On boundary value problem of implicit arbitrary orders differential equations in reflexive Banach spaces

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Abstract

Applying the technique associated with the fixed point theorem due to O'Regan, we prove the existence of a unique weak solution to the functional integral equation

$$y(t) = I^\beta f(t, \int_0^T G(t,s)y(s) \, ds, y(t)), \quad t \in I = [0,T], \quad \beta = 1 - \alpha.$$ 

In the last section, we study the pseudo and weakly differential solutions to the boundary value problem of implicit arbitrary orders differential equations with integral boundary conditions

$$x'(t) = f(t, x(t), D^\alpha x(t)), \quad 0 < \alpha \leq 1$$

$$x(T) = x(0) + \mu \int_0^T x(s) \, ds, \quad \mu \in R^*.$$ 

in reflexive Banach spaces.

Keywords: Fractional derivative, Pseudo solution, boundary value problem, fractional Pettis integral.
باستخدام شروط حدية تكاملية في فضاء بناء الانعكاسي.

الكلمات المفتاحية: التفاضل الكسري ، حلول زائفة ، مسألة القيمة الحدبة ، تكامل نتيس الكسري.
1 Introduction and Preliminaries

The existence of weak solutions of the integral and differential equations has studied in several papers (see for examples [11]-[22]).

In [18] and [20] the existence of weak solutions in the reflexive Banach space $E$ for the initial value problem of the arbitrary (fractional) orders differential equation

$$x'(t) = f(t, D^a x(t)), x(0) = x_0, t \in [0,1]$$

has been considered. The theory of boundary value problems is one of the most important and useful branches of mathematical analysis. Recently some theories for the boundary value problems of fractional differential equations has been discussed in (see for example [3], [4], [9], [13], [19], [21], [23] and [24]). For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to [7], [12] and the references therein.

Let $E$ be a reflexive Banach space with norm $\| \cdot \|$ and dual $E^*$. Denote by $C_\omega = C[I, E^\omega]$ the Banach space of weakly continuous functions $x : I \to E$ with

$$\|x\|_\omega = \sup_{t \in I} \|x(t)\|_E, t \in I = [0,T].$$

We recall the following definitions. Let $E$ be a Banach space and let $x : I \to E$, then

1. $x(\cdot)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\phi \in E^*$, $\phi(x(\cdot))$ is continuous (measurable) at $t_0$.

2. A function $h : E \to E$ is said to be weakly sequentially continuous if $h$ maps weakly convergent sequences in $E$ to weakly convergent sequences in $E$.

3. $x$ is said to be Pettis integrable on $I$ if and only if there is an element $x_I \in E$ corresponding to each $J \subset I$ such that
\[ \phi(x_j) = \int \phi(x(s)) \, ds \quad \text{for all } \phi \in E^*, \]

where the integral on the right is supposed to exist in the sense of Lebesgue. By definition

\[ \int x(s) \, ds = x_j. \]

**Definition 1.** Let \( x : I \rightarrow E \). The fractional Pettis integral of \( x \) of order \( \alpha > 0 \) is defined by (see [15])

\[ I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \, ds, \quad t > 0. \]

In the above definition the sign "\( \int \)" denotes the Pettis integral.

It is evident that in reflexive Banach spaces, both Pettis integrable functions and weakly continuous (strongly measurable) functions are weakly measurable. Moreover the weakly measurable function \( x(.) \) is Pettis integrable on \( I \) if and only if \( \phi(x(.)) \) is Lebesgue integrable on \( I \); for every \( \phi \in E^* \) ([5]).

For the properties of the fractional Pettis integral in reflexive Banach spaces (see [9],[18],[21]).

Now, we give the definition of the weak derivative of fractional order.

**Definition 2.** Let \( x : I \rightarrow E \) be a weakly differentiable function and \( x' \) is weakly continuous, then the weak derivative of \( x \) of order \( \beta \in (0,1] \) by

\[ D^\beta x(t) = I^1_{-\beta} DX(t) \]

where \( D \) the weakly differential operator.

