Contra gp*- Cleavability

Ghazeel Almahdi, Jalalah / Department of Mathematics / Faculty of Education / Sirte University / Libya

Munera Shaili Saki / Faculty of Engineering Technology / Zuwarah / Libya.
Abstract:

Different types of cleavability (originally named splitability) by Skii. A (1985) introduced as following:

A topological space $X$ is said to be cleavable over a class of spaces $\mathcal{P}$ if for $A \subset X$ there exists a continuous mapping $f: X \to Y \in \mathcal{P}$ such that $f^{-1}(f(A)) = A$, $f(X)=Y$.

In this paper, we used special functions called contra gp*-continuous function to study the concept of cleavability over these special topological spaces as called gp*-Hausdorff spaces, gp*-normal spaces and gp*connected spaces as following:

If $\mathcal{P}$ is a class of topological spaces with certain properties and if $X$ is cleavable over $\mathcal{P}$, then $X \notin \mathcal{P}$. also if $\mathcal{P}$ is a class of topological spaces with certain properties and if $Y$ is cleavable over $\mathcal{P}$ then $Y \notin \mathcal{P}$.

**Key words:** contra gp*– cleavable space, contra gp*-pointwise cleavable space, a contra - gp* - absolutely cleavable space, a contra gp*- double cleavable spaces.

Contra gp*- Cleavability

الملخص:

في هذا البحث درسنا حالة الانشقاق (الانشطار) باستخدام الدالة المتصلة الخاصة : contra gp* - continuous function (gp* Hausdroff, completely Hausdroff, gp*-normal, gp*-connected) spaces.

1 - فصل من الفضاءات الطوبولوجية بخصائص معينة وكانت قابلة للانشطار باستخدام الدالة المتصلة : $X$


خضرة في الفضاءات الطوبولوجية 

إذا كان $X$ من الفضاءات الطوبولوجية معينة وكانت $Y$ قابلة للانشطار باستخدام الدالة المتصلة $f$, فإن $Y$ من الفضاءات الطوبولوجية معينة.
1-Introduction

S. Sekar, P. Jayakumar [6,8] introduced the concept of contra gp*-continuous function via the notion of gp*-open set and studied new spaces called gp*-Hausdorff spaces, gp*-normal spaces and gp*-connected.

The aim of this paper to study the concept of cleavability by using this special continuous function over class of new topological spaces.

Throughout this paper (X, τ) and (Y, σ) represent the non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let A ⊂ X, the closure of A and interior of A will be denoted by cl(A) and int(A) respectively, The union of all preopen sets of X contained in A is called pre-interior of A and it is denoted by pint(A).

The intersection of all preclosed sets of X containing A is called pre-closure of A and it is denoted by pcl(A). The union of all gp*-open sets X contained in A is called gp*-interior of A and it is denoted by gp*-int(A), the intersection of all gp*-closed sets of X containing A is called gp*-closure of A and it is denoted by gp*-cl(A).

2-Preliminaries:

We recall the following definitions and notations, which are useful in this paper.

Definition12.

Let A subset A of a topological space (X, ), is called a generalized pre- closed set (briefly gp- closed) if pcl (A)⊆ U whenever A⊆U and U is open in X. The complement of gp-closed set is called gp-open. The family of all gp-open
[respectively gp-closed] sets of \((X, \tau)\) is denoted by \(gp-O(X, \tau)\) [respectively \(gp-CL(X, \tau)\)].

### 2.2. Definition (Jayakymar, Mariappa and Sekar) [2,4,6,8]

A subset \(A\) of a topological space \((X, \tau)\), is called \(gp^*\)-closed set if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(gp\)-open in \(X\). The complement of \(gp^*\)-closed set is called \(gp^*\)-open. The family of all \(gp^*\)-open [respectively \(gp^*\)-closed] sets of \((X, \tau)\) is denoted by \(gp^*-O(X,)\) [respectively \(gp^*-CL(X, \tau)\)].

### 2.3. Definition (Sekar and Jayakymar)[8].

The \(gp^*\)-closure of a set \(A\), denoted by \(gp^*-\text{Cl}(A)\) is the intersection of all \(gp^*\)-closed sets containing \(A\).

