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ON THE CARTESIAN PRODUCT INTUITIONISTIC FUZZY RELATION

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Abstract:

In this paper we study the intuitionistic fuzzy relation and the operations on the intuitionistic fuzzy relations. We first define theCartesian product of intuitionistic fuzzy sets, then introduce the projection and the cylindric extension of intuitionistic fuzzy relation and show the relation between them. The concept of a composition operator for the intuitionistic fuzzy relations, which represents a generalization for those of fuzzy sets is also discussed in this paper.

Key word: fuzzy set, intuitionistic fuzzy, fuzzy relation, intuitionistic fuzzy relation.

الضرب الكارتيزي للدالة الضبابية INTUITONISTIC

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الملخص :

ندرس في هذه الورقة العلاقة الحدسية الضبابية والعمليات التي تطرأ عليها. اولا نعرف حاصل الضرب الديكارتي للمجموعة الحدسية الضبابية ثم نقوم بتقديم الاسقاط والتمديد الاسطواني للعلاقات الحدسية الضبابية ونوضح العلاقات التي بينهما. وايضا سنقوم بمناقشة المفهوم للعامل الانشائي للعلاقات الحدسية الضبابية والذي بدوره يمثل تعميم لهذه المجموعة الضبابية.

الكلمات المفتاحية : المجموعة الضبابية، ضبابي حدسي، علاقه ضبابية، العلاقة الحدسية الضبابية .

1 Introduction

The concept of fuzzy sets was first introduced by Zadeh [4] who defined the concept of fuzzy relation, projections, cyindric extension and composition of the fuzzy set. Since then numerous studies have been conducted in this area due to the diverse applications ranging from engineering and computer science to social behaviour studies. The idea of an intuitionistic fuzzy set was presented by Atanassov[1] and Hosseini et.al [17] generalized some results on intuitionistic fuzzy space. Intuitionistic fuzzy sets play an important role in modern mathematics and in the field of fuzzy systems and controls. Recently Yao et.al[7] elaborated on fuzzy soft set and soft fuzzy set. Cagman et.al [15] proposed interesting applications of fuzzy soft set theory. Xu et.al [8] introduced the definition of intuitionistic fuzzy soft set. On the other hand, to make easy computation with the operations of fuzzy soft sets, the fuzzy soft matrix theory is presented and fuzzy soft max-min decision making method is set up by Cagman and Enginoglu [16]. Chaudhuri et.al[9] used fuzzy soft relation to solve decision making problems. Alkhazaleh et.al introduced possibility fuzzy soft set [12] and the concept of fuzzy parameterized interval-valued fuzzy soft set[13] and soft multi sets[11] as a generalization of [10]. Alhazaymeh et.al[14] then generalized the concept of soft intuitionistic fuzzy sets.

The objective of this article is to generalize the fuzzy relation concepts to intuitionistic fuzzy concept by defining a Cartesian product, projection, cylindric extension and composition for the case of intuitionistic fuzzy sets. We start this article by recalling some basic definitions related to fuzzy and intuitionistic fuzzy sets.

2. Preliminaries

Definition2.1(see[4])A fuzzy set in universe of discourse U is characterized by a membership function that takes values in the interval$[0,1]$.

Therefore a fuzzy set is a generalization of a classical set by allowing the membership function to take values in the interval $[0,1]$, instead of only two values zero and one.

Definition2.2(see [4]) A fuzzy set A in U is a function from U into $[0,1]$.

Definition 2.3(see[1]) An intuitionistic fuzzy set A in a nonempty set U(a universe of discourse) is an object having the form

$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$. where the function denotes the

degree of membership function and the degree of non-membership function of each element $x \in U$ to the set A respectively, such that $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all $x \in U$.

