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Approximate Solution for Solving Fractional order Lotka-Volterra Equation

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Research Summary:

A lot of work regarding prey-predator type Lotka-Volteara model has already been done by many researchers during the last century. However, the work devoted to the concept of model related to fractional order derivatives is still inadequate. We employ Homotopy perturbation Sumudu transform method (HPSTM) to obtain the approximate solutions of these equations. Homotopy perturbation Sumudu transform method gives a new approach to the solution of these kind of problems. And we plot figures to illustrate this technique and it is seen that the (HPSTM) is very useful and effective method to get the approximate solutions.

The most significant part of this study is the decay of both prey-predator populations with increase of the fractional time derivative. Another important part is increase of time taken for the meeting of prey-predator with increase of fractional time derivative. The article aims to motivate both engineers and biologists who are working on the pre-predator model to carry on further researches in this area.

Key words: fractional lotka-volterra model, predator-prey interactions, homotopy perturbation method, he's polynomials , Sumudu transform method.

الحل التقريبي لمعادلات لوتكا فولتيرا من الرتب الكسرية

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الملخص:

في هذا البحث استخدمنا طريقة الاضطراب التماثلي وتحويل Sumudu (HPSTM)، وهي تركيب لطريقة الاضطراب التماثلي (HP) وطريقة تحويل Sumudu (STM). للحصول على الحلول التقريبية لنظام الفريسة والمفترس من الرتب الكسرية (لوتكا فولتيرا).

هذه الطريقة تعطي الحلول التقريبية لإظهار الفعالية والتطبيق في حل هذا النوع من النموذج الرياضي (الفريسة والمفترس). لنستطيع بسهولة تحليل الحدود غير الخطية بمساعدة (HP) لإيجاد الحلول التقريبية بطريقة تيسّر حل هذا النظام المعقد.

وأخيرا، هذه التقنية تعطي نتيجة أسرع وأدق من الطرق السابقة في ايجاد الحل الرياضي للمسائل التطبيقية التي تعتمد على مثل هذا النموذج الرياضي.

الكلمات المفتاحية: نموذج لوتكا فولتيرا من الرتب الكسرية، تفاعل بين الفريسة والمفترس، طريقة الاضطراب التماثلي(كثيرات الحدود) وطريقة تحويل Sumudu .

Abstract: In this paper, we use homotopy perturbation Sumudu transform method (HPSTM), which is a combination of homotopy perturbation (HP) and Sumudu transform method (STM), to obtain the approximate solutions of nonlinear fractional- order prey-predator systems. We can easily decompose the nonlinear terms by the help of (HP) he's polynomials. This technique which converges fast to the accurate solution of the problems. Numerical illustrations are given to show the effectiveness and applicability of this method in solving these kind of system.

1. Introduction

We consider the following fractional-order lotka-volterra model (two-species) prey-predator system [1]

$$\begin{aligned} D_t^{\alpha} x(t) &= ax(t) - bx(t)y(t), \\ D_t^{\alpha} y(t) &= cx(t)y(t) - dy(t), \end{aligned} \qquad (1)$$

$\alpha \in (0,1]$ with the initial conditions $x(0) = x_0$ and $y(0) = y_0$, where $D_t^{\alpha} \equiv \frac{d^{\alpha}}{dt^{\alpha}}$. Where a, b, c and d are all a positive real parameters describing the interaction of the two species, and $x(t)$ is a population size of prey at time t and $y(t)$ is a population size of predator at time t, $D_t x(t)$ and $D_t y(t)$ represent the instantaneous growth rates of the two populations; frequently used to model the dynamics of ecological systems. This model is one of the most interesting and important application of stability theory involves the interactions between two or more biological populations. It shows the situation in which one species (the predator) preys on the other species (the prey), while the prey lives on a different source of food. These kinds of examples contain rather simple equations, but they characterize a wide class of problems of competing species [2,3].

Most fractional differential equations do not have exact solutions, so approximate and numerical techniques (see [4], [5], [6], [7], [8], [9]), must be used. Recently, several numerical and approximate method to solve the fractional differential equations have been given such as Sumudu transform method (STM) was first proposed by [10]. The method was successfully applied to various differential equations as an alternative to Laplace transform by [11, 12] and introduced fundamental properties of Sumudu transform [13, 14, 15, 16]. It was given that a number of studies in this fields [17, 18, 19, 20] [21] [22] [23] [24].