### 2.4. Definition (Sekar and Jayakymar)[8].

The \(gp^*\)-interior of a set \(A\), denoted by \(gp^*-\text{int}(A)\) is the union of all \(gp^*\)-open sets contained in \(A\)

### 2.5. Definition (Sekar and Jayakymar)[8].

A function \(f: (X,\tau) \rightarrow (Y, \sigma)\) is called contra \(gp^*\)-continuous if \(f^{-1}(V)\) is \(gp^*\)-closed in \((X,\tau)\) for every open set \(V\) in \((Y, \sigma)\).

### 2.6. Example (Sekar and Jayakymar)[8].

Let \(X = Y = \{a, b,c\}\) with \(\tau = \{X,\varnothing ,\{a\},\{b\},\{a, b\}\}\) and \(\sigma = \{Y,\varnothing ,\{a, b\}\}\). Define a function \(f: (X,\tau) \rightarrow (Y, \sigma)\) by \(f(a) = b, f(b) = c, f(c) = a\). Clearly \(f\) is contra \(gp^*\)-continuous.
2.7. Example
Let \( X = Y = \{a, b, c\} \) with \( \tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}\} \) and \( \sigma = \{Y, \varnothing, \{b, c\}\} \). Define a function \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = a, f(b) = b, f(c) = c \).
Clearly \( f \) is contra gp*-continuous.

2.7. Example
Let \( X = Y = \{a, b, c\} \) with \( \tau = \{X, \varnothing, \{a\}, \{a, b\}\} \) and \( \sigma = \{Y, \varnothing, \{a, c\}\} \). Define a function \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = c, f(b) = a, f(c) = b \), \( f \) is not contra gp*-continuous. Because \( f^{-1}(\{a, c\}) = \{a, b\} \) is not contra gp* closed in \((X, \tau)\) where \( \{a, c\} \) is open in \((Y, \sigma)\).

3. Contra gp*-Cleavability

3.1. Definition
A topological space \( X \) is said to be a contra gp*-cleavable over a class of spaces \( \mathcal{P} \), if for any subset \( A \) of \( X \), there exists a contra gp*-continuous mapping \( f: X \to Y \), such that \( f^{-1}(f(A)) = A \), and \( f(X) = Y \).

3.2. Definition
A topological space \( X \) is said to be a contra gp*-double cleavable over a class of spaces \( Y \) if for any subsets \( A \subset X \) and \( B \subset X \) there exists a contra gp*-continuous mapping \( f: X \to Y \) such that \( f^{-1}(f(A)) = A \) and \( f^{-1}(f(B)) = B \).

3.3. Definition
A topological spaces is said to be contra gp* point wise cleavable over a class of spaces \( \mathcal{P} \), if for every point \( x \in X \) there exists a contra gp*-continuous injective mapping \( f: X \to Y \), such that \( f^{-1}(\{x\}) = \{x\} \).
3.4. Remark

By a contra gp* -pointwise cleavable, we mean that a contra gp*-continuous function \( f: X \rightarrow Y \in \mathcal{P} \) is an injective and a contra gp*-continuous.

3.5. Definition

A topological space \( X \) is said to be a contra gp*- absolutely cleavable over a class of spaces \( \mathcal{P} \), if for any subset \( A \) of \( X \), there exists an injective a contra gp*-continuous mapping \( f : X \rightarrow Y \), such that \( f^{-1}(A) = A \).

3.6. Remark

If \( f \) is an open (resp. closed) a contra-gp*-continuous mapping, we shall say that \( X \) is open (resp. closed) a contra-gp*- absolutely cleavable over \( \mathcal{P} \).

3.7. Definition (Sekar and Jayakymar)[8].

A topological space \( (X, \tau) \) is said to be gp*-Hausdorff space if for each pair of distinct points \( x \) and \( y \) in \( X \) there exists \( U \in \text{gp}^*\text{-O}(X, x) \) and \( V \in \text{gp}^*\text{-O}(X, y) \) such that \( U \cap V = \emptyset \).

3.8. Example (Sekar and Jayakymar)[8].

Let \( X = \{a, b, c\} \) with \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\} \). Let \( x \) and \( y \) be two distinct points of \( X \), there exists an gp*-open neighbourhood of \( x \) and \( y \) respectively such that \( \{x\} \cap \{y\} = \emptyset \). Hence \( (X, \tau) \) is gp*-Hausdorff space.

