Multi-Point Nonlocal Problem for Linear Second-Order Differential Equations

Howaida. S. M. Salem \ Department of Mathematics, Faculty of Arts and Science, University of Benghazi, Benghazi, Libya \
Multi-Point Nonlocal Problem for Linear Second-Order Differential Equations

Summary:

The paper focuses on the study of problem-related to the linear second order boundary value problem (BVP). Three generalization styles of nonlocal boundary conditions of the Dirichlet type are discussed. In the first, the Non-local conditions at grid points are studied. In the second, the Non-local conditions at non-grid points are treated. In the third, the Non-local conditions at different points within the domain (grid point and non-grid point) are treated. The finite difference representation of the BVP is employed to reduce the problem into a system of algebraic equations, in addition, the interpolation technique to the nonlocal problem (non-grid point) is introduced.

Abstract: This paper is devoted to the numerical treatment of linear second-order ordinary differential equations, with multi-point nonlocal boundary conditions. The finite difference method of the (BVP)s is used as global technique within the overall available domain. A consistent system of algebraic equations corresponding to a standard BVP is generated within the overall domain, \([0,1]\). The solution technique is proposed of boundary conditions at a non-grid point based on the Lagrange interpolation method. Application of the technique is illustrated through a classical model second-order BVP’s. Simple numerical experiments confirming the applicability of the treatment are introduced. Numerical results obtained by present method show that the present method is simple and accurate for second-order multi-point nonlocal BVP’s,

Keywords: Second-order ordinary differential equations, Multi-point nonlocal boundary conditions, Finite difference method, Interpolation technique.
المسألة غير محلية متعددة النقاط للمعادلات التفاضلية الخطية من الدرجة الثانية

د. هويدا سالم منصور

الملخص:

يركز هذا البحث على دراسة المسألة المرتبطة بالمسألة الحدية الخطية من الرتبة الثانية ويناقش البحث ثلاثة أساليب تعميم الشروط الحدية غير المحلية من نوع Dirichlet في الأسلوب الأول تم دراسة الشروط غير المحلية عند نقاط مرتبطة بالشبكة وفي النقلان تم دراسة نفس المسألة مع الشروط غير المحلية عند نقاط غير مرتبطة بالشبكة وفي الثالث تم دراسة المسألة مع الشروط غير المحلية في نقاط مختلفة داخل المجال (نقاط مرتبطة بالشبكة ونقاط غير مرتبطة بالشبكة).

ولم تحل المسألة الحدية إلى نظام من المعادلات الجبرية باستخدام طريقة الفروق المنتهية، بالإضافة إلى تقديم أسلوب الاستيفاء عند حل المسألة الحدية غير المحلية (نقاط غير مرتبطة بالشبكة).

الكلمات المفتاحية: معادلة تفاضلية الخطية من الرتبة الثانية، الشروط الحدية غير محلية متعددة النقاط، طريقة فروق المنتهية، تقنية الاستيفاء.
1. Introduction

Consider the following second-order BVP for ordinary differential equation:

\[ u''(t) + F(t, u(t), u'(t)) = f(t), \quad (1.a) \]

\[ u(0) = \mu, \quad (1.b) \]

\[ u(1) = cu(\xi) + d. \quad (1.c) \]

Where \( f(t), d \) and \( \mu \) are given, \( c \) is number, and \( 0 < \xi < 1 \). The definite feature of this problem is that, instead of an ordinary boundary condition, it contains the nonlocal condition (1.c) is a simplest nonlocal condition of type Bitsadze-Samarskii, which involves the values of the unknown function at the boundary points as well as at interior points of the interval \([0, 1]\).

Problems with such nonlocal boundary conditions (1.b, c) for differential equations were considered in [1,2].

In general, the nonlocal boundary condition (1.c) can be written in a simple form

\[ u(1) = \sum_{i=1}^{m} c_i u(\xi_i) + d \quad (1.d) \]

Where \( 0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1 \) and constants \( c_1, c_2, \ldots, c_m \), and real numbers satisfying the inequalities \(-\infty < c_1 + c_2 + c_3 + \cdots + c_m \leq 1\).

