Using Frechet space to solve non linear functional integral equations

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Abstract

In this paper we have used Frechet space to study a unique solution on a semi-infinite integral equation by applying a nonlinear alternative of the Leray-Schauder type for contraction maps.

Keywords: quadratic integral equation, existence and uniqueness, fixed point, Leray-Schauder, Frechet space.
Introduction

Integral equations have a wide range applications in many fields of science, like physics, engineering economics and many other fields. (1,2,3,4,5)

Various techniques have been devised by many researchers to solve the integral equations. Fixed-point theorem is an effective tool and it has seen used as a powerful method for demonstrating the existence and uniqueness of solution to many types of nonlinear integral equations.(6,7,8,9)

In this paper we have used Leray-schauder alternatives to show the existence of a unique solution for nonlinear functional integral equations.(10)

\[ x(t) = c(t, s) + A(x(t)) \left( \int_{0}^{t} f(t, s, x(s))ds \int_{0}^{t} v(t, s, x(s))ds \right), \quad t \in I = [0, +\infty) \]

which defined on a semi-finite interval \( I = [0, +\infty) \),

where \( G: I \times I \rightarrow R, f: I \times [0, T] \times R \rightarrow R \) and \( v: I \times [0, T] \times R \rightarrow R \) are given functions and \( A: C(I, R) \rightarrow C(I, R) \) is an appropriate operator, here \( C(I, R) \) denotes the space of continuous functions \( x: I \rightarrow R \).

Enchohra and Darwish (11) studied the existence of a unique solution for the nonlinear quadratic equation of Urysohn type.

\[ x(t) = f(t) + (Ax)(t)\int_{0}^{t} u(t, s, x(s))ds, \quad t \in [0, +\infty) \]

Also, in Sadoon(12),authors showed the existence of a unique solution for the nonlinear quadratic integral equation of Fredholm - Volterra integral equation

\[ x(t) = f(t) + (Ax)(t)\left[ \int_{0}^{T} u(t, s, x(s)) \right] + \int_{0}^{T} g(t, s, x(s))ds, \quad t \in [0, +\infty) \]

It is clear that equation (1) is more comprehensive than equations 2 and 3. The required conditions to achieve the desired result have been established.

Section 2 of this paper represents the hypotheses that will be used in the later section. The results are explained in section 3.
Materials and methods

This section begins by introducing some notations, definitions, and theorems that will be used throughout the rest of this paper. Let \( X \) be a Frechet space with a family of semi-norms \( \{ \| \cdot \|_n \}_{n \in \mathbb{N}} \). Let \( Y \subset X \), we say that \( Y \) is bounded if for every \( n \in \mathbb{N} \), there exists \( M_n > 0 \) such that \( \| y \|_n \leq M_n \) for all \( y \in Y \).

**Theorem 1** [Enochora and Darwish\(^1\)]: Let \( \Omega \) be a closed subset of a Frechet space \( X \) such that \( 0 \in \Omega \) and \( F: \Omega \to X \) is a contraction such that \( F(\Omega) \) is bounded. Then either

1. \( F \) has a unique fixed point
2. There exists \( \lambda \in (0,1), n \in \mathbb{N} \) and \( u \in \partial \Omega^n \) such that \( \| u - \lambda F(u) \|_n = 0 \).

Where \( \partial \Omega^n \) is boundary of \( \Omega^n \).

Results and discussion

In this section, we assume that the following assumptions are satisfied:

a) \( G: I \times I \to \mathbb{R} \) is a continuous function, with \( B_n = \sup \{ E(t, s) \mid (t, s) \in I \times I \} \)

b) For each \( n \in \mathbb{N}, U_n > 0 \) s.t. \( |(Ax)(t) - (Ay)(t)| \leq U_n |x(t) - y(t)| \) for each \( x, y \in C(I, R) \) and \( t \in [0, n] \).

c) There exist nonnegative constants \( a, b \) such that \( |(Ax)(t)| \leq a + b |x(t)| \) for each \( x \in C(I, R) \) and \( t \in I \).

d) \( f: I \times [0, T] \times R \to R \) is continuous function and for each \( n \in \mathbb{N} \) there exist a constant \( L_n > 0 \) such that:

\[
f(t, s, x) - f(t, s, y) \leq L_n |x - y| \quad \text{for all } (t, s) \in [0, T] \text{ and } x, y \in R
\]

e) There exist a continuous nondecreasing function \( \psi: I \to (0, \infty) \) and \( p \in C(I, R^+) \) such that \( |f(t, s, x)| \leq p(s) \psi(|x|) \) for each \( (t, s) \in I \times [0, T] \), \( x \in R \)

f) \( \nu: I \times [0, T] \to R \) is continuous function and for each \( n \in \mathbb{N} \) there exist a constant \( H_n > 0 \) such that:

\[
\nu(t, s, x) - \nu(t, s, y) \leq H_n |x - y| \quad \text{for all } (t, s) \in [0, T] \text{ and } x, y \in R
\]

g) There exist a continuous nondecreasing function \( \delta: I \to (0, \infty) \) with:

\( q \in C(I, R^+) \) such that \( |\nu(t, s, x)| \leq q(s) \delta(|x|) \) for each \( (t, s) \in I \times [0, T], x \in R \)

h) moreover there exists constants \( M_n, n \in \mathbb{N} \) such that:
Theorem 2

Suppose that hypotheses (a-i) are satisfied. If

\[ T([\alpha + bM_n[b]]p'(s)\theta(M_n)H_n' + L_n') + q'(s)\delta(M_n)p'(s)\theta(M_n) \geq 1 \]  

Then Eq (1) has a unique solution.