Definition 2.4 (see [4]]) Let A, B be fuzzy sets in X and Y respectively. The cross (Cartesian) product $A \times B$ is a fuzzy set defined by $A \times B = \{\langle (x,y), \mu_{A\times B}(x,y)\rangle, (x,y) \in X \times Y\}$ where

$$\mu_{A\times B}(x,y) = min(\mu_A(x), \mu_B(x)).$$

Definition 2.5(see [1]]) Let A and B be intuitionistic fuzzy sets in X and Y respectively. The cross (Cartesian) product $A \times B$ is an intuitionistic fuzzy set defined by

$$A \times B = \{\langle (x,y), \mu_{A\times B}(x,y), \nu_{A\times B}(x,y)\rangle : (x,y) \in X \times Y\}$$

where $\mu_{A\times B}(x,y) = min(\mu_A(x), \mu_B(y))$

$$\nu_{A\times B}(x,y) = max(\nu_A(x), \nu_B(y))$$

Definition2.6 (see[4]) Let Q be a fuzzy relation in $U_1 \times U_2 \times \ldots \times U_n$ andlet$\{i_1, \ldots, i_k\}$ be a subsequence of$\{1,2, \ldots n\}$.Then the projection of Q on$U_{i_1} \times \ldots \times U_{i_k}$is a fuzzy relation Q_P in ,

$\mu_{Q_P}(u_{i_1}, \ldots, u_{i_k}) = \max \mu_Q(u_1, \ldots, u_n)$,such that $u_{j1} \in U_{ji,\ldots} u_{j(n-k)} \in U_{j(n-k)}$

where $\{u_{j1}, \ldots, u_{j(n-k)}\}$ is the complement of $\{u_{i_1}, \ldots, u_{i_k}\}$ with respect to$\{u_1, \ldots, u_n\}$.

As a special case, if Q is a binary relation in $X \times Y$, then the projection of Q on X, denoted by$Q_x$is a fuzzy set in$U$ defined by

$$\mu_{Q_x} = \max_{y \in Y}(\nu_Q(x,y)).$$

Definition 2.7(see [3]) let Q_P be a fuzzy relation in $U_{i_1} \times \ldots \times U_{i_k}$ and$\{i_1, \ldots i_k\}$ a subsequence $\{1,2, \ldots n\}$ then the cylindric extension of Q_P to $U_1 \times U_2 \times \ldots \times U_n$ is a fuzzy relation Q_E in $U_1 \times U_2 \times \ldots \times U_n$ defined by $\mu_{Q_{PE}}(u_1, \ldots, u_n) = \mu_{Q_P}(u_{i_1}, \ldots, u_{i_k})$.

As a special case, if Q is a fuzzy set in U then the cylindric extension of Q to $U \times V$ is fuzzy relation Q_E in $U \times V$ that is $\mu_{Q_{1E}}(x,y) = \mu_{Q_1}(x)$.

Definition 2.8 (see[3], [5],[6])The composition of fuzzy relation $P(U,V)$and $Q(V,W)$ denoted by $P \circ Q$ is defined as a fuzzy relation in $U \times W$ whose membership function is $\mu_{P \circ Q}(x,z) = max_{y\in v}\, t(\mu_p(x,y), \mu_Q(y,z))$for any $(x,z) \in U \times W$ where t is any t-norm.

There are two most popularly used compositions namely max-min composition and, max-product composition which are defined as follows:

- The max-min composition of fuzzy relation $P(U,V)$and $Q(V,W)$ is fuzzy relation $P \circ Q$in V,W defined by the membership function

$\mu_{P \circ Q}(x,z) = \max min_{y\in v}(\mu_p(x,y), \mu_Q(y,z)$ where $(x,z) \in U \times W$.

- The max-product composition of fuzzy relation $P(U,V)$ and $Q(V,W)$ is a fuzzy relation$P \circ Q$ in $U \times W$ defined by the membership function

$$\mu_{P \circ Q}(x,z) = max_{y\in v}(\mu_p(x,y)\mu_Q(y,z)$$

where $(x,z) \in U \times W$.

Below are some operations on intuitionistic fuzzy set as noted by Mohammad Fathi [2]) which we will be using.