**Definition 3.** A function \( x(.) \) is said to be pseudo-differentiable on \( I \) to a function \( y(.) \) if for every \( \phi \in E^* \), there exists a null set \( N(\phi) \) (i.e. \( N \) is depending
on $\phi$ and $\text{mes}(N(\phi) = 0)$ such that the real function $t \to \phi(x(t))$ is differentiable a.e. on $I$ and

$$\phi(x'(t)) = \phi(y(t)), \ t \in I \setminus N(\phi).$$

The function $y(.)$ is called a pseudo-derivative of $x(.)$

**Proposition 1.** [17] Let $x(.) : I \to E$ be a weakly measurable function.

(A) If $x(.)$ is Pettis integrable on $I$, then the indefinite Pettis integral

$$y(t) = \int_0^t x(s)ds, \ t \in I,$$

is absolutely continuous on $I$ and $x(.)$ is a pseudo-derivative of $y(.)$.

(B) If $y(.)$ is an absolutely continuous function on $I$ and it has a pseudo-derivative $x(.)$ on $I$, then $x(.)$ is Pettis integrable on $I$ and

$$y(t) = y(0) + \int_0^t x(s)ds, \ t \in I.$$

**Definition 4.** A function $x : I \to E$ is a pseudo solution of the problem (1) if $x \in C_\alpha$ has fractional derivative of order $\alpha \in (0,1]$, $x(T) = x(0) + \mu \int_0^T x(s) ds$ and satisfies

$$\phi(x'(t)) - \phi(f(t, x(t), D^\alpha x(t))) = 0 \ a.e. \ on \ I, \ for \ all \ \phi \in E^*.$$

Also, we have the following Fixed point theorem, due to O’Regan, in reflexive Banach space (see [16]) and some propositions which will be used in the sequel (see [18],[22]).

**Theorem 1. (O’Regan Fixed point theorem)**

Let $E$ be a Banach space and let $Q$ be a nonempty, bounded, closed and convex subset of the space $E$ and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in $E$ for each $t \in I$. Then, $F$ has a fixed point in the set $Q$. 


Proposition 2. A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.

The following result follows directly from the Hahn-Banach theorem.

Proposition 3. Let $E$ be a normed space with $x_0 \neq 0$. Then there exists a $\phi \in E^*$ with $\| \phi \| = 1$ and $\phi(x_0) = \| x_0 \|$.

Here, we studied the existence of solution $x \in C_\omega$ for the boundary value problem of the implicit arbitrary (fractional) orders nonlinear differential equation

$$\begin{cases} 
x'(t) = f(t, x(t), D^\alpha x(t)), \\
0 < \alpha \leq 1, t \in I = [0, T] \\
x(T) = x(0) + \mu \int_0^T x(s) ds, \mu \in \mathbb{R}^*. 
\end{cases}$$

The main results

Consider the boundary value problem (1). Operating by $I^{1-\alpha}$ on both sides we obtain

$$D^\alpha x(t) = I^{1-\alpha} f(t, x(t), D^\alpha x(t)).$$

(2)

Let $D^\alpha x(t) = y(t) \in C_\omega$, then for the existence results on the initial value problem (1) we need the following lemma [4].

Lemma 1. Let $0 < \alpha \leq 1$ and let $y \in C_\omega$ be a given function, then the boundary value problem

$$D^\alpha x(t) = y(t)$$

(3)

$$x(T) = x(0) + \mu \int_0^T x(s) ds, \mu \in \mathbb{R}^*$$

(4)

has a unique solution given by

$$x(t) = \int_0^T G(t, s)y(s) ds$$

(5)

where $G(t, s)$ is the Green's function defined by
Notes that, the function \( t \to \int_0^T G ( t, s ) \, ds \) is continuous on \( I \), and hence is bounded. Let

\[
\bar{G} = \sup \left\{ \int_0^T | G(t,s) | \, ds , \ t \in I \right\}
\]

\[
x(t) = x(0) + I^\alpha y(t) = x(0) + I^\alpha f(t,x(0) + I^\alpha y(t),y(t)), \quad (7)
\]

\[
y(t) = I^\beta f(t, \int_0^T G ( t, s ) y(s ) \, ds , y(t) ), \quad t \in I = [0,T], \ \beta = 1 - \alpha. \quad (8)
\]

So, we have prove the following lemma.

**Lemma 2.** The solution of the boundary value problem \( (1) \), if it exists, then it can be represented by the solution of the nonlinear integral equation \( (8) \), this solution is given by \( (7) \).