3.9. Definition (Mashhour, Abd El-Monsef, and El-Deeb)[4].
A topological space $X$ is said to be completely Hausdorff–space if for every two distinct points $x$ and $y$ there exist two disjoint open sets $U$ and $V$ such that $x \in U$, $y \in V$, $y \notin U$, $y \in V$, $x \notin V$, and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

3.10. Example
The countable extension topology is the topology on the real line generated by the union of the usual Euclidean topology and the countable topology. Sets are open in this topology if and only if they are of the form $U \setminus A$ where $U$ is open in the Euclidean topology and $A$ is countable. This space is completely Hausdorff.

3.11. Proposition
Let space $X$ be a closed contra gp*- pointwise cleavable over a class of completely Hausdorff space $\mathcal{P}$, then $X$ is a gp*-Hausdorff space.

Proof
Let $x_1$, $x_2$ be any points of $X$, with $x_1 \neq x_2$, then there exists a closed contra- gp*- continuous mapping $f: X \to Y$, such that $f^{-1}\{x_1\} = \{x_1\}$, $f^{-1}\{x_2\} = \{x_2\}$, then $f(x_1)$, $f(x_2)$ are distinct points of $Y$, since $Y$ is completely Hausdorff, so there exist open sets $G$, $H$ such that $f(x_1) \in G$, $f(x_2) \in H$, $\text{cl}(G) \cap \text{cl}(H) = \emptyset$, $f^{-1}(x_1) \in f^{-1}(G)$.
and \( f^{-1}(x_2) \in f^{-1}(H) \), this implies that \( x_1 \in f^{-1}(G) \) , \( x_2 \in f^{-1}(H) \), since \( f \) is a contra- \( gp^* \)- continuous , so \( f^{-1}(G) \), \( f^{-1}(H) \) are \( gp^* \)- open sets of \( X \) and \( cl(f^{-1}(G)) \bigcap cl(f^{-1}(H)) \subseteq f^{-1}(cl(G)) \bigcap f^{-1}(cl(H)) = f^{-1}(cl(G)) \bigcap cl(H)) = f^{-1}(\emptyset) = \emptyset \). Hence \( X \) is \( gp^* \)-Hausdorff space.

3.12.**Proposition**

Let \( X \) be a completely Hausdorff space is closed contra \( gp^* \)-pointwise cleavable over a class of spaces \( Y \), then \( Y \) is \( gp^* \)-Hausdorff.

**Proof:**

Let \( y_1 \), \( y_2 \) be any distinct points of \( Y \), there exists \( x_1 \), \( x_2 \) such that \( f^{-1}(y_1) = x_1 \), \( f^{-1}(y_2) = x_2 \) and a closed contra \( gp^* \)- continuous mapping \( f: X \rightarrow Y \) with \( f^{-1}(f^{-1}(y_1)) = f^{-1}(y_1) \), \( f^{-1}(f^{-1}(y_2)) = f^{-1}(y_2) \), \( f^{-1}(y_1) \) and \( f^{-1}(y_2) \) are two distinct points of \( X \), but \( X \) is completely Hausdorff , so there exist open sets \( U, V \) Such that \( f^{-1}(y_1) \subseteq U \), \( f^{-1}(y_2) \subseteq V \), \( cl(U) \bigcap cl(V) = \emptyset \), then \( f^{-1}(y_1) \notin X \setminus U \), this implies that \( ff^{-1}(y_1) \notin f(X \setminus U) \), \( y_1 \in Y \setminus f(U) \) so \( y_1 \in f(U) \)in \( Y \), similarly \( f^{-1}(y_2) \notin X \setminus V \) this implies that \( ff^{-1}(y_2) \notin f(X \setminus V) \) , then \( y_2 \notin Y \setminus f(V) \), so \( y_2 \notin f(V) \),and since \( f \) is contra \( gp^* \)-continuous so \( cl(f(U)) \bigcap cl(f(V)) \subseteq f(cl(U)) \bigcap f(cl(V)) = f(cl(U) \bigcap cl(V)) = f(\emptyset) = \emptyset \). then \( Y \) is \( gp^* \)-Hausdorff.

3.13.**Definition** [7].

