Existence and uniqueness of monotonic positive solution of a functional differential equation with nonlocal conditions

NAJAH S ABDALLA & El-kadeky. Kh. W & SALIMA SAEID ABDALLA
Mathematical Department, Faculty of Arts and Science Al-Marj, University of Benghazi-Libya
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Abstract:
In this paper we study the existence and uniqueness of at least one monotonic positive solution of a nonlocal two-point, with parameters, boundary value problem of the second order functional differential equation.

Keywords: Functional differential equation, nonlocal condition, fixed point theorem, existence and uniqueness of a solution.
1. Introduction

The nonlocal boundary value problems of ordinary differential equations arise in a variety of different areas of applied mathematics and physics.

The study of nonlocal boundary value problems was initiated by II’ in and Moiseev [1] and [2]. Since then, for the nonlocal boundary value problems have been studied by several authors considered in [3-6], [8-16] and and references therein.

In [7], the authors studied the existence of monotonic positive solution for the nonlocal problem

\[ x''(t) = f(t, x(x(t))), \quad t \in (0,1), \]

with the nonlocal condition

\[ \sum_{k=1}^{n} a_k x(\tau_k) = x_0, \quad x'(0) + \sum_{j=1}^{m} b_j x(\eta_j) = x_1 \]

where \( \tau_k \in (a, d) \subset (0,1), \eta_j \in (c, e) \subset (0,1), \) and \( x_0, x_1 > 0. \)

Consider the nonlocal two-point boundary value problem with parameters \( \alpha \) and \( \beta; \)

\[ x''(t) = f(t, x(x(t))), \quad t \in (0,1), \]
\[ \alpha x(\tau) = x_0, \quad x'(0) + \beta x'(\eta) = x_1 \]

where \( \tau \in (a, d) \subset (0,1), \eta \in (c, e) \subset (0,1), \) and \( x_0, x_1 > 0. \)

Here, we study the existence of at least one monotonic positive solution \( x \in C[0,1] \) for the nonlocal problem (1)-(2). Also the unique solution \( x \in C[0,1] \) will be studied.

2. Integral Equation Representation:

Consider the functional differential equation (1) with the nonlocal condition (2) with the following assumptions.

(i) \( f: [0,1] \times R^+ \to R^+ \) is measurable in \( t \in [0,1] \) for all \( x \in R^+ \) and continuous in \( x \in R^+ \) for almost all \( t \in [0,1] \) and there exists an integrable function \( a \in L_1[0,1] \) and constant \( b > 0 \) such that a

\[ |f(t,x)| \leq |a(t)| + b|x|, \quad \forall (t,x) \in [0,1] \times D, \quad D \subseteq R^+. \]

(ii) \( \varphi: (0,1) \to (0,1) \) is continuous.

(iii) \( B = (\beta + 1)^{-1} b < \frac{1}{3-B} \)

(iv) \( \beta > 0 \alpha > 0 \)

Now, we have the following Lemma.
Lemma 1.

The solution of the nonlocal problem (1)-(2) can be expressed by the integral equation

\[ x(t) = A\{x_0 - \alpha \int_0^t (t-s) f(s, x(\varphi(s))) \, ds \} \]

\[ + B(t - \tau) \left\{ x_1 - \beta \int_0^{\eta} f\left(s, x(\varphi(s)) \right) \, ds \right\} \]

\[ + \int_0^t (t-s) f\left(s, x(\varphi(s)) \right) \, ds. \] \hspace{1cm} (3)

Where \( A = \alpha^{-1}, B = (\beta + 1)^{-1}. \)

Proof.

Integrating (1), we get

\[ x'(t) = x'(0) + \int_0^t f\left(s, x(\varphi(s)) \right) \, ds. \] \hspace{1cm} (4)

Integrating (4), we obtain

\[ x(t) = x(0) + x'(0)t + \int_0^t (t-s) f\left(s, x(\varphi(s)) \right) \, ds. \] \hspace{1cm} (5)

Let \( t = \tau, \) in (5), we get

\[ \alpha x(\tau) = \alpha x(0) + \alpha \tau x'(0) + \alpha \int_0^{\tau} (\tau-s) f\left(s, x(\varphi(s)) \right) \, ds, \]

And we deduce that

\[ x(0) = A \left\{ x_0 - \alpha x'(0) - \alpha \int_0^{\tau} (\tau-s) f\left(s, x(\varphi(s)) \right) \, ds \right\}. \] \hspace{1cm} (6)

Substitute from (6) into (5). We obtain

\[ x(t) = A \left\{ x_0 - \alpha \int_0^t (t-s) f\left(s, x(\varphi(s)) \right) \, ds \right\} + x'(0)(t-\tau) \]

\[ + \int_0^t (t-s) f\left(s, x(\varphi(s)) \right) \, ds. \] \hspace{1cm} (7)