Homotopy perturbation method (HPM) was first proposed by [25, 26, 27, 28] HPM can be an attractive in order to reach the analytical and approximate solutions in applied mathematics and engineering.

In this paper, we used HPSTM including STM and HPM in order to find solution of non-linear fractional- order prey- predator model.

2. Basic Definitions Of Fractional Calculus And The Sumudu Transform Method

We describe some necessary definitions and mathematical preliminaries of the fractional calculus theory and the Sumudu transform method which will be used further in this work.

Definition 1. The Riemann – Liouiville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_\mu$ and $\mu \geq -1$; for $t > 0$ is defined as [29] :

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f(\zeta) d\zeta \,, \tag{2}$$

$$J^0 f(t) = f(t) \tag{3}$$

The Riemann- Liouville derivative has certain disadvantage when trying to model real- world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by M. Caputo in his work on the theory of viscoelasticity [30].

Definition 2. The Caputo fractional derivative operator D^α of order α is defined in the following form [31]:

$$D^\alpha \left[f(t) \right] = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\zeta)^{m-\alpha-1} f(\zeta)^{(m)} d\zeta \,, \tag{4}$$

For $m-1 < \alpha \leq m,\ m \in N, t > 0$

For the Riemann- Lioville fractional integral and the caputo fractional derivative, we have the following relation:

$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!} \,. \tag{5}$$

Definition 3. The Sumudu transform is defined over the set of functions [10] :

$$A=\left\{f\,(t):\exists M\,,\ \tau_1,\ \tau_2>0\,,|f\,(t)|<Me^{\frac{|t|}{\tau_j}}\,,\ if\ t\in(-1)^j\times[0,\infty)\right\}$$

by the following formula :

$$\bar{f}(u)=S\left[f\,(t)\right]=\int_0^t f\,(ut)e^{-t}dt\ \ ,u\in(-\tau_1,\tau_2). \tag{6}$$

Some special properties of the Sumudu transform are as follows:

1. $S\left[c\right]=c$
2. $S\left[\dfrac{t^m}{\Gamma(m+1)}\right]=u^m\ ;\ m>0\ ,$

For further detail and properties of this transform can be found in [13] .

Definition 4. The Sumudu transform of Caputo fractional derivative is defined as follows [32] :

$$S\left[D_t^{\alpha}f\,(t)\right]=\frac{S\left[f\,(t)\right]}{u^{\alpha}}-\sum_{k=0}^{m-1}\frac{f^{(k)}(0)}{u^{\alpha-k}}\ ,m-1<\alpha\le m\ . \tag{7}$$

3. Homotopy Perturbation Sumudu Transform Method (HPSTM).

To illustrate the basic idea of this method, we are taking the sumudu transform on both sides of Eq.(1) , we get [33,34] :

$$\begin{aligned}&S\left[D_t^{\alpha}x\,(t)\right]=S\left[ax\,(t)-bx\,(t)y\,(t)\right]\\&S\left[D_t^{\alpha}y\,(t)\right]=S\left[cx\,(t)y\,(t)-dy\,(t)\right].\end{aligned} \tag{8}$$

Using the differentiation property of the Sumudu transform and a above initial conditions , we have

$$\begin{aligned}&S\left[x\,(t)\right]=x\,(0)+u^{\alpha}S\left[ax\,(t)-bx\,(t)y\,(t)\right]\\&S\left[y\,(t)\right]=y\,(0)+u^{\alpha}S\left[cx\,(t)y\,(t)-dy\,(t)\right]\end{aligned} \tag{9}$$

Now, applying the inverse of Sumudu transform on both sides of Eq.(8) , we get

$$\begin{aligned}&x\,(t)=x_0+S^{-1}\left[u^{\alpha}S\left[ax\,(t)-bx\,(t)y\,(t)\right]\right]\\&y\,(t)=y_0+S^{-1}\left[u^{\alpha}S\left[cx\,(t)y\,(t)-dy\,(t)\right]\right]\end{aligned} \tag{10}$$

Now, we apply the homotopy perturbation method HPM [35] and assuming that the solution of Eq.(10) is in the form

$$x(t)=\sum_{n=0}^{\infty}p^n x_n(t),$$
$$y(t)=\sum_{n=0}^{\infty}p^n y_n(t), \tag{11}$$

