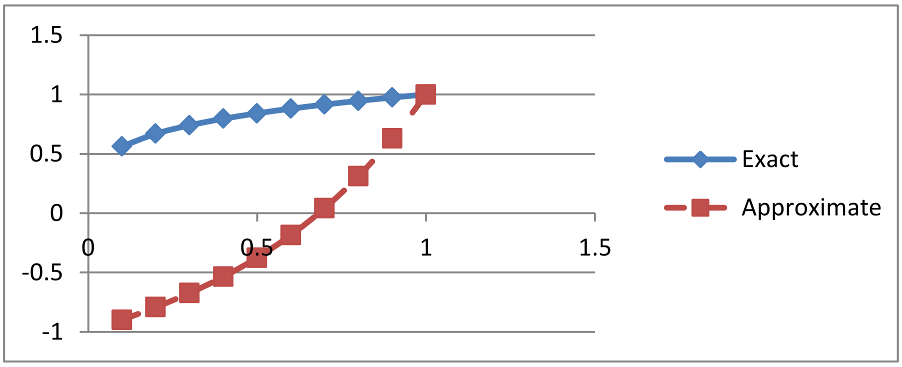

Fig 2: Numerical result for Example 5.2

Example5. 3.

Consider the second Kind Abel integral equation [2,7,15]

$$y(x) = x^3 + \frac{32}{35}x^{7/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}dt$$

which has the exact solution $y(x) = x^3$.

- **Solution by using collocation method**

  For this problem we use equation (15) with $\alpha = {}^{1}/_{2}$, $\lambda = 1$ *and* $n = 3$. We obtain

$$A_0 = 0, c_0 = \frac{1}{4}\,, c_1 = \frac{9}{20}\,, c_2 = \frac{1}{4}\,, c_3 = \frac{1}{2o}.$$

Therefore using (8) we have

$$y_3(x) = \frac{1}{4}p_0(x) + \frac{9}{20}p_1(x) + \frac{1}{4}p_2(x) + \frac{1}{20}p_3(x)$$

$$y_3(x) \;= \frac{1}{4}(1) + \frac{9}{20}\left(\frac{2}{a}x - 1\right) + \frac{1}{4}\left(\frac{6}{a^2}x^2 - \frac{6}{a}x + 1\right) + \frac{1}{20}\left(\frac{20}{a^3}x^3 - \frac{30}{a^2}x^2 + 12x - 1\right)$$

at $a = 1$

$$y_3(x) = \frac{1}{4}(1) + \frac{9}{20}(2x - 1) + \frac{1}{4}(6x^2 - 6x + 1) + \frac{1}{20}(20x^3 - 30x^2 + 12x - 1)$$

$$=x^3$$

which is the exact solution.

- **Solution by using Chebyshev polynomial**

  With the exact solution x^3 . Since the exact solution is a polynomial of degree 3 this method gives the exact solution for $n \geq 3$.

Example 5.4.

Consider the second Kind Abel integral equation **[**2,8]

$$y(x) = x^{1/3} + \frac{2\pi}{3\sqrt{3}}x - \int_0^x \frac{y(t)}{(x-t)^{1/3}}dt$$

which has the exact solution$y(x) = x^{1/3}$.

- **Solution by using collocation method**

For this problem we use equation (15) , with $\alpha = {}^{1}/_{3}$,$\lambda = 1$ and $n = 3$. We obtain

$$A_0 = 1, c_0 = 0, c_1 = 0\,, c_2 = 0\,, c_3 = 0\,.$$

Therefore using (8) we have

$$y_3(x) = x^{1/3}\,,$$

which is the exact solution.

- **Solution by using Chebyshev polynomial**

Chebyshev polynomial at $n = 3$			
x	**Exact**	**Approximate**	**Abs. Error**
0.1	**0.215443469**	**0.2467281434**	**0.0312846744**
0.2	**0.3419951893**	**0.3437134437**	**0.0017182544**
0.3	**0.4481404747**	**0.4410256129**	**0.0071148618**
0.4	**0.5428835233**	**0.5367341973**	**0.006149326**
0.5	**0.6299605249**	**0.6290361225**	**0.0009244024**
0.6	**0.7113786609**	**0.7161597818**	**0.0047811209**
0.7	**0.7883735163**	**0.7963392216**	**0.0079657053**
0.8	**0.861773876**	**0.8678069132**	**0.0060330372**
0.9	**0.9321697518**	**0.9287925457**	**0.0033772061**
1.0	**1.0**	**0.9775241999**	**0.0224758001**

Table 3: Estimate the exact solution, approximate solution and error of Example 5.4

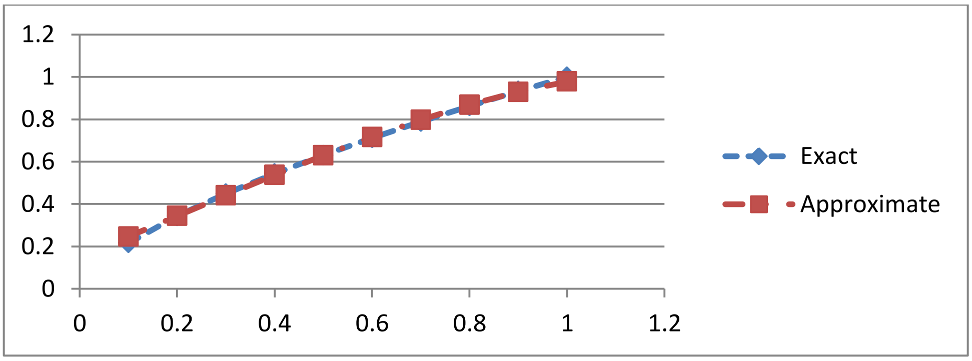

Fig 3 : Numerical results for Example 5.4

Note that: we have computed the numerical results by Maple 16 programming interesting applications of some integral equations are given in [9-11].

Conclusion

This study showed that for most problems, the collocation method is better than Chebyshev polynomials method.

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