In multi-point boundary value problems, some of the given conditions are given as combination of the unknown function at different points within the domain of the problem.

Linear second-order ordinary differential equations, with multipoint boundary value problems, were established in [2-4].

The existence and uniqueness of solutions to multi-point nonlocal boundary value problem have been investigated by many authors, among them [2,4].

The solutions to multipoint boundary value problems for linear PDEs was presented in [5].

The approximate solution of the linear three-point boundary value problems has been presented in [6]. The solvability of some nonlocal problem of differential equations subject to multi-point nonlocal conditions was presented in [7]. Also, multi-point boundary value
problems for linear and nonlinear equations of the second and higher orders was appeared in [8].

Although, the finite difference is considered as the oldest numerical technique used in the approximation of differential equations it is still the most applicable method due to its simplicity. The philosophy of the finite difference method is the replacement of the continuous domain by a discrete set of grid points as in figure (1) and the replacement of the derivatives appears in the differential equation by a corresponding difference representation at the grid points. Accordingly, a set of algebraic relations are defined at the grid points. Finite difference treatment for linear boundary value problems requires solution of structured large linear systems of algebraic equations.

Our main objective is to introduce the solution of (1. a, b) subject to the nonlocal boundary condition (1. d) by the finite difference method, a numerical example is given in different points within the domain of the problem, to demonstrate the efficiency of the present method. This objective can be achieved by splitting the problem into three tracks. In the first track the included classical second order differential equation with nonlocal boundary condition containing two grid points of the form

\[ u''(t) + p(t)u'(t) + q(t)u(t) = f(t), \quad 0 < t < 1 \]  
\[ u(0) = \mu, \quad u(1) = c_1u(\xi_1) + c_2u(\xi_2) + d, \quad 0 < \xi_1 < \xi_2 < 1, \text{and} \quad -\infty < c_1 + c_2 \leq 1 \]  

(2.a) \hspace{1cm} (2.b)

In the second track the differential equation (1.a) with nonlocal boundary condition containing two non-grid points of the form

\[ u''(t) + p(t)u'(t) + q(t)u(t) = f(t), \quad 0 < t < 1 \]  
\[ u(0) = \mu, \quad u(1) = c_1u(\xi_1) + c_2u(\xi_2) + d, \quad t_{i_1} \leq \xi_1 \leq t_{i_1+1}, \quad t_{i_2} \leq \xi_2 \leq t_{i_2+1}, \text{and} \quad -\infty < c_1 + c_2 \leq 1 \]  

(3.a) \hspace{1cm} (3.b)

In the third track the differential equation (1.a) with nonlocal boundary condition is given as combination of non-grid point and grid point of the form
1.1 The Finite Difference Method, [9,10]

In the finite difference method, the continuous domain [0, 1] is superimposed by a set of discrete points \( P_N = \{t_0, t_1, \ldots, t_N\} \), known as the finite difference grid as shown in figure (1).

![Figure 1 the grid imposed on the interval [0, 1]](image)

\[ u''(t) + p(t)u'(t) + q(t)u(t) = f(t), \quad 0 < t < 1 \quad (4.a) \]
\[ u(0) = \mu, \quad u(1) = c_1u(\xi_1) + c_2u(\xi_2) + d, \quad (4.b) \]
\[ 0 < \xi < 1, \quad t_i \leq \xi_i \leq t_{i+1}, \text{and} \quad -\infty < c_1 + c_2 \leq 1 \]

It is natural to use the notation \( u(t_i) = u_i \), the central difference approximation for the classical first order derivative \( u'(t_i) = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2) \), the central difference approximation for the classical second order derivative \( u''(t_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2) \).