Let \( t = \eta, \) in (4), we obtain

\[ \beta x'(\eta) = \beta x'(0) + \beta \int_0^{\eta} f\left(s, x(\varphi(s)) \right) \, ds, \]

\[ x_1 - x'(0) = \beta x'(0) + \beta \int_0^{\eta} f\left(s, x(\varphi(s)) \right) \, ds, \]

and we deduce that

\[ x'(0) = B \left( x_1 - \beta \int_0^{\eta} f\left(s, x(\varphi(s)) \right) \, ds \right), \] \hspace{1cm} (8)

\[ B = (\beta + 1)^{-1}. \]
Substitute from (8) into (7). We obtain
\[ x(t) = A\{x_0 - \alpha \int_0^t (t - s)f(s, x(\varphi(s))) \, ds\} \
+ B(t - \tau) \left\{ x_1 - \beta \int_0^\eta f(s, x(\varphi(s))) \, ds \right\} \
+ \int_0^\tau (t - s)f(s, x(\varphi(s))) \, ds. \]

Which proves that the solution of the nonlocal problem (1)–(2) can be expressed by the integral equation (3).

3. Existence of the solution:

We study here the existence of at least one monotonic nondecreasing solution \( x \in C[0, 1] \) for the integral equation (3).

**Theorem 1.**

Let the assumptions (i)-(iv) be satisfied. Then the nonlocal problem (1)–(2) has at least one solution \( x \in C[0, 1] \).

**Proof.**

Define the subset \( Q_r \subset C(0,1) \) by
\[ Q_r = \left\{ x \in C : |x(t)| \leq r, r = (Ax_0 + Bx_1 + \frac{(3-B)a}{1-(3-B)b}) \right\}. \]

Clear the set \( Q_r \), which is nonempty, closed, and convex.

Let \( F \) be an operator defined by
\[ (Fx)(t) = A \left\{ x_0 - \alpha \int_0^t (t - s)f(s, x(\varphi(s))) \, ds \right\} \
+ B(t - \tau) \left\{ x_1 - \beta \int_0^\eta f(s, x(\varphi(s))) \, ds \right\} \
+ \int_0^\tau (t - s)f(s, x(\varphi(s))) \, ds. \]

Let \( x \in Q_r \), then
\[ |(Fx)(t)| \leq A \left\{ x_0 - \alpha \int_0^t (t - s)|f(s, x(\varphi(s)))| \, ds \right\} \
+ B(t - \tau) \left\{ x_1 - \beta \int_0^\eta |f(s, x(\varphi(s)))| \, ds \right\} \
+ \int_0^\tau (t - s)|f(s, x(\varphi(s)))| \, ds \]
\[ \leq A \left\{ x_0 + \alpha \int_0^t |a(s)| \, ds \right\} + B(t - \tau) \left\{ x_1 + \beta \int_0^\eta |b(s)| \, ds \right\} + \int_0^\tau (t - s)|f(s, x(\varphi(s)))| \, ds \]

\[ \leq A \left\{ x_0 + \alpha \int_0^t |a(s)| \, ds \right\} + B(t - \tau) \left\{ x_1 + \beta \int_0^\eta |b(s)| \, ds \right\} + \int_0^\tau (t - s)|f(s, x(\varphi(s)))| \, ds \]
Then and is uniformly bounded in $0$. Also for $f_{t_1}^t$ such that $x(t)$, we have

$$\begin{aligned}
(Fx)(t_2) - (Fx)(t_1) &= B(t_2 - t_1) \left[ x_1 - \beta \int_0^t f(s, x(\varphi(t))) \, ds \
+ \int_{t_1}^{t_2} (t_2 - s)f(s, x(\varphi(t))) \, ds 
- B(t_1 - t_1) \left[ x_1 - \beta \int_0^t f(s, x(\varphi(t))) \, ds \
- \int_{t_1}^{t_1} (t_2 - t_1)f(s, x(\varphi(t))) \, ds 
+ \int_{t_1}^{t_2} (t_2 - t_1)f(s, x(\varphi(t))) \, ds 
\right] 
\right].
\end{aligned}$$

Then

$$
|(Fx)(t_2) - (Fx)(t_1)| \leq B|t_2 - t_1| \left[ |x_1 + \beta \int_0^t [l(a(s)) + b|x(\varphi(s))]| \right] ds 
+ |t_2 - t_1| \int_0^t [l(a(s)) + b|x(\varphi(s))]| \, ds 
+ \int_{t_1}^{t_2} (t_2 - s)[l(a(s)) + b|x(\varphi(s))]| \, ds 
\leq B|t_2 - t_1| \left[ |x_1 + \beta[l(a) + b] \right] 
+ |t_2 - t_1|[l(a) + b] + \int_{t_1}^{t_2} |a| ds + b[r(t_2 - t_1)].
$$

