On boundary value problem of implicit arbitrary orders differential equations in reflexive Banach spaces

‡Aziza.A.H. Abd EL-Mwla   &   ◦Ebtisam Ali Haribash
‡Faculty of Science, University of Derna, Derna, Libya
◦Faculty of Science, Al-Mergib University, Al-Khums, Libya

On boundary value problem of implicit arbitrary orders differential equations in reflexive Banach spaces

Abstract

Applying the technique associated with the fixed point theorem due to O'Regan, we prove the existence of a unique weak solution to the functional integral equation

$$y(t) = I^\beta f\left(t, \int_0^T G(t,s)y(s)\,ds, y(t)\right), \ t \in I = [0,T], \ \beta = 1 - \alpha.$$

In the last section, we study the pseudo and weakly diffierential solutions to the boundary value problem of implicit arbitrary orders diffierential equations with integral boundary conditions

$$x'(t) = f\big(t, x(t), D^\alpha x(t)\big), \qquad 0 < \alpha \le 1$$

$$x(T) = x(0) + \mu \int_0^T x(s)\,ds, \qquad \mu \in R^*.$$

in reflexive Banach spaces.

Keywords : Fractional derivative, Pseudo solution, boundary value problem, fractional Pettis integral.

حول مشكلة القيمة الحدية للمعادلات التفاضلية الضمنية من الرتبة التعسفية في فضاءات بناخ

د. عزيزة عبد الحفيظ حسين   &   د. إبتسام علي حريش

الملخص:

نطبق نظرية النقطة الثابتة المشهورة بنظرية اوريجون لنثبت وجود حل ضعيف وحيد للمعادلة التكاملية

$$y(t) = I^\beta f\left(t, \int_0^T G(t,s)y(s)\,ds, y(t)\right), \ t \in I = [0,T], \ \beta = 1 - \alpha.$$

و باستخدام هذا الحل قمنا في القسم الاخير بدراسة الحلول الزائفة و الحلول الضعيفة القابلة للتفاضل لمشكلة القيمة الحدية

$$\begin{cases} x'(t) = f\big(t, x(t), D^\alpha x(t)\big), \\ \quad 0 < \alpha \le 1, t \in I = [0,t] \\ x(T) = x(0) + \mu \int_0^t x(s)\,ds, \mu \in R^*. \end{cases}$$

العدد السابع والخمسون / مارس / 2022

بإستخدام شروط حديه تكاملية في فضاء بناخ الانعكاسي.

الكلمات المفتاحية: التفاضل الكسري ، حلول زائفة ، مسألة القيمة الحدبة ، تكامل بتيس الكسري

1 Introduction and Preliminaries

The existence of weak solutions of the integral and diffierential equations has studied in several papers (see for examples [11]-[22]).

In [18] and [20] the existence of weak solutions in the reflexive Banach space E for the initial value problem of the arbitrary (fractional) orders diffierential equation

$$x'(t) = f\big(t, D^\alpha x(t)\big), x(0) = x_0, t \in [0,1]$$

has been considered. The theory of boundary value problems is one of the most important and useful branches of mathematical analysis. recently some theories for the boundary value problems of fractional differential equations has been discussed in (see for example [3], [4], [9], [13], [19], [21], [23] and [24]). For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to [7], [12] and the references therein.

Let E be a reflexive Banach space with norm $\|.\|$ and dual E^*. Denote by $\mathcal{C}_\omega = C[I, E_\omega]$ the Banach space of weakly continuous functions $x: I \to E$ with

$$\|x\|_0 = \sup \|x(t)\|_E, t \in I = [0,T].$$

We recall the following definitions. Let E be a Banach space and let $x: I \to E$, then

1. $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0.

2. A function $h: E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

3. x is said to be Pettis integrable on I if and only if there is an element $x_J \in E$ corresponding to each $J \subset I$ such that

University of Benghazi
Faculty of Education Almarj

المجلة الليبية العالمية
Global Libyan Journal

جامعة بنغازي
كلية التربية – المرج
ISSN 2518-5845

Global Libyan Journal

المجلة الليبية العالمية

العدد السابع والخمسون / مارس / 2022

$$\phi(x_J) = \int_J \phi(x(s)) \, ds \quad for \; all \;\; \phi \in E^*,$$

where the integral on the right is supposed to exist in the sense of Lebesgue. By definition

$$\int_J x(s) \, ds = x_J.$$

Definition 1. *Let* $x: I \to E$. *The fractional Pettis integral of* x *of order*

$\alpha > 0$ *is defined by (see [15])*

$$I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \, ds, \;\; t > 0.$$

In the above definition the sign $"\int"$ denotes the Pettis integral.