It is generally accepted that every differential equation can be approximated by a corresponding finite difference scheme by replacing the derivative terms by their corresponding finite difference approximation at each grid point. Accordingly, equation (1.a) can be written in the discrete form

\[ \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + P(t_i)\frac{u_{i+1} - u_{i-1}}{2h} = f(t_i), \quad i = 1, \ldots, N - 1 \quad (5) \]

Accordingly, a system of algebraic equations is obtained the solution of the algebraic system gives approximation to the solution of the given boundary value problem.

Thus, the classical two point Dirichlet boundary value problems

\[ u''(t) + p(t)u'(t) - q(t)u(t) = f(t), \quad 0 < t < 1 \quad (6.a) \]
\[ u(0) = r_1, \quad u(1) = r_2, \quad (6.b) \]
is approximated by the algebraic system,

\[(1 - \frac{h}{2}p_i)u_{i-1} - (2 + h^2q_i)u_i + (1 + \frac{h}{2}p_i)u_{i+1} = h^2f_i, \quad i = 1, 2, \ldots, N - 1 \quad (7)\]

It is well known that the finite difference scheme must pass through some tests of consistency, stability and convergence in order to give reliable results [9,10,11].

The finite difference equation (7) is written in matrix form as

\[
\begin{pmatrix}
    s_1 & d_1 \\
    z_1 & \ddots & \ddots \\
    & \ddots & \ddots & d_{N-2} \\
    & & z_{N-1} & s_{N-1}
\end{pmatrix}
\begin{pmatrix}
    u_1 \\
    u_2 \\
    \vdots \\
    u_{N-2} \\
    u_{N-1}
\end{pmatrix}
= \begin{pmatrix}
    h^2f_1 - z_1 p_1 \\
    h^2f_2 \\
    \vdots \\
    h^2f_{N-2} \\
    h^2f_{N-1} - d_{N-1} p_{N-1}
\end{pmatrix}
\]

where

\[z_i = \left(1 - \frac{h}{2}p_i \right), \quad s_i = -(2 + h^2q_i), \quad d_i = \left(1 + \frac{h}{2}p_i \right) \quad \text{for} \quad i = 1, \ldots, N - 1\]

The coefficient square matrix \(A\), is of order \((N - 1) \times (N - 1)\), the unknown vector \(U\) and the right-hand side \(F\) are two \((N - 1)\) columns vectors, the coefficient matrix, \(A = (a_{ij})\) is diagonally dominant, positive definite and tridiagonal.

2. Material and Methods

In nonlocal boundary value problem (NBVP) some of the given boundary conditions (Dirichlet or Neumann) are given at grid point \(\xi\).

![Figure 2](image)

Figure 2 a nonlocal boundary condition in a different positioning’s.

Also, sometimes the given conditions are given at a point which is not a grid point as shown in figure (2), the point \(\xi_l\) lies between the grid points \(t_i\) and \(t_{i+1}\), one has to approximate such conditions by function values at grid points (interpolation techniques).
The interpolation techniques presented in [4], can use to solve boundary value problems (1. a, b) with nonlocal boundary condition (1. d).

We will consider \( h \) as a mesh-step, where \( h \) is less than a half of the smallest segments \([0, \xi_1], [\xi_1, \xi_2], ..., [\xi_m, \xi_1] \), let \( i \) be defined by the condition such as \((i)h \leq \xi_i \leq (i + 1)h\).

Where \( t_i = (i)h \), \( t_{i+1} = (i + 1)h \)

Since \( u(1) = \sum_{i=1}^{m} c_i \), \( u(\xi_i) + d \), we apply Lagrange (Linear) interpolation of the points \((i)h, (i + 1)h\).

\[
\begin{align*}
    u(t) & \approx L_t = y_i t + \frac{t - (i + 1)h}{(i)h - (i + 1)h} + \frac{t - (i)h}{(i + 1)h - (i)h} \\
    & = y_i t - \frac{h}{h} + y_{i+1} \frac{h}{h} \\
    & = y_i \left( (i + 1)h - t \right) + y_{i+1} \left( t - (i)h \right)
\end{align*}
\]