Then, we obtain

$$
|(Fx)(t_2) - (Fx)(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1.
$$
Thus \( \{FX(t)\} \) is equicontinuous. From Arzela-Ascoli theorem [8], \( \{FX(t)\} \) is relatively compact and \( F:Q_r \rightarrow Q_r \) is compact. Since all conditions of Schauder theorem[11] hold, then \( F \) has a fixed point in \( Q_r \) which proves the existence of at least one solution \( x \in C[0,1] \) of the integral equation (3), where

\[
\lim_{t \to 0^+} x(t) = A \left\{ x_0 - \alpha \int_0^\tau (\tau - s)f(s,x(\varphi(s))) \, ds \right\} - B\tau \left\{ x_1 - \beta \int_0^\eta f(s,x(\varphi(s))) \, ds \right\} = x(0),
\]

\[
\lim_{t \to 1^-} x(t) = A \left\{ x_0 - \alpha \int_0^\tau (\tau - s)f(s,x(\varphi(s))) \, ds \right\} + B(1 - \tau) \left\{ x_1 - \beta \int_0^\eta f(s,x(\varphi(s))) \, ds \right\} + \int_0^1 (1 - s)f(s,x(\varphi(s))) \, ds = x(1).
\]

To complete the proof, we prove that the integral equation (3) satisfies nonlocal problem (1)-(2).

Differentiating (3), we get

\[
x'(t) = B \left\{ x_1 - \beta \int_0^\eta f(s,x(\varphi(s))) \, ds \right\} + \int_0^\tau f(s,x(\varphi(s))) \, ds,
\]

\[
x''(t) = f(t,x(\varphi(t))).
\]  

Let \( t = \tau \) in (3), we obtain

\[
x(t) = A \left\{ x_0 - \alpha \int_0^\tau (\tau - s)f(s,x(\varphi(s))) \, ds \right\} + \int_0^\tau (\tau - s)f(s,x(\varphi(s))) \, ds.
\]

Which proves

\[\alpha x(\tau) = x_0.\]

Also let \( t = \eta \) in (9), we obtain

\[
x'(\eta) = B \left\{ x_1 - \beta \int_0^\eta f(s,x(\varphi(s))) \, ds \right\} + \int_0^\eta f(s,x(\varphi(s))) \, ds
\]

Then

\[
\beta x'(\eta) = B \beta \left\{ x_1 - \beta \int_0^\eta f(s,x(\varphi(s))) \, ds \right\} + \beta \int_0^\eta f(s,x(\varphi(s))) \, ds.
\]

(10)

Let \( t = 0 \) in (9), we obtain

\[
x'(0) = B \left\{ x_1 - \beta \int_0^\eta f(s,x(\varphi(s))) \, ds \right\}.
\]  

(11)
Adding (10) and (11), we obtain

\[ x'(0) + \beta x'(\eta) = x_1 \]

This implies that there exists at least one solution \( x \in C[0,1] \) of the nonlocal problem (1) and (2). This completes the proof.

**Corollary 1.**

The solution of the problem (1) and (2) is monotonic nondecreasing.

**Proof.**

Let \( x \) be a solution of the integral equation (3). Then for \( t_1 < t_2 \), we have

\[
x(t_1) = A \left\{ x_0 - \alpha \int_0^t (t-s)f(s,x(\varphi(s)))\,ds \right\} + B(t_1 - \tau) \left\{ x_1 - \beta \int_0^t f(s,x(\varphi(s)))\,ds \right\} + \int_0^{t_1} (t_1-s)f(s,x(\varphi(s)))\,ds < A \left\{ x_0 - \alpha \int_0^t (t-s)f(s,x(\varphi(s)))\,ds \right\} + B(t_2 - \tau) \left\{ x_1 - \beta \int_0^t f(s,x(\varphi(s)))\,ds \right\} + \int_0^{t_2} (t_2-s)f(s,x(\varphi(s)))\,ds = x(t_2).
\]

Which proves that the solution \( x \) of the problem (1)-(2) is monotonic nondecreasing.

**Corollary 2.**

Let the assumptions (i)-(iv) of Theorem 1 be satisfied. Then the solution of second order functional differential equation (1) with nonlocal condition

\[ \alpha x(t) = x_0, \quad x'(0) = 0. \quad (12) \]

is positive for \( t \in [d,1] \).