Yager sum and product:

$$S_w^{\mu}(a,b) = min(1,(a^w + b^w)^{\frac{1}{w}}),\ T_w^{\nu}(a,b) = 1 - min\left[1,((a')^w + (b')^w)^{\frac{1}{w}}\right] \text{where } w \in (0,\infty) .$$

Drastic sum and product:

$$S_{ds}^{\mu} = \begin{cases} a \ \ if \ \ b = 0 \\ b \ \ if \ \ a = 0 \\ 1 \quad otherwise \end{cases} \quad T_{dp}^{V} = \begin{cases} \grave{a} \ \ if \ \ \grave{b} = 0 \\ \grave{b} \ \ if \ \ \grave{a} = 0 \\ 1 \quad otherwise \end{cases}$$

Einstein sum and product:

$$S_{es}^{\mu}(a,b) = \frac{a+b}{1+ab},\ T_{ep}^{\nu}(a',b') = 1 - \frac{(1-a')(1-b')}{a'+b'+(1-a')(1-b')}$$

3. Intuitionistic Fuzzy Relation

In this section we introduce operators of intuitionistic fuzzy relation, which are projection, cylindric and composition of intuitionistic fuzzy relation

We now propose the following definition with respect to cross products of intuitionistic fuzzy sets.

Definition3.1 An intuitionistic fuzzy relation is an intuitionistic fuzzy set in the Cartesian product of crisp sets $U_1, U_2, \dots U_n$ with the representation scheme$A = \{\langle \mu_A(x), \nu_A(x) : x \in U \rangle\}$.

An intuitionistic fuzzy relation Q in $U_1 \times U_2 \times \dots \times U_n$is defined as intuitionistic fuzzy set

$$Q = \{\langle (u_1, u_2, \dots, u_n),\ \mu_Q(u_1, u_2, \dots, u_n), \nu_Q(u_1, u_2, \dots, u_n) \rangle : (u_1, u_2, \dots, u_n) \in U_1 \times U_2 \times \dots \times U_n\}$$

such that $\mu_Q : U_1 \times U_2 \times \dots \times U_n \to [0,1], \nu_Q : U_1 \times U_2 \times \dots \times U_n \to [0,1]$.

As a special case, a binary intuitionistic fuzzy relation is an intuitionistic fuzzy set defined in the Cartesian product of two crisp sets.

We now propose the following definition of the projection of intuitionistic fuzzy sets.

Definition3.2 Let Q be an intuitionistica fuzzy relation in $U_1 \times U_2 \times \dots \times U_n$ and let$\{i_1, \dots, i_k\}$ be a subsequence of $\{1,2, \dots n\}$ Then the projection of Q on $U_{i1} \times \dots U_{ik}$ is an intuitionistic fuzzy relation Q_P in $U_{i_1} \times \dots \times U_{i_k}$ defined by member ship function and non-membership function

$\mu_{Q_P}(u_{i_1}, \dots, u_{i_k}) = \max \mu_Q(u_1, \dots, u_n)$ where $u_{j1} \in U_{j1}, \dots, u_{j(n-k)} \in U_{j(n-k)}$ and

$\nu_{Q_P}(u_{i_1}, \dots, u_{i_k}) = \min \nu_Q(u_1, \dots, u_n)$ where $u_{j1} \in U_{j1}, \dots, u_{j(n-k)} \in U_{j(n-k)}$

such that$\{u_{j1}, \dots, u_{j(n-k)}\}$ is the complement of $\{u_{i_1}, \dots, u_{i_k}\}$ with respect to$\{u_1, \dots, u_n\}$.

As a special case, if Q is an intuitionistic fuzzy relation in $X \times Y$ then the intuitionistic fuzzy projection of Q on X, denoted by Q_X , is an intuitionistic fuzzy set in X defined by

$$\mu_{Q_x} = \max_{y \in Y}(\mu_Q(x, y)) \quad \text{and} \quad \nu_{Q_x} = \min_{y \in Y}(\nu_Q(x, y)).$$

Example3.3 Consider the universe of discourse $X = \{1, 2\}$ and$Y = \{a, b\}$. Let Q be

An intuitionistic fuzzy relation in $X \times Y$ defined by

$$Q = \{\langle (1, a), 0, 0.2 \rangle, \langle (1, b), 0.4, 0 \rangle, \langle (2, a), 0.5, 0 \rangle, \langle (2, b), 0, 0.1 \rangle\}.$$

We find the projection of Q on X and on $Y$by using

i) Drastic sum and product, and ii) Einstein sum and product.