Where \( t = \xi_i \)

\[
\begin{align*}
    u(\xi_i) & \approx y_i \left( \frac{(i + 1)h - \xi_i}{h} \right) + y_{i+1} \left( \frac{\xi_i - (i)h}{h} \right)
\end{align*}
\]

In order to find the solution of the difference equation, we will approximate the boundary conditions (1.b, d), by the following:

\[
\begin{align*}
    y_0 = \mu, \quad L(h)y = \sum_{i=1}^{m} c_i u(\xi_i) - y_N + d = 0 \\
    y_m = \mu, \quad L(h)y = \sum_{i=1}^{m} c_i \left( y_i \left( \frac{(i + 1)h - \xi_i}{h} \right) + y_{i+1} \left( \frac{\xi_i - (i)h}{h} \right) \right) - y_N + d = 0
\end{align*}
\]

Equivalently

\[
\begin{align*}
    y_0 = \mu, \quad y_N = \sum_{i=1}^{m} c_i \left( y_i \left( \frac{(i + 1)h - \xi_i}{h} \right) + y_{i+1} \left( \frac{\xi_i - (i)h}{h} \right) \right) + d
\end{align*}
\]

3. Error Estimates

In order to estimate the accuracy of the obtained solution define the global error estimate and local error estimate as follows
1. \( U = [U_1, U_2, ..., U_n]^T \) denotes the approximate solution generated by some FD scheme with no round-off errors and \( u = [u(t_1), u(t_2), ..., u(t_n)] \) is the exact solution at the grid points \( t_1, t_2, ..., t_n \), then the global error vector is defined as \( E = U - u \).

2. The local truncation error refers to the difference between the original differential equation and its FD approximation at a grid point.

\[ T_i = L_h w(t_i) - L w(i), \quad i = 1, ..., N, \text{ where } w \text{ is a smooth function on } I. \]

4. Numerical Experiments

To illustrate the theoretical results described above simple numerical example is considered, and use the central-difference scheme, which provides second-order accuracy, for the approximation of the solution. The three cases described of nonlocal Dirichlet boundary conditions at (grid point \( \xi \) case1, non-grid point \( \xi_i \) case 2 and the combination of different points within the domain case 3) different step size \( h \) for the grid are considered.

Example: Consider the following differential equation

\[
 u''(t) + 2u'(t) - 3u(t) = 2te^{-\xi}, \quad 0 < t < 1
\]  

Subject to nonlocal Dirichlet boundary condition

case I

\[ u(0) = 0.3, \quad u(1) = -0.3u(0.2) - 0.7u(0.6) + d \]

where \( u(t) = -0.138076e^{-3t} + 0.438076e^{-5t} - 0.5te^{-\xi} \), is the exact solution. Since \( c_1 = -0.3, c_2 = -0.7 \) and \( \xi_1 = 0.2, \xi_2 = 0.6 \) both are grid point, and the nonlocal boundary conditions \( u(\xi_i) = -0.138076e^{-3\xi_i} + 0.438076e^{5\xi_i} - 0.5\xi_i e^{-\xi_i}; 0 < \xi_i \leq 1 \) Since \( d = u(1) + 0.3u(0.2) + 0.7u(0.6) \).

Discretizing equation (10), and using the central-difference scheme on an equidistant grid, we obtain the finite-difference equation

\[
 (1 + h)u_{i+1} - (2 + 3h^2)u_i + (1 - h)u_{i-1} = 2ih^2e^{-ih}, \quad i = 1, ..., N - 1
\]

The last equation of linear system is

\[
 u_N + 0.3u_{N-8} + 0.7u_{N-4} = d
\]

Table 1, represents the results obtained at different step size \( h \) for the grid (\( h = 0.1, \ h = 0.02 \)).
Table 1: the results of exact and approximation solutions at $\xi_1 = 0.2, \xi_2 = 0.6$.