**Proof.**

Letting \( \beta = 0 \) and \( x_1 = 0 \) in the integral equation (3) and the nonlocal condition (2), then the solution of the nonlocal problem (1)-(12) will be given by the integral equation

\[
x(t) = A \left\{ x_0 - \alpha \int_0^t (t-s)f(s,x(\varphi(s)))\,ds \right\} + \int_0^t (t-s)f(s,x(\varphi(s)))\,ds.
\]

where \( A = \alpha^{-1} \).
Let $t \in [d, 1]$, then $\tau \leq t$ and
\[
\int_0^\tau (t-s)f(s, x(\varphi(s))) \, ds \leq \int_0^\tau (t-s)f(s, x(\varphi(s))) \, ds,
\]
\[
\alpha \int_0^\tau (t-s)f(s, x(\varphi(s))) \, ds \leq \alpha \int_0^\tau (t-s)f(s, x(\varphi(s))) \, ds.
\]
Multiplying by $A = \alpha^{-1}$, we obtain
\[
A\alpha \int_0^\tau (t-s)f(s, x(\varphi(s))) \, ds \leq A\alpha \int_0^\tau (t-s)f(s, x(\varphi(s))) \, ds
\]
\[
= \int_0^\tau (t-s)f(s, x(\varphi(s))) \, ds.
\]
Which proves that the solution (3) is positive for $t \in [d, 1].$

Example 1.
Consider the nonlocal problem of the second order functional differential equation (1) with nonlocal condition
\[
(13)
\]
Then the nonlocal problem (1) - (13) has at least one monotonic nondecreasing solution $x \in C[0, 1]$ represented by the integral equation
\[
\int_0^\tau (t-s)f(s, x(\varphi(s))) \, ds + \int_0^\tau (t-s)f(s, x(\varphi(s))) \, ds.
\]
This the solution is positive with $\tau < t.$

. **Uniqueness of the solution:**
Consider the problem (1)-(2) under the following assumption
\[
(i^*) f: [0, 1] \times R^+ \rightarrow R^+ \text{ is measurable in } t \in [0, 1] \text{ for all } x \in R^+ \text{ satisfied the Lipschitz condition}
\]
\[
\text{with positive constant } L \text{, such that}
\]
\[
|f(t, x) - f(t, y)| \leq L|x - y| \quad \text{and} \quad |f(t, 0)| = |a(t)|
\]

**Theorem 2.**
Let the assumption (i*) be satisfied. If $(2 + \beta B)L < 1.$ Then the solution of the problem (1)-(2) is a unique.
From assumption (i*) we have
\[
|f(t, x)| - |f(t, 0)| \leq |f(t, x) - f(t, 0)| \leq b|x|
\]
and
\[
|f(t, x)| \leq |f(t, 0)| + b|x| = |a(t)| + b|x|\]
Then the assumption (i) is satisfied, so there exist at least one solution $x \in C[0,1]$ of the problem (1)-(2).

Let $x$ and $y$ be two solutions of the problem (1)-(2), then we have

$$|x(t) - y(t)| = \left| A \left( x_0 - \alpha \int_0^t (t-s)f(s, x(\varphi(s))) \, ds \right) + B(t-t) \left( x_1 - \beta \int_0^\eta f(s, x(\varphi(s))) \, ds \right) + \int_0^t (t-s)f(s, x(\varphi(s))) \, ds \right|$$

$$- A \left( x_0 - \alpha \int_0^t (t-s)f(s, y(\varphi(s))) \, ds \right) - B(t-t) \left( x_1 - \beta \int_0^\eta f(s, y(\varphi(s))) \, ds \right) - \int_0^t (t-s)f(s, y(\varphi(s))) \, ds \right|$$

$$\leq \int_0^\eta \left| f(s, x(\varphi(s))) - f(s, y(\varphi(s))) \right| \, ds + \beta B \int_0^\eta \left| f(s, x(\varphi(s))) - f(s, y(\varphi(s))) \right| \, ds +$$

$$+ \int_0^\eta \left| f(s, x(\varphi(s))) - f(s, y(\varphi(s))) \right| \, ds$$

$$\leq L \int_0^\eta |x(\varphi(s)) - y(\varphi(s))| \, ds$$

$$+ \beta BL \int_0^\eta |x(\varphi(s)) - y(\varphi(s))| \, ds L \int_0^\eta |x(\varphi(s)) - y(\varphi(s))| \, ds$$

$$\leq L \int_0^1 |x(\varphi(s)) - y(\varphi(s))| \, ds$$

$$+ \beta BL \int_0^1 |x(\varphi(s)) - y(\varphi(s))| \, ds L \int_0^1 |x(\varphi(s)) - y(\varphi(s))| \, ds$$

$$\leq L \|x - y\| + \beta BL \|x - y\| + L \|x - y\|$$

Then $\|x - y\| \leq (2 + \beta B)L\|x - y\|.$

And $(1 - (2 + \beta B)L)\|x - y\| \leq 0.$

Since $(2 + \beta B)L < 1$, then $x(t) = y(t).$

Hence the solution of the problem (1)-(2) is unique.
References


