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On Canonical Hypergroup and Congruence of Semihypergroup

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On Canonical Hypergroup and Congruence of Semihypergroup

Abstract:

In previous studies on hypergroup and canonical hypergroup, the normal subhypergroup is defined [3]. In this paper, we proved that the hyper sum of two subcanonical hypergroups is normal provided that the first is normal.

We also introduced definition congruence on a semihypergroup [4],and we prove the composition two congruences is congruence on a semihypergroup.

Key Words: semihypergroup, quasihypergroup, hypergroup, canonical hypergroup, and congruence on a semihypergroup.

حول الزمرة الفائقة القانونية وعلاقة التطابق لنصف الزمرة الفائقة

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الملخص:

في دراسات سابقة أجريت على الزمرة الفائقة والزمرة الفائقة القانونية تم تعريف الزمرة الفائقة الجزئية الناظمية [3]، وفي هذه الورقة أثبتنا أن المجموع الفائق لأثنين من الزمر الفائقة القانونية الجزئية يكون ناظمي بشرط أن تكون الأولى ناظمية. كما أننا ذكرنا تعريف العلاقة التطابق على نصف الزمرة الفائقة وأثبتنا أن تركيب علاقتي تطابق هو أيضا علاقة تطابق.

الكلمات المفتاحية: نصف الزمرة الفائقة، شبه الزمرة الفائقة، الزمرة الفائقة، الزمرة الفائقة القانونية، علاقة التطابق على نصف الزمرة الفائقة.

1. Introduction

The concept of hyperstructure, was introduced by Marty in 1934 . Hyperstructures have many applications to other areas of various sciences. Many books and papers have been published related to the applications of hyperstructures in the fields of geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, probabilistic, etc, for example, see [1-3]. Canonical hypergroup as a special kind of hypergroups. The congruence relation was studied for its important role in the study of the quotient for hyper-structures.

2. Preliminaries

In this section, we introduced all definitions and basic properties we require of canonical hypergroups .

Definition 2.1. [1] Let H be a non empty set. The operation $\circ : H \times H \rightarrow P^*(H)$ is called a ***hyperoperation*** and $(H , \circ)$ is called a ***hypergroupoid***, where $P^*(H)$ is the collection of all non empty subsets of H . In this case, for $A , B \subseteq H$,

$$A \circ B = \cup \{a \circ b \mid a \in A, b \in B\}.$$

Definition 2.2.[1] A hypergroupoid $(H , \circ)$ is called a ***semihypergroup*** if

$$(a \circ b) \circ c = a \circ (b \circ c) \ , \ \forall a , b , c \in H \quad \text{(Associtivity)}$$

Which means that :

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v$$

The hypergroupoid $(H , \circ)$ is called ***quasihypergroup*** if

$$a \circ H = H = H \circ a \ , \ \forall a \in H \qquad \text{(Reproduction Axiom)}$$

A hypergroupoid $(H , \circ)$ is called a ***hypergroup*** if it is both a semihypergroup and quasihypergroup .

Example 2.3.[6] Let $H = \{a, b, c\}$ Define the hyperoperation $*$ on H by the following table.

$*$	a	b	c
a	a	$\{a, b\}$	$\{a, c\}$
b	a	b	c
c	a	b	c

Then $(H, *)$ is semihypergroup, but not quasihypergroup .

Definition 2.4.[5] A non-empty subset K of a hypergroup (H, o) is called a ***subhypergroup*** if it is a hypergroup .

i.e for all $a \in K$ we have , $a \circ K = K = K \circ a$

Definition 2.5.[5] We say that a hypergroup H is ***canonical*** if

1) it is commutative ,

2) it has a scalar identity (also called scalar unit), which means that

$$\exists\, e \in H, \forall x \in H,\ x \circ e = e \circ x = x,$$

3) every element has a unique inverse, which means that for all $x \in H$, there exists unique inverse $x^{-1} \in H$, such that

$$e \in x \circ x^{-1} \cap x^{-1} \circ x,$$

4) it is reversible , which means that if $x \in y \circ z$, then there exists inverse y^{-1} of y and z^{-1} of z , such that $z \in y^{-1} \circ x$ and $y \in x \circ z^{-1}$.

Definition 2.6.[3] A non-empty subset N of (H, o) is called a ***canonical subhypergroup*** of H, denoted by $N \leq H$ if it is a canonical hypergroup itself .

Definition 2.7.[3] A canonical subhypergroup N of H is said to be ***normal*** if for all $x \in H$, $x + N - x \subseteq N$.