It is evident that in reflexive Banach spaces, both Pettis integrable functions and weakly continuous (strongly measurble) functions are weakly measurable. Moreover the weakly measurable function $x(.)$ is Pettis integrable on I if and only if $\phi(x(.))$ is Lebesgue integrable on I; for every $\phi \in E^*$ ([5]).

For the properties of the fractional Pettis integral in reflexive Banach spaces (see [9],[18],[21]).

Now, we give the definition of the weak derivative of fractional order.

Definition 2. *Let* $x: I \to E$ *be a weakly differentiable function and* x' *is weakly continuous, then the weak derivative of* x *of order* $\beta \in (0,1]$ *by*

$$D^\beta x(t) = I_+^{1-\beta} Dx(t)$$

where D *the weakly differential operator.*

Definition 3. *A function* $x(.)$ *is said to be pseudo-differentiable on* I *to a function* $y(.)$ *if for every* $\phi \in E^*$, *there exists a null set* $N(\phi)$ *(i.e.* N *is depending*

University of Benghazi

Faculty of Education Almarj

Global Libyan Journal

جامعة بنغازي

كلية التربية – المرج

ISSN 2518-5845

المجلة الليبية العالمية

العدد السابع والخمسون / مارس / 2022

on ϕ and $mes(N(\phi) = 0)$) such that the real function $t \rightarrow \phi(x(t))$ is differentiable a.e. on I and

$$\phi(x'(t)) = \phi(y(t)), \ \ t \in I \setminus N(\phi).$$

The function $y(.)$ is called a pseudo-derivative of $x(.)$

Proposition 1. *[17] Let $x(.): I \rightarrow E$ be a weakly measurable function.*

(A) *If $x(.)$ is Pettis integrable on I, then the indefinite Pettis integral*

$$y(t) = \int_0^t x(s)ds, \ \ t \in I,$$

is absolutely continuous on I and $x(.)$ is a pseudo-derivative of $y(.)$.

(B) *If $y(.)$ is an absolutely continuous function on I and it has a pseudo-derivative $x(.)$ on I, then $x(.)$ is Pettis integrable on I and*

$$y(t) = y(0) + \int_0^t x(s)ds, \ \ t \in I.$$

Definition 4. *A function $x: I \rightarrow E$ is a pseudo solution of the problem (1) if $x \in \mathcal{C}_\omega$ has fractional derivative of order $\alpha \in (0,1]$, $x(T) = x(0) + \mu \int_0^T x(s) \, ds$ and satisfies*

$\phi(x'(t)) - \phi(f(t, x(t), D^\alpha x(t))) = 0$ *a.e. on I, for all $\phi \in E^*$.*

Also, we have the following Fixed point theorem, due to O'Regan, in reflexive Banach space (see [16]) and some propositions which will be used in the sequel (see [18],[22]).

Theorem 1. *(O'Regan Fixed point theorem)*

Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of the space E and let $F: Q \rightarrow Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in E for each $\in I$. Then, F has a fixed point in the set Q.

University of Benghazi
Faculty of Education Almarj

المجلة الليبية العالمية
Global Libyan Journal

جامعة بنغازي
كلية التربية – المرج
ISSN 2518-5845

Global Libyan Journal

المجلة الليبية العالمية

العدد السابع والخمسون / مارس / 2022

Proposition 2. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

The following result follows directly from the Hahn-Banach theorem.

Proposition 3. *Let E be a normed space with $x_0 \neq 0$. Then there exists a $\phi \in E^*$ with $\| \phi \| = 1$ and $\phi(x_0) = \| x_0 \|$.*

Here, we studied the existence of solution $x \in \mathcal{C}_\omega$ for the boundary value problem of the implicit arbitrary (fractional) orders nonlinear diffierential equation

$$\begin{cases} x'(t) = f\big(t, x(t), D^\alpha x(t)\big), \\ \quad 0 < \alpha \leq 1, t \in I = [0, T] \\ x(T) = x(0) + \mu \int_0^T x(s)\,ds, \mu \in R^*. \end{cases} \qquad (1)$$

2 The main results

Consider the boundary value problem (1). Operating by $I^{1-\alpha}$ on both sides we obtain

$$D^\alpha x(t) = I^{1-\alpha} f\big(t, x(t), D^\alpha x(t)\big). \qquad (2)$$

Let $D^\alpha x(t) = y(t) \in \mathcal{C}_\omega$, then for the existence results on the initial value problem (1) we need the following lemma [4].