The projection of Q on X:

i) Drastic sum and product

$$S_{ds}\left((a,a'),(b,b')\right)=\left(S^{\mu}_{ds}(a,b),S^{\nu}_{ds}(a',b')\right)$$

where

$$S^{\mu}_{ds}=\begin{cases} a & if \quad b=0 \\ b & if \quad a=0 \\ 1 & otherwise \end{cases} \qquad T^{V}_{dp}=\begin{cases} \grave{a} & if \quad \grave{b}=0 \\ \grave{b} & if \quad \grave{a}=0 \\ 1 & otherwise \end{cases}$$

we obtain

$$\begin{aligned} Q_X=&\{\langle 1,S^{\mu}_{ds}\left(\mu_Q(1,a),\mu_Q(1,b)\right)\rangle,\ T^{\nu}_{d\rho}\left(\nu_Q(1,a),\nu_Q(1,b)\right)\rangle,\\ &\langle 2,S^{\mu}_{ds}\left(\mu_Q(2,a),\mu_Q(2,b)\right)\rangle,T^{\nu}_{d\rho}\left(\nu_Q(2,a),\nu_Q(2,b)\right)\rangle\}\\ =&\{\langle 1,0.4,0.2\rangle,\langle 2,0.5,0.1\rangle\}. \end{aligned}$$

The second part Q_y (projection of intuitionistic fuzzy relation Q on Y)

$$\begin{aligned} Q_y=&\{\langle a,S^{\mu}_{ds}(\mu_Q(1,a),\mu_Q(2,a))\ ,T^{\nu}_{d\rho}(\nu_Q(1,a),\nu_Q(2,a))\rangle,\\ &\langle b,S^{\mu}_{ds}(\mu_Q(1,b),\mu_Q(2,b))\ ,T^{\nu}_{d\rho}(\nu_Q(1,b),\nu_Q(2,b))\rangle\}\\ =&\{\langle a,0.5,0.2\rangle,\langle b,0.4,0.1\rangle\}\ . \end{aligned}$$

ii) Einstein sum and product

$$S^{\mu}_{es}(a,b)=\frac{a+b}{1+ab} \quad T^{\nu}_{ep}(a',b')=1-\frac{(1-a')(1-b')}{a'+b'+(1-a')(1-b')}$$

The first part Q_x (projection of intuitionistic fuzzy relation Q on X)

$$\begin{aligned} Q_X=&\{\langle 1,S^{\mu}_{es}\left(\mu_Q(1,a),\mu_Q(1,b)\right)\rangle,\ T^{\nu}_{e\rho}\left(\nu_Q(1,a),\nu_Q(1,b)\right)\rangle,\\ &\langle 2,S^{\mu}_{es}\left(\mu_Q(2,a),\mu_Q(2,b)\right)\rangle,T^{\nu}_{e\rho}\left(\nu_Q(2,a),\nu_Q(2,b)\right)\rangle\} \end{aligned}$$

$$\begin{aligned} Q_X=&\left\{\langle 1,\frac{0+0.4}{1+(0\times 0.4)},1-\frac{(1-0.2)(1-0)}{0.2+0+(1-0.2)(1-0)}\rangle,\right.\\ &\left.\langle 2,\frac{0.5+0}{1+(0.5\times 0)},1-\frac{(1-0)(1-0.1)}{0+0.1+(1-0.1)(1-0.1)}\rangle\right\}\\ =&\{\langle 1,0.4,0.2\rangle,\langle 2,0.5,0.1\rangle\}. \end{aligned}$$

The second part Q_y (projection of intuitionistic a fuzzy relation Q on Y)

$$Q_y = \{ \langle a, S^{\mu}_{es}(\mu_Q(1,a), \mu_Q(2,b)) \rangle, T^{\nu}_{ep}(\nu_Q(1,a), \nu_Q(2,b)) \rangle,$$

$$\langle b, S^{\mu}_{es}(\mu_Q(1,b), \mu_Q(2,b)) \rangle, T^{\nu}_{ep}(\nu_Q(1,b), \nu_Q(2,b)) \rangle \}$$

$$= \{ \langle a, 0.5, 0.2 \rangle, \langle b, 0.4, 0.1 \rangle \}.$$

We now propose the definition of cylindric of intuitionistic fuzzy relation.