<table>
<thead>
<tr>
<th>$t_1$</th>
<th>Exact</th>
<th>App.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>0.1</td>
<td>0.336618</td>
<td>0.336685</td>
</tr>
<tr>
<td>0.2</td>
<td>0.377416</td>
<td>0.377527</td>
</tr>
<tr>
<td>0.3</td>
<td>0.424081</td>
<td>0.424217</td>
</tr>
<tr>
<td>0.4</td>
<td>0.477881</td>
<td>0.478028</td>
</tr>
<tr>
<td>0.5</td>
<td>0.539824</td>
<td>0.539968</td>
</tr>
<tr>
<td>0.6</td>
<td>0.610759</td>
<td>0.610886</td>
</tr>
<tr>
<td>0.7</td>
<td>0.691464</td>
<td>0.691558</td>
</tr>
<tr>
<td>0.8</td>
<td>0.783699</td>
<td>0.783745</td>
</tr>
<tr>
<td>0.9</td>
<td>0.885257</td>
<td>0.885231</td>
</tr>
<tr>
<td>1.0</td>
<td>1</td>
<td>0.999882</td>
</tr>
</tbody>
</table>

Figure 3: Exact solution and approximate solution, when $h=0.1$
case II

We consider equation (10), with nonlocal Dirichlet boundary condition

\[ u(0) = 0.3, \quad u(1) = u(0.75) - 2u(0.85) + d \]

where \( u(t) = -0.138076e^{-3t} + 0.438076e^t - 0.5te^{-t} \), is the exact solution. Since \( c_1 = 1, c_2 = -2 \) and \( \xi_{i_1} = 0.75, \xi_{i_2} = 0.85 \) both are non-grid point, and the nonlocal boundary conditions \( u(\xi_{i_k}) = -0.138076e^{-3\xi_{i_k}} + 0.438076e^{\xi_{i_k}} - 0.5\xi_{i_k}e^{-\xi_{i_k}} \);

\[ t_{i_1} \leq \xi_{i_2} \leq t_{i_2+1}. \] Since

\[ d = u(1) - u(0.75) + 2u(0.85). \]

Discretizing equation (10), and using the central-difference scheme on an equidistant grid, we obtain the finite-difference equation

\[ (1 + h)u_{i+1} - (2 + 3h^2)u_i + (1 - h)u_{i-1} = 2ih^3e^{-ih}, i = 1, ..., N - 1 \]

The last algebraic equation of linear system is

\[ -0.5u_{N-3} + 0.5u_{N-2} + u_{N-1} + u_N = d \]

Table 2, represents the results obtained at different step size h for the grid (h= 0.1, h =0.02)
Table 2: the results of exact and approximation solutions at $\xi_{10} = 0.75, \xi_{12} = 0.85$.

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>Exact</th>
<th>App.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>0.1</td>
<td>0.336618</td>
<td>0.336586</td>
</tr>
<tr>
<td>0.2</td>
<td>0.377416</td>
<td>0.377345</td>
</tr>
<tr>
<td>0.3</td>
<td>0.424081</td>
<td>0.423961</td>
</tr>
<tr>
<td>0.4</td>
<td>0.477881</td>
<td>0.477705</td>
</tr>
<tr>
<td>0.5</td>
<td>0.539824</td>
<td>0.539581</td>
</tr>
<tr>
<td>0.6</td>
<td>0.610759</td>
<td>0.610437</td>
</tr>
<tr>
<td>0.7</td>
<td>0.691464</td>
<td>0.691045</td>
</tr>
<tr>
<td>0.75</td>
<td>0.735716</td>
<td>0.735716</td>
</tr>
<tr>
<td>0.8</td>
<td>0.782699</td>
<td>0.782168</td>
</tr>
<tr>
<td>0.85</td>
<td>0.832511</td>
<td>0.832511</td>
</tr>
<tr>
<td>0.9</td>
<td>0.885257</td>
<td>0.884583</td>
</tr>
<tr>
<td>1.0</td>
<td>1</td>
<td>0.999158</td>
</tr>
</tbody>
</table>