Remark 2.8.[9] Let H be a canonical hypergroup, and let N be a subcanonical hypergroup of H. We denote the subset $\{ x \in H : x - x \subseteq N\}$ of H by S_N.

Proposition 2.9.[9] Let H be a canonical hypergroup, and let N be a subcanonical hypergroup of H. Then, N is normal if and only if $S_N = H$.

Proof.

Let N be normal. Then, for $x \in H$, $x + 0 - x \subseteq N$. That is, $x \in S_N$. Hence, $S_N = H$. Conversely, if $S_N = H$, then for $x \in H$, we get $x + N - x = x - x + N \subseteq N + N = N$. Thus, N is normal. ∎

Proposition 2.10. Let A, B be subcanonical hypergroups of a canonical hypergroup (H,+) such that A is normal, then the subcanonical hypergroup A + B is also normal.

Proof.

Let $A + B = \bigcup_{\substack{a \in A \\ b \in B}} \{a + b\}$

$\because\ 0 \in A, 0 \in B$ (since A, B are subcanonical Hypergroups of H)

$\Longrightarrow 0 + 0 = 0 \in A + B$

$\therefore A + B \neq \emptyset$.

Let $x, y \in A + B$

$\Longrightarrow \exists\ a, a_1 \in A$ and $b, b_1 \in B$ such that

$x \in a + b,\ y \in a_1 + b_1$

consider $x - y \subseteq (a + b) - (a_1 + b_1) = (a - a_1) + (b - b_1)$

$\because\ a - a_1 \subseteq A,\ b - b_1 \subseteq B$ (since A, B are subcanonical Hypergroups of H)

Thus $x - y \subseteq A + B$

$\therefore$ A + B is a subcanonical Hypergroup of H.

Now to prove A + B is a normal subcanonical Hypergroup of H .

Let x ∈ H , consider $x + (A + B) - x = (x - x) + (A + B)$

Since A is normal

$\Longrightarrow S_A = H \Longrightarrow x \in S_A$ means that $x - x \subseteq A$

$\therefore (x - x) + (A + B) \subseteq A + B$

Thus $x + (A + B) - x \subseteq A + B$

A + B is a normal subcanonical Hypergroup of H. ∎

3. congruence on semihypergroup.

Definition 3.1.[7] Let (H , ·) be a semihypergroup and R be an equivalence relation on H. If A and B are non-empty subsets of H, then we define

$A\bar{R}B$ means that $\forall a \in A$, $\exists b \in B$ such that aRb and

$\forall b \in B$, $\exists a \in A$ such that aRb;

$A\bar{\bar{R}}B$ means that $\forall a \in A$, $\forall b \in B$, we have aRb.

Definition 3.2.[4] An equivalence relation R on H is called

(i) ***congruence*** if

$$aRb \text{ and } cRd \quad \Rightarrow a \cdot c \,\bar{R}\, b \cdot d , \forall\, a,b,c,d \in \mathrm{H}$$

(ii) ***strong congruence*** if

$$aRb \text{ and } cRd \quad \Rightarrow a \cdot c \,\bar{\bar{R}}\, b \cdot d , \forall\, a,b,c,d \in \mathrm{H}$$

Definition 3.3.[4] If R and S are binary relations on $(H,*)$ then R o S, the composition of R and S, is defined as usual by

$$R \circ S = \{(x, y) \mid \exists\, z \in H : (x, z) \in R, (z, y) \in S\}$$

Proposition 3.4. Let R and S be (resp. strong) congruence on a semihypergroup H. Then R o S the composition of R and S is a (resp. strong) congruence on $(H,*)$.

Proof . Let K= $R \circ S = \{(x, y) \mid \exists\, z \in H : (x, z) \in R, (z, y) \in S\}$

Let (x, y) , $(x_1, y_1) \in$ K and $t \in x * x_1$

$\Rightarrow \exists\, z , z_1 \in H$ s.t. $(x, z) \in R, (z, y) \in S$ and $(x_1, z_1) \in R, (z_1, y_1) \in S$ (by Def.3.1)

Since R and S are congruence on H then

$x * x_1 \,\bar{R}\, z * z_1$ and $z * z_1 \,\bar{S}\, y * y_1$

$\Rightarrow \exists\, a \in z * z_1$ s.t. $t R a$ and $\exists\, b \in y * y_1$ s.t. $a S b$, thus $(t, b) \in K$.

Therefore R o S is a strong congruence on H. ■

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