Definition3.4 Let Q_P be an intuitionistic fuzzy relation in $U_{i_1} \times \ldots \times U_{i_k}$ and $\{i_1, \ldots i_k\}$ is subsequence of $\{1,2,\ldots n\}$ then the cylindric extension of Q_P to $U_1 \times U_2 \times \ldots \times U_n$ is intuitionistic fuzzy relation Q_{PE} in $U_1 \times U_2 \times \ldots \times U_n$ defined by

$\mu_{Q_{PE}}(u_1,\ldots,u_n) = \mu_{Q_P}(u_{i_1},\ldots,u_{i_k})$ and $\nu_{Q_{PE}}(u_1,\ldots,u_n) = \nu_{Q_P}(u_{i_1},\ldots,u_{i_k})(u_{i_1},\ldots,u_{i_k})$.

As a special case, if Q is a fuzzy set in U, then the cylindric extension of Q to $U \times V$ is an intuitionistic fuzzy relation in $U \times V$ and (we denote it by $\hat{Q}$) defined by

$\mu_{\hat{Q}}(x,y) = \mu_Q(x)$, $\nu_{\hat{Q}}(x,y) = \nu_Q(x)$,

Example 3.5. Consider the universe of discourse X={1,2} and Y={a,b} and let A be an intuitionistic fuzzy relation on $X \times Y$ where

$A = \{ \langle (1,a), 0.2, 0 \rangle, \langle (1,b), 0.5, 0.3 \rangle, \langle (2,a), 0.1, 0.1 \rangle, \langle (2,b), 0.8, 0 \rangle \}$.

We want to find the projection of *A* on *X*, then cylindric extension of the projection of *A* on *X*. The first part is to find the projection of *A*, by using the algebraic sum on the membership values of the elements of *A* and using the algebraic product on the non-membership values of the elements of *A*.We know that

$X \times Y = \{(1,a),(1,b),(2,a),(2,b)\}$

and the intuitionistic algebraic sum and product is

$S^{\mu}_{ds}(a,b) = a + b - ab$ $T^{\nu}_{ap}(a',b') = 1-(1-a')(1-b')$

We start by the element 1 and we find

$S^{\mu}_{as}(\mu_A(1,a), \mu_A(1,b))$, $T^{\nu}_{as}(\nu_A(1,a), \nu_A(1,b))$

For this

element $\mu_A(1,a) = 0.2$ $\mu_A(1,b) = 0.5$, $S^{\mu}_{as}(0.2, 0.5) = 0.2 + 0.5 - (0.2 \times 0.5) = 0.6$.

For $\nu_A(1,a)=0 \quad \nu_A(1,b)=0.3,\ \mathrm{T}^{\nu}_{as}(0,0.3)-1-(1-0)(1-0.3)=1-0.7=0.3$.

Hence the element 1 belongs to A_X with respect to the intuitionistic algebraic sum with 0.6 degree of membership and 0.3 degree of non-membership.

For the element 2 with $\mu_A(2,a)=0.1,\ \mu_A(2,b)=0.8.$

$$\mathrm{S}^{\mu}_{as}(0.1,0.8)=0.1+0.8-(0.1\times 0.8)=0.82$$

And for $\nu_A(2,a)=0.1, \nu_A(2,b)=0,\ \mathrm{T}^{\mu}_{as}(0.1,0)=1(1-0.1)(1-0)=0.1$

Hence the element 2 belongs to A_X with 0.82 degree of membership and 0.1 degree of non-membership. Thus $A_X=\{\langle 1,0.6,0.3\rangle,\langle 2,0.82,0.1\rangle\}$.