Figure 5: Exact solution and approximate solution, when $h=0.1$
Figure 6: Exact solution and approximate solution, when h=0.02

**case III**

We consider equation (10), with nonlocal Dirichlet boundary condition

\[ u(0) = 0.3, \quad u(1) = u(0.5) - 2u(0.83) + d \]

where \( u(t) = -0.138076e^{-0.5t} + 0.438076e^{0.5} - 0.5te^{-0.5t} \), is the exact solution. Hence

\[ c_1 = 1, \quad c_2 = -2 \quad \text{and} \quad \xi_1 = 0.5, \quad \xi_{i_2} = 0.83 \]

where \( \xi_1 \) is grid point and \( \xi_{i_2} \) is non-grid point, and the nonlocal boundary conditions

\[ u(\xi_1) = -0.138076e^{-0.5\xi_1} + 0.438076e^{\xi_1} - 0.5\xi_1e^{-0.5\xi_1} ; \quad 0 < \xi_1 \leq 1 \]

\[ u(\xi_{i_2}) = -0.138076e^{-0.5\xi_{i_2}} + 0.438076e^{\xi_{i_2}} - 0.5\xi_{i_2}e^{-0.5\xi_{i_2}} ; \quad t_{i_{n-1}} \leq \xi_{i_2} \leq t_{i_{n+1}} \]

Since

\[ d = u(1) - u(0.5) + 2u(0.83) \]

Discretizing equation (10), and using the central-difference scheme on an equidistant grid, we obtain the finite-difference equation

\[ (1 + h)u_{i+1} - (2 + 3h^2)u_i + (1 - h)u_{i-1} = 2ih^3e^{-ih}, \quad i = 1, \ldots, N - 1 \]

The last algebraic equation of linear system is

\[ -u_{N-1} + 1.4u_{N-2} + 0.6u_{N-1} + u_N = d \]

Table 3, represents the results obtained at different step size h for the grid \( h= 0.1, \ h=0.02 \)
Table 3: the results of exact and approximation solutions at $\xi_1 = 0.5 , \xi_{\infty} = 0.83$.

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>Exact</th>
<th>App.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>0.1</td>
<td>0.336618</td>
<td>0.336541</td>
</tr>
<tr>
<td>0.2</td>
<td>0.377416</td>
<td>0.377262</td>
</tr>
<tr>
<td>0.3</td>
<td>0.424081</td>
<td>0.423845</td>
</tr>
<tr>
<td>0.4</td>
<td>0.477881</td>
<td>0.477559</td>
</tr>
<tr>
<td>0.5</td>
<td>0.539824</td>
<td>0.539406</td>
</tr>
<tr>
<td>0.6</td>
<td>0.610759</td>
<td>0.610233</td>
</tr>
<tr>
<td>0.7</td>
<td>0.691464</td>
<td>0.690812</td>
</tr>
<tr>
<td>0.8</td>
<td>0.782699</td>
<td>0.781901</td>
</tr>
<tr>
<td>0.83</td>
<td>0.812224</td>
<td>0.81224</td>
</tr>
<tr>
<td>0.9</td>
<td>0.885257</td>
<td>0.884289</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>0.998831</td>
</tr>
</tbody>
</table>

Figure 7: Exact solution and approximate solution, when $h=0.1$. 
6. Conclusion

In this paper, the finite difference method is used to study Linear second-order ordinary
differential equations, with multipoint nonlocal conditions. The efficiency of the interpolation
technique appears when the BVP contains nonlocal Dirichlet boundary conditions at non-grid
points, where approximate these conditions by known function values at grid points to
exclude introducing new unknowns. Also, can apply the technique of BVP subject to
nonlocal Dirichlet boundary condition at a non-grid point near the right limit of the domain
$t_n$. The efficient use of the numerical method appears in two forms the first is the accuracy of
the results compared to the exact solutions and the second considers reducing the
computational work needed to obtain solutions. It is worthy to note that this method can be
generalized to higher-order BVPs. All computations are performed by Mathematica 11.0.
References