Lemma 1. *Let $0 < \alpha \leq 1$ and let $y \in \mathcal{C}_\omega$ be a given function, then the boundary value problem*

$$D^\alpha x(t) = y(t) \qquad (3)$$

$$x(T) = x(0) + \mu \int_0^T x(s)\,ds, \mu \in R^* \qquad (4)$$

has a unique solution given by

$$x(t) = \int_0^T G(t, s)y(s)\,ds \qquad (5)$$

where $G(t, s)$ is the Green's function defined by

University of Benghazi

Faculty of Education Almarj

Global Libyan Journal

جامعة بنغازي

كلية التربية – المرج

ISSN 2518-5845

المجلة الليبية العالمية

العدد السابع والخمسون / مارس / 2022

$$G(t,s) = \begin{cases} \dfrac{-(T-s)^\alpha + \alpha T(t-s)^{\alpha-1}}{T\Gamma(\alpha+1)} + \dfrac{(T-s)^{\alpha-1}}{T\mu\Gamma(\alpha)} & if \quad 0 \le s < t \\ \dfrac{-(T-s)^\alpha}{T\Gamma(\alpha+1)} + \dfrac{(T-s)^{\alpha-1}}{T\mu\Gamma(\alpha)} & if \quad t \le s < T, \quad t,s \in [0,T]. \end{cases} \tag{6}$$

Notes that, the function $t \to \int_0^T G(t,s)\,ds$ is continuous on I, and hence is bounded. Let

$$\tilde{G} = \sup\{\int_0^T |G(t,s)|\,ds, \ t \in I\}$$

$$x(t) = x(0) + I^\alpha y(t) = x(0) + I^1 f(t, x(0) + I^\alpha y(t), y(t)), \tag{7}$$

$$y(t) = I^\beta f(t, \int_0^T G(t,s)y(s)\,ds, y(t)), \quad t \in I = [0,T], \ \beta = 1-\alpha. \tag{8}$$

So, we have prove the following lemma.

Lemma 2. *The solution of the boundaty value problem (1), if it exists, then it can be represented by the solution of the nonlinear integral equation (8), this solution is given by (7).*

2.1 Functional integral equation

The integral equation (8) will be investigated under the assumptions:

i.   For each $t \in I$, $f_t = f(t,.,.): I \times E \times E \to E$ is weakly-weakly sequentially continuous.

ii. For each $x,y \in \mathcal{C}_\omega$, $f(.,x(.),y(.))$ is strongly measurable on I.

iii. For any $r > 0$, $\phi \in E^*$, there exists the function $a: I \to E$ is bounded and measurable such that

$$|\phi(f(t,x,y))| \le a(t) + b\|x\|, \ b > 0, \ \forall \, t \in I.$$

Where $a = \sup\{|a(t)| : t \in I\}$.

Definition 5. *By a weak solution to (8) we mean a function $y \in \mathcal{C}_\omega$ which satisfies weakly, the integral equation (8). This is equivalent to finding*

$y \in \mathcal{C}_\omega$ *with*

$$\phi(y(t)) = \phi(I^\beta f(t, \int_0^T G(t,s)y(s)\,ds, y(t))),\ t \in I,$$

for all $\phi \in E^*$.

Theorem 1. *Let the assumptions (i)-(iii) be satisfied, If $b\tilde{G}T^\beta < \Gamma(\beta+1)$, then the nonlinear integral equation (8) has at least one weak solution $y \in \mathcal{C}_\omega$.*

Proof. Let F be an operator defined by

$$Fy(t) = I^\beta f(t, \int_0^T G(t,s)y(s)\,ds, y(t)),\ t \in I = [0,T].$$

For any $y \in \mathcal{C}_\omega$, $I^\alpha y \in \mathcal{C}_\omega$, $f(., x(0) + I^\alpha y(.), y(.))$ is strongly measurable on I (assumption (ii)). By (Lemma 19 in [21]) $f(., x(0) + I^\alpha y(.), y(.))$ is Pettis integrable for all $t \in I$, we have that, by (Theorem 8 in [21]) , $f(., x(0) + I^\alpha y(.), y(.))$ is fractionally Pettis integrable for all $t \in I$ and thus the operator F makes sense.