And the nonlinear term can be decomposed as

$$Nx(t)=\sum_{n=0}^{\infty}p^n H_n(x), \tag{12}$$

For some He's polynomials $H_n(x)$ (see [36, 37]) that are given by

$$\mathrm{H}_n(x_0,x_1,\ldots,x_n)=\frac{1}{n!}\frac{d^n}{dp^n}\left[N\left(\sum_{i=0}^{\infty}p^i x_i\right)\right]_{p=0},\ n=0,1,2,3,\ldots\ldots \tag{13}$$

The first few components of Eq.(13) are given by

$$H_0=x_0y_0\ ,$$
$$H_1=x_0y_1+x_1y_0\ ,$$
$$H_2=x_0y_2+x_1y_1+x_2y_0\ ,$$
$$\vdots$$

Substituting Eq.(11) and (12) in Eq.(10), we get

$$\sum_{n=0}^{\infty}P^n x_n(t)=x_0+P\left(S^{-1}\left[u^{\alpha}S\left[a\sum_{n=0}^{\infty}P^n x_n(t)-b\sum_{n=0}^{\infty}P^n H_n(x)\right]\right]\right)$$
$$\sum_{n=0}^{\infty}P^n y_n(t)=y_0+P\left(S^{-1}\left[u^{\alpha}S\left[c\sum_{n=0}^{\infty}P^n H_n(x)-d\sum_{n=0}^{\infty}P^n y_n(t)\right]\right]\right) \tag{14}$$

Which is the coupling of the Sumudu transform and the homotopy perturbation method using He's polynomials.

Comparing the coefficient of like powers of P, the following approximations are obtained.

$$P^0:\begin{cases}x_0(t)=x_0\ ,\\ y_0(t)=y_0\ ,\end{cases}$$

$$P^1:\begin{cases}x_1(t)=S^{-1}\left[u^{\alpha}S\left[ax_0-bH_0\right]\right],\\ y_1(t)=S^{-1}\left[u^{\alpha}S\left[cH_0-dy_0\right]\right],\end{cases}$$

$$P^2:\begin{cases}x_2(t)=S^{-1}\left[u^{\alpha}S\left[ax_1-bH_1\right]\right],\\ y_2(t)=S^{-1}\left[u^{\alpha}S\left[cH_1-dy_1\right]\right],\end{cases}$$

$$P^3:\begin{cases} x_3(t)=S^{-1}\left[u^{\alpha}S\left[ax_2-bH_2\right]\right],\\ y_3(t)=S^{-1}\left[u^{\alpha}S\left[cH_2-dy_2\right]\right],\end{cases}$$

$$\vdots$$

$$P^{k+1}:\begin{cases} x_{k+1}(t)=S^{-1}\left[u^{\alpha}S\left[ax_k-bH_k\right]\right],\\ y_{k+1}(t)=S^{-1}\left[u^{\alpha}S\left[cH_k-dy_k\right]\right],\end{cases} \tag{15}$$

4. Applications

Numerical example :

Let us consider the parameters with values $a=2, b=c=1, d=3$ with initial condition $x(0)=1,\ y(0)=2$ in Eq.(15) we obtained [38]:

$$P^0:\begin{cases} x_0=1\\ y_0=2\end{cases}$$

$$H_0=2$$

$$P^1:\begin{cases} x_1=0\\ y_1=-4\dfrac{t^{\alpha}}{\Gamma(\alpha+1)}\end{cases}$$

$$H_1=-4\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$P^2:\begin{cases} x_2=4\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ y_2=8\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}\end{cases}$$

$$H_2=16\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$P^3:\begin{cases} x_3=-8\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\ y_3=-8\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}\end{cases}$$

If we continue the iteration we get the series form of solution as :

$$x(t)=1+4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\dots$$

$$y(t)=2-4\frac{t^{\alpha}}{\Gamma(\alpha+1)}+8\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\dots$$

5. Numerical results and discussion

In this section numerical results of the prey-predator populations. For the standard population size $\alpha=1$ are calculated for various values of time t .

$$x(t)=1+2t^2-\frac{4}{3}t^3+\dots$$

$$y(t)=2-4t+4t^2-\frac{4}{3}t^3+\dots$$

These results are presented graphically through Figure1. The relations between the number of prey and predators in time. As the graph states the number of predators increase , as the number of preys decrease.