**2.1 Functional integral equation**

The integral equation \( (8) \) will be investigated under the assumptions:

i. For each \( t \in I, \ f_t = f(t, , , ) : I \times E \times E \to E \) is weakly-weakly sequentially continuous.

ii. For each \( x, y \in C_\omega, \ f( , x( ), y( ) ) \) is strongly measurable on \( I \).

iii. For any \( r > 0, \ \phi \in E^*, \) there exists the function \( a:I \to E \) is bounded and measurable such that

\[
| \phi(f(t,x,y)) | \leq a(t) + b \| x \|, \ b > 0, \ \forall \ t \in I.
\]

Where \( a = \sup \{ |a(t)| : t \in I \} \).
Definition 5. By a weak solution to (8) we mean a function $y \in \mathcal{C}_\omega$ which satisfies weakly, the integral equation (8). This is equivalent to finding

$$y \in \mathcal{C}_\omega \text{ with }$$

$$\phi(y(t)) = \phi(I^\beta f(t, \int_0^T g(t, s)y(s) \, ds, y(t))), \quad t \in I,$$

for all $\phi \in E^*$.

Theorem 1. Let the assumptions (i)-(iii) be satisfied. If $bG_T \beta < \Gamma(\beta + 1)$, then the nonlinear integral equation (8) has at least one weak solution $y \in \mathcal{C}_\omega$.

Proof. Let $F$ be an operator defined by

$$Fy(t) = I^\beta f(t, \int_0^T g(t, s)y(s) \, ds, y(t))), \quad t \in I = [0, T].$$

For any $y \in \mathcal{C}_\omega$, $I^\alpha y \in \mathcal{C}_\omega$, $f(\cdot, x(0) + I^\alpha y(\cdot), y(\cdot))$ is strongly measurable on $I$ (assumption (ii)). By (Lemma 19 in [21]) $f(\cdot, x(0) + I^\alpha y(\cdot), y(\cdot))$ is Pettis integrable for all $t \in I$, we have that, by (Theorem 8 in [21]) $f(\cdot, x(0) + I^\alpha y(\cdot), y(\cdot))$ is fractionally Pettis integrable for all $t \in I$ and thus the operator $F$ makes sense.

Now, define the subset $Q_r$ of $\mathcal{C}_\omega$ by

$$Q_r = \{y \in \mathcal{C}_\omega : \|y\|_0 \leq r \text{ and }$$

$$\|y(t_2) - y(t_1)\| \leq \left(\frac{a}{\Gamma(\beta + 1)} + \frac{brG}{\Gamma(\beta + 1)}\right)\left[2(t_2 - t_1)^\beta + |t_2^\beta - t_1^\beta|\right]\}.$$
Note that $Q_r$ is nonempty, bounded, closed and convex subset of $C_\omega$.

We shall show that $F$ satisfies the assumptions of O’Regan fixed point theorem \cite{16}.

The operator $F$ maps $Q_r$ into itself. To see this, take $y \in Q_r$, $\| y \|_0 \leq r$; without loss of generality; assume $Fy(t) \neq 0$, then there exists $\phi \in E^*$ with $\| \phi \| = 1$ and $\| Fy(t) \| = \phi(Fy(t))$. By the Assumption (iii), we obtain

\[
\| Fy(t) \| = \phi(Fy(t)) = \phi\left( I^\beta f \left( t, \int_0^T G(t, s)y(s) \, ds, y(t) \right) \right)
= \int_0^T \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} | \phi \left( f \left( s, \int_0^T G(s, \tau)y(\tau) \, d\tau, y(s) \right) \right) | \, ds
\leq \int_0^T \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left( \sup_{s \in I} |a(s)| + b \sup_{s \in I} \left[ \int_0^T |G(s, \tau)| \, d\tau \| y \|_0 \right] \right) \, ds
\leq \frac{aT^\beta}{\Gamma(\beta+1)} + \frac{b\tilde{c}T^\beta}{\Gamma(\beta+1)}.
\]

From the last estimate, we deduce that

\[
r = \left( \frac{aT^\beta}{\Gamma(\beta+1)} \right) \left( 1 - \frac{b\tilde{c}T^\beta}{\Gamma(\beta+1)} \right)^{-1}
\]

Therefore, $\| Fy \|_0 = \sup_{t \in I} \| Fy(t) \| \leq r$.