Now, define the subset Q_r of $\mathcal{C}_\omega$ by

$Q_r = \{y \in \mathcal{C}_\omega : \| y \|_0 \le r$ and

$$\| y(t_2) - y(t_1) \| \le (\frac{a}{\Gamma(\beta+1)} + \frac{br\tilde{G}}{\Gamma(\beta+1)})[2(t_2 - t_1)^\beta + | t_2^\beta - t_1^\beta |]$$

University of Benghazi
Faculty of Education Almarj

جامعة بنغازي
كلية التربية – المرج
ISSN 2518-5845

Global Libyan Journal

المجلة الليبية العالمية

العدد السابع والخمسون / مارس / 2022

Note that Q_r is nonempty, bounded, closed and convex subset of $\mathcal{C}_\omega$.

We shall show that F satisfies the assumptions of O'Regan fixed point theorem [16].

The operator F maps Q_r into itself. To see this, take $y \in Q_r$, $\| y \|_0 \leq r$; without loss of

generality; assume $Fy(t) \neq 0$, then there exists $\phi \in E^*$

with $\| \phi \| = 1$ and $\| Fy(t) \| = \phi(Fy(t))$. By the Assumption (iii), we obtain

$$\| Fy(t) \| = \phi(Fy(t)) = \phi\left(I^\beta f\left(t, \int_0^T G(t,s)y(s)\, ds, y(t)\right)\right)$$

$$= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mid \phi\left(f\left(s, \int_0^T G(s,\tau)y(\tau)\, d\tau, y(s)\right)\right) \mid ds$$

$$\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(\sup_{s\in I}|a(s)| + b[\sup_{s\in I}\int_0^T |G(s,\tau)| \| y \|_0 \, d\tau]\right) ds$$

$$\leq \frac{aT^\beta}{\Gamma(\beta+1)} + \frac{br\tilde{G}T^\beta}{\Gamma(\beta+1)}.$$

From the last estimate, we deduce that

$$r = \left(\frac{aT^\beta}{\Gamma(\beta+1)}\right)\left(1 - \frac{b\tilde{G}T^\beta}{\Gamma(\beta+1)}\right)^{-1}$$

Therefore, $\| Fy \|_0 = \sup_{t\in I} \| Fy(t) \| \leq r$.

Now, let $t_1, t_2 \in I$, $t_2 > t_1$, without loss of generality, assume
$Fy(t_2) - Fy(t_1) \neq 0$ we get

$$\| Fy(t_2) - Fy(t_1) \| = \phi(Fy(t_2) - Fy(t_1)).$$

$$= \phi(I^\beta f(t_2, \int_0^T G(t_2,s)y(s)\, ds, y(t_2)) - I^\beta f(t_1, \int_0^T G(t_1,s)y(s)\, ds, y(t_1)))$$

University of Benghazi
Faculty of Education Almarj

جامعة بنغازي
كلية التربية – المرج
ISSN 2518-5845

Global Libyan Journal

المجلة الليبية العالمية

العدد السابع والخمسون / مارس / 2022

$$= \phi\left(\int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} f\left(s, \int_0^T G(s,\tau)y(\tau)\,d\tau, y(s)\right) ds\right.$$

$$\left. - \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} f\left(s, \int_0^T G(s,\tau)y(\tau)\,d\tau, y(s)\right) ds\right)$$

$$\leq \left| \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \phi\left(f\left(s, \int_0^T G(s,\tau)y(\tau)\,d\tau, y(s)\right)\right) ds\right.$$

$$\left. - \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \phi\left(f\left(s, \int_0^T G(s,\tau)y(\tau)\,d\tau, y(s)\right)\right) ds \right|$$

$$\leq \left| \int_0^{t_1} \left(\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\right) \phi\left(f\left(s, \int_0^T G(s,\tau)y(\tau)\,d\tau, y(s)\right)\right) ds \right|$$

$$+ \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \phi\left(f\left(s, \int_0^T G(s,\tau)y(\tau)\,d\tau, y(s)\right)\right) ds \right|.$$

$$\leq \int_0^{t_1} \left| \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \right|$$

$$\left| \phi\left(f\left(s, \int_0^T G(s,\tau)y(\tau)\,d\tau, y(s)\right)\right) \right| ds$$