Predators will reach their maximum as the preys reach their minimum. Since the number of preys has decreased, there are not enough food for predators. So their number would decrease and so on .

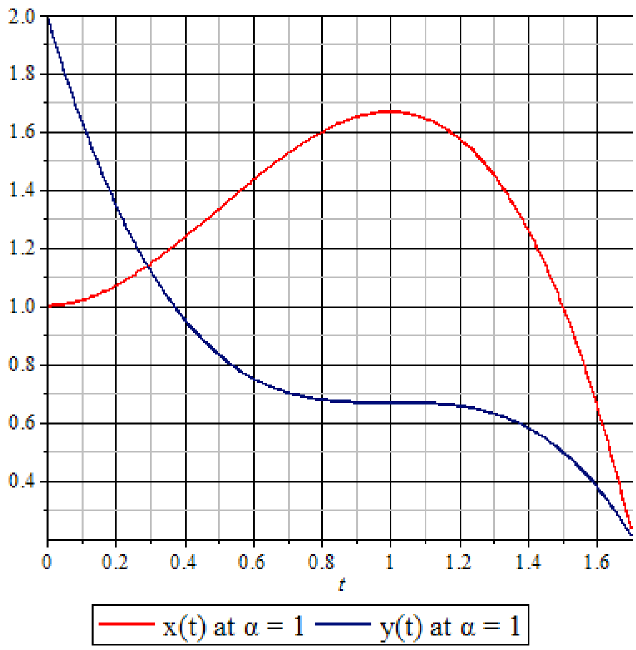

Figure 1: Approximate solutions $x(t)$ and $y(t)$ for $\alpha=1$.

The prey-predator population for different fractional , $\alpha = 0.9$, $\alpha = 0.95$ It is seen from Fig(2).

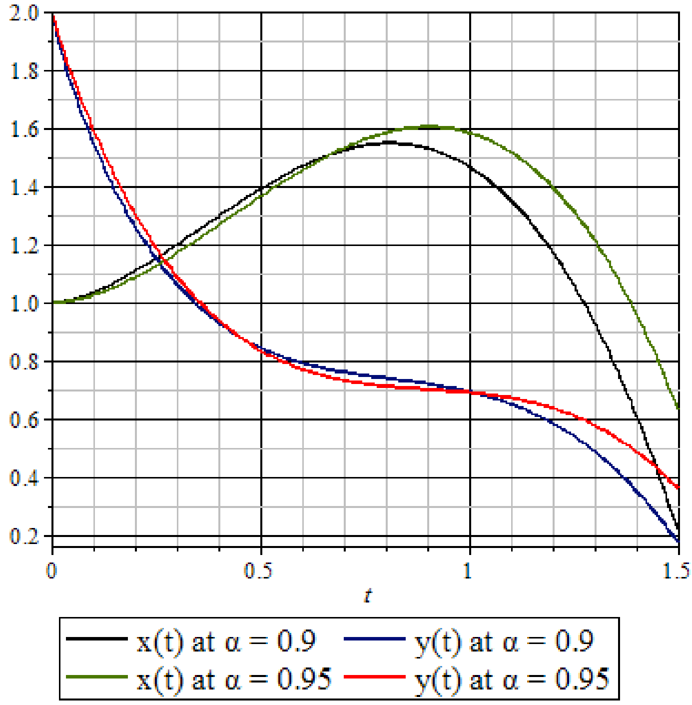

Figure 2: Approximate solutions x(t) and y(t) for $\alpha = 0.9$, $\alpha = 0.95$.

Conclusions

In this work, we successfully apply the Homotopy perturbation Sumudu Transform Method (HPSTM) to find approximations solution for a fractional Lotka-Volterra model. This method reduces the computational difficulties of other traditional difficulties of other traditional methods. Therefore, given a new approach to these kinds of problems. We solve one example and plot figures to illustrate this technique, not that the behavior of solutions depicted in figure1 and figure 2 and it is seen that the (HPSTM) is very useful and effective method to get the approximate solutions. Thus, it can be applied to many complicated fractional- order linear and nonlinear equation reliably.

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