Proof: For every \( n \in \mathbb{N} \), we define in \( C(I,\mathbb{R}) \) the semi-norms by

\[ \| y \|_n = \sup \{|y(t)|: t \in [0,n]\} \]

then \( C(I,\mathbb{R}) \) is a Frechet space with the family of semi-norms \( \{\|\|_n\}_{n \in \mathbb{N}} \) see Enchohra and Darwish11

Transform the Eq (1) into a fixed point problem. Consider the operator

\[ F: C(I,\mathbb{R}) \to C(I,\mathbb{R}) \]

defined by

\[ (F\varphi)(t) = G(t,s) + A(\varphi(t)) \left( \int_0^T f(t,s,\varphi(s))ds \int_0^T \nu(t,s,\varphi(s))dg \right), t \in I \]

Let \( \varphi \) be a possible solution of the Eq (1). Given \( n \in \mathbb{N} \), then with the view of (i),(d),(f) and (h) we have:

\[ |\varphi(t)| \leq |G(t,s)| + |A(\varphi(t))| \left( \int_0^T |f(t,s,\varphi(s))|ds \int_0^T |\nu(t,s,\varphi(s))|dg \right) \]

\[ \leq B_n + (\alpha + b|\varphi(t)|) \left( \int_0^T p'(s)\theta(|\varphi(s)|)ds \int_0^T q(s)\delta(|\varphi(s)|)ds \right) \]

\[ \|\varphi\|_n \leq B_n + (\alpha + b\|\varphi\|_n)\|p\|_n^2\|q\|_n^2(\theta(\|\varphi\|_n)\delta(\|\varphi\|_n), \|\varphi\|_n)) \]

then

\[ \|\varphi\|_n \leq \frac{B_n + (\alpha + b\|\varphi\|_n)\|p\|_n^2\|q\|_n^2(\theta(\|\varphi\|_n)\delta(\|\varphi\|_n))}{B_n + (\alpha + b\|\varphi\|_n)\|p\|_n^2\|q\|_n^2(\theta(\|\varphi\|_n)\delta(\|\varphi\|_n))} \leq 1 \]

From (3) it follows that for each \( n \in \mathbb{N} \), \( \|\varphi\|_n \neq M_n \)

Now, set

\[ \Omega = \{\varphi \in C(I,\mathbb{R}): \|\varphi\|_n \leq M_n \text{ for all } n \in \mathbb{N}\} \]

Clearly, \( \Omega \) is a closed subset of \( C(I,\mathbb{R}) \), we shall show that \( F: \Omega \to C(I,\mathbb{R}) \) is a contraction operator,

consider \( \varphi, \psi \in \Omega \) for each \( t \in [0,n] \) and \( n \in \mathbb{N} \) from (b)-(i) we have:

Let \( \varphi, \psi \in \Omega \), then
By adding and subtracting \( A(\varphi(t)) \left( \int_0^T f(t, s, \varphi(s)) \, ds \int_0^T v(t, g, \psi(g)) \, dg \right) \) we get

\[
F\varphi(t) - F\psi(t) = A(\varphi(t)) \left( \int_0^T f(t, s, \varphi(s)) \, ds \int_0^T v(t, g, \psi(g)) \, dg \right)
- A(\psi(t)) \left( \int_0^T f(t, s, \psi(s)) \, ds \int_0^T v(t, g, \psi(g)) \, dg \right)
\]

Furthermore, by adding and subtracting \( A(\psi(t)) \int_0^T f(t, s, \varphi(s)) \, ds \) we get

\[
= A(\varphi(t)) \left[ \int_0^T f(t, s, \varphi(s)) \, ds \int_0^T v(t, g, \varphi(g)) - v(t, g, \psi(g)) \right] \, dg
+ \int_0^T v(t, g, \varphi(g)) \, dg \left[ \int_0^T f(t, s, \varphi(s)) \, ds \left[ A(\varphi(t)) - A(\psi(t)) \right] + A(\psi(t)) \left[ f(t, s, \varphi(s)) - f(t, s, \psi(s)) \right] \right] \, ds
\]

Therefore,

\[
|F\varphi(t) - F\psi(t)| = 
A(\varphi(t)) \left[ \int_0^T f(t, s, \varphi(s)) \, ds \int_0^T \left| v(t, g, \varphi(g)) - v(t, g, \psi(g)) \right| \, dg \right]
+ \int_0^T v(t, g, \varphi(g)) \, dg \left[ \int_0^T f(t, s, \varphi(s)) \, ds \left[ A(\varphi(t)) - A(\psi(t)) \right] + A(\psi(t)) \left[ f(t, s, \varphi(s)) - f(t, s, \psi(s)) \right] \right] \, ds
\]

Hence,

\[
|F\varphi(t) - F\psi(t)| \leq 
A(\varphi(t)) \left[ \int_0^T \left| f(t, s, \varphi(s)) \right| \, ds \int_0^T \left| v(t, g, \varphi(g)) - v(t, g, \psi(g)) \right| \, dg \right]
\]
According to (5) \( F \) is a contraction for all \( n \in N \). Based on the selection of \( \Omega \) is no \( \varphi \in \partial \Omega \) such that \( \varphi = \lambda F(\varphi) \) for some \( \lambda \in (0,1) \). Then the statement 2* in the theorem (3) does not hold. The nonlinear alternative of Leray-Schauder type demonstrates that statement 1* holds, and hence the operator \( F \) has a unique fixed point \( y \) in \( \Omega \) which is a solution of Eq (1).

**Conclusions**

This paper concentrated on a nonlinear alternative for the Leray-schauder type for condition maps. Section 3 shows that we have provided appropriate conditions to achieve the theory conditions. As mentioned previously equation (1) is more general than equation (2) and (3).

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