$$+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \left| \phi\left(f\left(s, \int_0^T G(s,\tau)y(\tau)\,d\tau, y(s)\right)\right) \right| ds.$$

$$\leq \int_0^{t_1} \left| \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \right| \left(\sup_{s\in I}|a(s)| + b\left[\sup_{s\in I}\int_0^T |G(s,\tau)| \, \| y \|_0 \, d\tau\right]\right) ds$$

$$+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \left(\sup_{s\in I}|a(s)| + b\left[\sup_{s\in I}\int_0^T |G(s,\tau)| \, \| y \|_0 \, d\tau\right]\right) ds.$$

$$\leq a\left(\int_0^{t_1} \left| \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \right| ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \, ds \right)$$

$$+ br\tilde{G}\left(\int_0^{t_1} \left| \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \right| ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \, ds \right)$$

$$\leq \left(\frac{a}{\Gamma(\beta+1)} + \frac{br\tilde{G}}{\Gamma(\beta+1)} \right) [2(t_2-t_1)^{\beta} + | t_2^{\beta} - t_1^{\beta} |].$$

University of Benghazi
Faculty of Education Almarj

Global Libyan Journal

جامعة بنغازي
كلية التربية – المرج
ISSN 2518-5845

المجلة الليبية العالمية

العدد السابع والخمسون / مارس / 2022

Hence

$$\| Fy(t_2) - Fy(t_1) \| \le (\frac{a}{\Gamma(\beta+1)} + \frac{br\ddot{G}}{\Gamma(\beta+1)})[2(t_2 - t_1)^\beta + | t_2^\beta - t_1^\beta |]. \qquad (9)$$

Hence $FQ_r \subset Q_r$. This means that the class FQ_r is weakly equi-continuous, then by Arzela-Ascoli Theorem the closure of FQ_r is weakly relatively compact. To show that the operator F is weakly sequentially continuous, let $\{y_n\}$ be sequence in Q_r converges weakly to y on I, since Q_r is closed then $y \in Q_r$. Fix $t \in I$ By the Lebesgue Dominated Convergence Theorem ([8]) we have $\int_0^T G(t,s)y_n(s)\, ds$ is weakly convergent to $\int_0^T G(t,s)y(s)\, ds$, since f is weakly-weakly sequentially continuous, we have $f(t, \int_0^T G(t,s)y_n(s)\, ds, y_n(t)))$ converges weakly to $f(t, \int_0^T G(t,s)y(s)\, ds, y(t)))$; hence again the Lebesgue Dominated Convergence Theorem (see assumption (iii)) for Pettis integral yields $F: Q_r \to Q_r$ is weakly sequentially continuous. Since all conditions of O'Regan fixed point theorem ([16]) are satisfied, then the operator F has at least one fixed point $y \in Q_r$, then the nonlinear integral equation of fractional order (8) has at least one weak solution $y \in \mathcal{C}_\omega$. ∎

2.1.1 Uniqueness of solution

For the uniqueness of the weak solution $x \in \mathcal{C}_\omega$ for the nonlinear integral equation of fractional order (8)

Theorem 3. *Let the assumptions of Theorem 2 be satisfied and replace the assumption (i) by*

(1°) *For each $t \in I$, $f(t,.,.)$ satisfies Lipschitz condition*

$$\phi(f(t,x_1,y_1) - f(t,x_2,y_2)) \le l[\phi(x_1 - x_2) + \phi(y_1 - y_2)]$$

with Lipschitz constant l. If M (Which we will know in (10))< 1, then the nonlinear integral equation of fractional order (8) has a unique weak solution $y \in \mathcal{C}_\omega$.

Proof. let y_1, y_2 be any two weak solutions in C_ω for the nonlinear integral equation of fractional order (8), then

$$\| y_2(t) - y_1(t) \| = \emptyset(y_2(t) - y_1(t))$$

$$= \phi\left(I^\beta f\left(t, \int_0^T G(t,s)y_2(s)\, ds, y_2(t)\right)\right.$$

$$\left. - I^\beta f\left(t, \int_0^T G(t,s)y_1(s)\, ds, y_1(s)ds, y_1(t)\right)\right)$$

$$= I^\beta \phi\left(f\left(t, \int_0^T G(t,s)y_2(s)\, ds, y_2(t)\right)\right.$$