Now, let $t_1, t_2 \in I$, $t_2 > t_1$, without loss of generality, assume $Fy(t_2) - Fy(t_1) \neq 0$ we get

\[
\| Fy(t_2) - Fy(t_1) \| = \phi(Fy(t_2) - Fy(t_1))
= \phi(I^\beta f(t_2, \int_0^T G(t_2, s)y(s) \, ds, y(t_2)) - I^\beta f(t_1, \int_0^T G(t_1, s)y(s) \, ds, y(t_1)))
\]
\[
\phi\left(\int_0^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} f(s, \int_0^T G(s, \tau) y(\tau) \, d\tau, y(s)) \, ds\right)
- \int_0^{t_1} \frac{(t_1 - s)^{\beta - 1}}{\Gamma(\beta)} f(s, \int_0^T G(s, \tau) y(\tau) \, d\tau, y(s)) \, ds
\]
\[
\leq \phi\left(\int_0^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} f(s, \int_0^T G(s, \tau) y(\tau) \, d\tau, y(s)) \, ds\right)
\]
\[
- \int_0^{t_1} \frac{(t_1 - s)^{\beta - 1}}{\Gamma(\beta)} f(s, \int_0^T G(s, \tau) y(\tau) \, d\tau, y(s)) \, ds
\]
\[
\leq \phi\left(\int_0^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} - \frac{(t_1 - s)^{\beta - 1}}{\Gamma(\beta)} f(s, \int_0^T G(s, \tau) y(\tau) \, d\tau, y(s)) \, ds\right)
\]
\[
+ \int_0^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} f(s, \int_0^T G(s, \tau) y(\tau) \, d\tau, y(s)) \, ds
\]
\[
\leq \phi\left(\int_0^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} - \frac{(t_1 - s)^{\beta - 1}}{\Gamma(\beta)} f(s, \int_0^T G(s, \tau) y(\tau) \, d\tau, y(s)) \, ds\right)
\]
\[
+ \int_0^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} f(s, \int_0^T G(s, \tau) y(\tau) \, d\tau, y(s)) \, ds
\]
\[
\leq \sup_{s \in I} \alpha(s) + \sup_{s \in I} \int_0^T G(s, \tau) \|y\|_0 \, d\tau\]
\[
+ \int_0^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} f(s, \int_0^T G(s, \tau) y(\tau) \, d\tau, y(s)) \, ds
\]
\[
\leq a \left(\int_0^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} - \frac{(t_1 - s)^{\beta - 1}}{\Gamma(\beta)} \right) \, ds
+ \int_0^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} \right) \, ds
\]
\[
+ b \int_0^{t_1} \frac{(t_2 - s)^{\beta - 1}}{\Gamma(\beta)} \right) \, ds
\]
\[
\leq \frac{a}{\Gamma(\beta + 1)} + \frac{b r G}{\Gamma(\beta + 1)} \right) [2(t_2 - t_1)^{\beta} + 1].
\]
Hence

\[ \| Fy(t_2) - Fy(t_1) \| \leq \left( \frac{a}{r(\beta + 1)} + \frac{b\gamma}{r(\beta + 1)} \right) [2(t_2 - t_1)\beta + |t_2^\beta - t_1^\beta|]. \] (9)

Hence \( FQ_r \subset Q_r \). This means that the class \( FQ_r \) is weakly equi-continuous, then by Arzela-Ascoli Theorem the closure of \( FQ_r \) is weakly relatively compact.

To show that the operator \( F \) is weakly sequentially continuous, let \( \{y_n\} \) be sequence in \( Q_r \) converges weakly to \( y \) on \( I \), since \( Q_r \) is closed then \( y \in Q_r \). Fix \( t \in I \) By the Lebesgue Dominated Convergence Theorem (8) we have \( \int_0^T G(t,s)y_n(s) \, ds \) is weakly convergent to \( \int_0^T G(t,s)y(s) \, ds \), since \( f \) is weakly-weakly sequentially continuous, we have \( f(t, \int_0^T G(t,s)y_n(s) \, ds, y_n(t)) \) converges weakly to \( f(t, \int_0^T G(t,s)y(s) \, ds, y(t)) \): hence again the Lebesgue Dominated Convergence Theorem (see assumption (iii)) for Pettis integral yields \( F: Q_r \to Q_r \) is weakly sequentially continuous. Since all conditions of O’Regan fixed point theorem (16) are satisfied, then the operator \( F \) has at least one fixed point \( y \in Q_r \), then the nonlinear integral equation of fractional order (8) has at least one weak solution \( y \in C_\omega \).