A topological space \((X, \tau)\) is said to be ultra -normal if each pair of non-empty disjoint closed sets in \((X, \tau)\) can be separated by disjoint clopen sets in \((X, \tau)\).
3.14. **Definition** (Sekar and Jayakymar)[8].

A topological space \((X, \tau)\) is said to be gp*-normal if each pair of non-empty disjoint closed sets in \((X, \tau)\) can be separated by disjoint gp*-open sets in \((X, \tau)\).

3.15. **Proposition**

Let space \(X\) be a closed contra gp* -absolutely double cleavable over a class of normal spaces , then \(X\) is normal.

**Proof:**

Suppose \(F_1, F_2\) be two disjoint closed sets of \(X\), then there exists an injective closed contra gp* -continuous mapping \(f: X \rightarrow Y\) such that 

\[ f^{-1}(F_1)=F_1, f^{-1}(F_2)=F_2. \]

Since \(f\) is closed , then \(f(F_1), f(F_2)\) are two disjoint closed sets of \(Y\), since \(Y\) is ultra normal space ,then

\[ f(F_1) \subset U, f(F_2) \subset V, \]

so there exist two closed sets \(U, V\) such that and \(U \cap V = \emptyset\) then \(f^{-1}(F_1) \subset f^{-1}(U), f^{-1}(F_2) \subset f^{-1}(V)\). This implies that \(F_1 \subset f^{-1}(U), F_2 \subset f^{-1}(V)\). Since \(f\) is contra gp* continuous , then \(f^{-1}(U), f^{-1}(V)\) are gp*-open sets of \(X\) and

\[ f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset. \]

Hence \(X\) is normal space .

3.16. **Definition**

A topological space \(X\) is said to be connected if \(X\) is not the union of two disjoint non empty open sets .i.e let \(G,H\) be open sets of \(X\),

\[ X \neq G \cup H, G \neq \emptyset, H \neq \emptyset, G \cap H = \emptyset. \]

3.17. **Example**

Let \(X = \{a, b, c\}\) and let \(= \{X, \emptyset \\{b\}, \{a, b\}, \{b, c\}\}\). \(X\) is connected .

**Definition:** 3.1(Sekar and Jayakymar)[6].
A topological space X is said to be gp*-connected if X cannot be expressed as a disjoint of two non-empty gp*-open sets in X. A sub set of X is gp*-connected if it is gp*-connected as a subspace.

3.18. Example (Sekar and Jayakymar)[6].
Let X = {a, b} and let = {X, Ø, {b}}. It is gp*-connected.

3.19. Proposition
Let X be gp*- connected contra – gp* - absolutely cleavable space over a class of spaces P, then Y is connected space.

Proof:
Suppose Y is not connected space , then Y=UU V , where U , V are disjoint non empty open sets of Y ,so U and V are clopen sets of Y, then there exists an injective contra - gp*- continuous mapping f: X→Y , such that f−1(f−1(U)) = f−1(U) , f−1f(f−1(V)) = f−1(V) since Y=UU V, then f−1(Y)=f−1(UU V ) → X=f−1(U) U f−1(V) since f is a contra- gp*- continuous function, then f−1(U) , f−1(V) are non empty disjoint gp*-open sets in X , which contradicts that X is gp*- connected . Therefore Y is connected space.

CONCLUSION
In this paper we have studied and proved these cases:
1) If P is a class of completely Hausdorff spaces with certain properties and if X is a contra- gp* - pointwise cleavable over P , then X is gp*-- Hausdorff .Hence X ∉ P, also if P is a class of gp*-- Hausdorff spaces with certain properties and if Y is - contra- gp*- pointwise cleavable over P , then Y is completely Hausdorff space .Hence Y ∉ P.
contra gp*-absolutely double cleavable over a class of normal spaces, then $X$ is normal.

2) if $\mathcal{P}$ is a class of gp*-normal spaces with certain properties and if $Y$ is contra–gp*-absolutely double cleavable over $\mathcal{P}$, then $Y$ is normal. Hence $Y \notin \mathcal{P}$.

3) if $\mathcal{P}$ is a class of gp*-connected spaces with certain properties and if $Y$ is contra–gp*-absolutely cleavable over $\mathcal{P}$, then $Y$ is connected. Hence $Y \notin \mathcal{P}$.
REFERENCES