$$\left. - f\left(t, \int_0^T G(t,s)y_1(s)\, ds, y_1(t)\right)\right)$$

$$\leq l I^\beta \left[\int_0^T G(t,s)\phi(y_2(s) - y_1(s))\, ds\right.$$

$$+ \phi(y_2(t) - y_1(t)) l I^\beta \left[\int_0^T G(t,s)\phi(y_2(s) - y_1(s))\, ds + \phi(y_2(t) - y_1(t))\right]$$

$$\leq I^\beta \left[\tilde{G} \sup_{t \in I} \| y_2(t) - y_1(t) \| + \sup_{t \in I} \| y_2(t) - y_1(t) \|\right]$$

$$\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left[\tilde{G} \| y_2 - y_1 \| + \| y_2 - y_1 \|\right] ds$$

$$\leq \left[\| y_2 - y_1 \| \frac{T^\beta \tilde{G}}{\Gamma(\beta+1)} + \frac{T^\beta}{\Gamma(\beta+1)} \| y_2 - y_1 \|\right]$$

$$\leq \left[\frac{l T^\beta \tilde{G}}{\Gamma(\beta+1)} + \frac{l T^\beta}{\Gamma(\beta+1)}\right] \| y_2 - y_1 \|]$$

We choose

$$M = \frac{l T^\beta}{\Gamma(\beta+1)} [\tilde{G} + 1] < 1 \qquad (10)$$

(by assumption), therefore

$$\| y_2 - y_1 \| < \| y_2 - y_1 \|,$$

University of Benghazi
Faculty of Education Almarj

Global Libyan Journal

جامعة بنغازي
كلية التربية – المرج
ISSN 2518-5845

المجلة الليبية العالمية

العدد السابع والخمسون / مارس / 2022

which implies that

$$y_1 = y_2$$

and the weak solution $y \in \mathcal{C}_\omega$ of (8) is an unique. ∎

2.2 Boundary Value Problem

Now from Lemma 2 and Theorem 3 we can prove the following corollary for the boundary value problem (1).

Corollary 1. *Let the assumptions of Theorem 3 be satisfied. Then the problem (1) has a unique weak solution $x \in \mathcal{C}_\omega$.*

2.2.1 Pseudo-solutions

Theorem 4. *Let the assumption of Theorem 2 be satisfies, then the boundary value problem (1) has a pseudo-solution.*

Proof. Since $f(., x(0) + I^\alpha y(.), y(.))$ is Pettis integrable and weakly measurable function, then the solution

$$x(t) = x_0 + I^1 f(t, \int_0^T G(t,s) y(s) \, ds, y(t)), \ t \in I$$

of (1) is absolutely continuous (proposition [17]). Thus for any $\phi \in E^*$ we have

$$\frac{d}{dt} \phi(x(t)) = \phi(f(t, \int_0^T G(t,s) y(s) \, ds, y(t))), \ a.e \ t \in I.$$

$$x'(t) = f(t, \int_0^T G(t,s) y(s) \, ds, y(t)) = f(t, x(t), D^\alpha x(t)).$$

2.2.2 Weakly differentiable solutions

Theorem 5. *Let the assumptions of Theorem 2 be satisfied and replace the assumption (ii) by*

العدد السابع والخمسون / مارس / 2022

(ii)* For each $x, y \in C_\omega$, $f(., x(.), y(.))$ is weakly continuous on I.

Then the boundary value problem (1) has at least one weakly differentiable solution $x \in C_\omega$.

Proof. From lemma 2, the solution of the boundary value problem (1) is given by

$$x(t) = x(0) + I^\alpha y(t) = x(0) + I^1 f(t, \int_0^T G\,(t,s)y(s)\,ds, y(t)),$$

since f is weakly continuous on I, then the integral of f is weakly differentiable with respect to the right end point of the integration interval and its derivative equals the integrand at that point ([14]), therefore $x(.)$ is weakly differentiable and

$$x'(t) = f(t, \int_0^T G\,(t,s)y(s)\,ds, y(t)) = f(t, x(t), D^\alpha x(t)).$$

References

[1] R.P. Agarwal, M. Benchohra, S. Hamani, boundary value problems for fractional differential equations, *Georgian Mathematical Journal*, 16(2009) 401-411.

[2] M. Benchohra, J.R. Graef, F.Z. Mostefai, Weak solutions for nonlinear fractional differential equations on reflexive Banach spaces, *Electron. J. Qual. Theory Differ. Equ*, 54(2010) 1-10.