2.1.1 Uniqueness of solution

For the uniqueness of the weak solution \( x \in C_\omega \) for the nonlinear integral equation of fractional order (8)

**Theorem 3.** Let the assumptions of Theorem 2 be satisfied and replace the assumption (i) by

(1°) For each \( t \in I \), \( f(t, \ldots) \) satisfies Lipschitz condition

\[ \phi(f(t,x_1,y_1) - f(t,x_2,y_2)) \leq |l[\phi(x_1 - x_2) + \phi(y_1 - y_2)]] \]

with Lipschitz constant \( l \). If \( M \) (Which we will know in (10)) < 1, then the nonlinear integral equation of fractional order (8) has a unique weak solution \( y \in C_\omega \).
Proof. Let \( y_1, y_2 \) be any two weak solutions in \( C_\omega \) for the nonlinear integral equation of fractional order (8), then

\[
\| y_2(t) - y_1(t) \| = \Phi(y_2(t) - y_1(t)) \\
geq \phi(t, \int_0^T G(t,s)y_2(s)\,ds, y_2(t)) \\
- \phi(t, \int_0^T G(t,s)y_1(s)\,ds, y_1(t)) \\
= I^{\beta} \phi(f(t, \int_0^T G(t,s)y_2(s)\,ds, y_2(t))) \\
- f(t, \int_0^T G(t,s)y_1(s)\,ds, y_1(t)) \\
\leq I^{\beta} \left[ \int_0^T G(t,s) \phi(y_2(s) - y_1(s))\,ds \right] \\
+ \phi(y_2(t) - y_1(t)) I^{\beta} \left[ \int_0^T G(t,s) \phi(y_2(s) - y_1(s))\,ds + \phi(y_2(t) - y_1(t)) \right] \\
\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left[ \int_0^T G(t,s) \phi(y_2(s) - y_1(s))\,ds + \phi(y_2(t) - y_1(t)) \right] \,ds \\
\leq \left[ \| y_2 - y_1 \| \frac{T^{\beta}}{\Gamma(\beta+1)} + \frac{T^{\beta}}{\Gamma(\beta+1)} \| y_2 - y_1 \| \right] \\
\leq \left[ \frac{1T^{\beta}}{\Gamma(\beta+1)} + \frac{T^{\beta}}{\Gamma(\beta+1)} \right] \| y_2 - y_1 \|
\]

We choose

\[
M = \frac{1T^{\beta}}{r(\beta+1)} [\tilde{G} + 1] < 1 \tag{10}
\]

(by assumption), therefore

\[
\| y_2 - y_1 \| < \| y_2 - y_1 \|,
\]
which implies that

\[ y_1 = y_2 \]

and the weak solution \( y \in C_\omega \) of (8) is unique.

### 2.2 Boundary Value Problem

Now from Lemma 2 and Theorem 3 we can prove the following corollary for the boundary value problem (1).

**Corollary 1.** Let the assumptions of Theorem 3 be satisfied. Then the problem (1) has a unique weak solution \( x \in C_\alpha \).

#### 2.2.1 Pseudo-solutions

**Theorem 4.** Let the assumption of Theorem 2 be satisfies, then the boundary value problem (1) has a pseudo-solution.

**Proof.** Since \( f(, x(0) + I^a y(., y(.))) \) is Pettis integrable and weakly measurable function, then the solution

\[ x(t) = x_0 + I^1 f(t, \int_0^T G(t, s)y(s) \, ds, y(t)), \quad t \in I \]

of (1) is absolutely continuous ( proposition [17] ). Thus for any \( \phi \in E^* \) we have

\[ \frac{d}{dt} \phi(x(t)) = \phi(f(t, \int_0^T G(t, s)y(s) \, ds, y(t))), \quad a.e \ t \in I. \]

\[ x'(t) = f(t, \int_0^T G(t, s)y(s) \, ds, y(t)) = f(t, x(t), D^a x(t)). \]

#### 2.2.2 Weakly differentiable solutions

**Theorem 5.** Let the assumptions of Theorem 2 be satisfied and replace the assumption (ii) by
(ii) For each \( x, y \in C_\omega \), \( f(t, x(t), y(t)) \) is weakly continuous on \( I \).

Then the boundary value problem (1) has at least one weakly differentiable solution \( x \in C_\omega \).

**Proof.** From lemma 2, the solution of the boundary value problem (1) is given by

\[
 x(t) = x(0) + \int_0^t f(s, y(s)) \, ds,
\]

since \( f \) is weakly continuous on \( I \), then the integral of \( f \) is weakly differentiable with respect to the right end point of the integration interval and its derivative equals the integrand at that point ([14]), therefore \( x(t) \) is weakly differentiable and

\[
 x'(t) = f(t, \int_0^t G(t, s)y(s) \, ds, y(t)) = f(t, x(t), D^\alpha x(t)).
\]
References