[3] M. Benchohra, J.R. Graef, F.Z. Mostefai, Weak Solutions for Boundary Value Problems with Nonlinear Fractional Diffierential Inclusions, *Nonlinear Dynamics and Systems Theory*, 11(3)(2011) 227-237.

[4] M. Benchohra, F.Z. Mostefai , weak solutions for nonlinear fractional differential equations with integral boundary conditions in Banach spaces, , *Opuscula Mathematica*, 32(2012).

[5] J. Diestel, J.J. Uhl, Jr, Vector Measures, Math. Surveys 15, *Amer. Math. Soc., Providence, R.I.*, (1977).

[6] N. Dinculeanu, On Kolmogorov-Tamarkin and M. Riesz compactness criteria in function spaces over a locally compact group, *Journal of Mathematical Analysis and Applications*, 89(1)(1982) 67-85.

[7] J. M. Gallardo, Second-order di_erential operators with integral boundary conditions and generation of analytic semigroups, *The Rocky Mountain Journal of Mathematics,* 30(4)(2000) 12651291.

[8] R.F. Geitz, Pettis integration, *Proc. Amer. Math. Soc,* 82(1981) 81-86.

[9] A.M.A. El-Sayed, Sh.A. Abd El-Salam, Weak solutions of a fractional-order nonlocal boundary value problem in reflexive Banach spaces, *Differential Equation and Control Processes*, 4(2008).

[10] E. Hille, R.S. Phillips, Functional Analysis and Semi-groups, *Amer. Math. Soc. Colloq. Publ.* vol. 31, Amer. Math. Soc., Providence, R.I, (1957).

[11] I. Kubiaczyk, On fixed point theorem for weakly sequentially continuous mappings, *Discuss. Math. Differ. Incl*. 15(1995) 15-20.

[12] G.L. Karakostas and P.C. Tsamatos, Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems, *Electronic Journal of Diffierential Equations*, 30(2002) 117.

[13] Y. Khouni, weak solutions of first-order diffierential inclusions in Banach space, *Journal of Engineering Technology and Applied Sciences*, 1(2016) 1-11.

[14] A.R. Mitchell, CH. Smith, An existence theorem for weak solutions of differential equations in Banach spaces, in: *Nonlinear Equations in Abstract Spaces, V. Lakshmikantham (ed.)*, (1978) 387-403.

[15]   K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, *John Wiley, New York* (1993).

[16] D. O'Regan, Weak solutions of ordinary differential equations in Banach spaces, *Appl. Math. Lett*, 12(1999) 101-105.

[17] B.J. Pettis, On integration in vector spaces, *Trans. Amer. Math. Soc*. 44(1938) 277-304.

[18] H.A.H. Salem, A.M.A. El-Sayed, Weak solution for fractional order integral equations in reflexive Banach spaces, *Math. Slovaca*. 55(2005) 169-181.

[19] H.A.H. Salem, On the nonlinear Hammerstein integral equations in Banach spaces and application to the boundary value problem of fractional order, *Mathematical and Computer Modelling*, 48(2008) 1178-1190.

[20]   H.A.H. Salem, A.M.A. El-Sayed, A note on the fractional calculus in Banach spaces, *Studia Scientiarum Mathematicarum Hungarica*. 42(2)(2005) 115-130.

[21]   H.A.H. Salem, M. Cichoń, On Solutions of Fractional Order Boundary Value Problems with Integral Boundary Conditions in Banach Spaces, *Journal of Function Spaces and*

University of Benghazi
Faculty of Education Almarj

Global Libyan Journal

جامعة بنغازي

كلية التربية – المرج

ISSN 2518-5845

المجلة الليبية العالمية

العدد السابع والخمسون / مارس / 2022

Applications. vol. 2013(2013) Article ID 428094, 13 pages (http://dx.doi.org/10.1155/2013/428094).

[22]   A. Szep, Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces, *Studia Sci. Math. Hungar*. 6(1971) 197-203.

[23] S. Weng, F. Wang, Existence of weak solutions for a boundary value problem of a second order ordinary diffierential equation, *Boundary Value Problems,* 9(2018)(https://doi.org/10.1186/s13661-018-0929-7).

[24]   S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional diffrential equations, *Electron. J. Differential Equations,* 36(2006) 1-12.