The second part we want to find the cylindric extension of A_X

$$\hat{A}_x=\{\langle (x,y),\mu_{A_x}(x),\nu_{A_x}(x)\rangle:(x,y)\in X\times Y\}$$

$$=\{\langle (1,a),\mu_{A_x}(1),\nu_{A_x}(1)\rangle,\langle (1,b),\mu_{A_x}(1),\nu_{A_x}(1)\rangle,$$

$$\langle (2,a),\mu_{A_x}(2),\nu_{A_x}(2)\rangle,\langle (2,b),\mu_{A_x}(2),\nu_{A_x}(2)\rangle\}$$

$$\langle (2,b),0.82,0.1\rangle\}.$$

Now we propose the relation between the Cartesian product and intersection of cylindric fuzzy sets by the following lemma.

Lemma3.6 Let Q be an intuitionistic fuzzy relation in $U_1\times U_2\ldots\times U_n$ and $Q_1,\ldots,Q_n$ be its projection on $U_1,\ldots,U_n$ respectively. Then $Q\subset Q_1\times Q_2,\ldots,\times Q_n$ we use minimum (denoted by *) for T-norm in

$$\mu_{Q_1\times Q_2\cdots Q_n}(u_1,\ldots,u_n)=\mu_{Q_1}(u_1)*\ldots*\mu_{Q_n}(u_n)$$

and use maximum for s-norm (denoted by∗)

$$\nu_{Q_1\times Q_2\cdots Q_n}(u_1,\ldots,u_n)=\nu_Q(u_1)*\ldots*\nu_{Q_n}(u_n) \text{ of } Q_1\times\ldots\times Q_n$$

Proof

From the definition of the projection of an intuitionistic fuzzy relation we have

$$\mu_{Q_P}(u_{i_1},\ldots,u_{i_k})=\max\mu_Q(u_1,\ldots,u_n) \text{ and } \nu_{Q_P}(u_{i_1},\ldots,u_{i_k})=\min\nu_Q(u_1,\ldots,u_n)(1)$$

where $\{u_{j1},\ldots,u_{j(n-k)}\}$ is the complement of $\{u_{i_1},\ldots,u_{i_k}\}$ with respect to $\{u_1,\ldots,u_n\}$.

$$\nu_{Q_P}(u_{i_1},\ldots,u_{i_k})=\min\nu_{Q_P}(u_1,\ldots,u_n)$$

Also from the definition of the cylindric extension of an intuitionistic fuzzy set we have

$\mu_{Q_{PE}}(u_1,\dots,u_n) = \mu_{Q_P}(u_{i_1},\dots,u_{i_k})$ and $v_{Q_{PE}}(u_1,\dots,u_n) = v_{Q_p}(u_{i_1},\dots,u_{i_k})(u_{i_1},\dots,u_{i_k})(2)$.

Substituting (1) into (2) we have

$\mu_{Q_{PE}}(u_1,\dots,u_n) = \max \mu_Q(u_1,\dots,u_n)$ and $v_{Q_{PE}}(u_1,\dots,u_n) = \min v_Q(u_1,\dots,u_n)$

$u_{j1} \in U_{ji,\dots,} u_{j(n-k)} \in U_{j(n-k)}$.

Hence, $Q \subset Q_{iE}$ for all i=1, 2…,n where Q_{iE} is the cylindric extension of Q_i to $U_1 \times U_2 \dots \times U_n$

Therefore by writing minimum-maximum for intersection, we have

$$Q \subset Q_{1E} \cap Q_{2E} \dots \cap Q_{nE} = Q_1 \times Q_2, \dots, \times Q_n.$$

Proposition3.7 Let A and B be normal intuitionistic fuzzy sets (is an intuitionistic fuzzy sets which have height equal one and depth equal to zero), in the universe of discourse X and Y respectively. Then the projection of intuitionistic fuzzy relation $(A \times B)$ on X is$Q_x(A \times B) = A$, and the projection

of intuitionistic fuzzy relation $(A\times B)$ on Y is $Q_y(A\times B) = B$.

Proof:

$$\mu_{Q_x}(x) = \max_{y \in Y}(\mu_{A\times B}(x,y)) = \max_{y \in Y}(\mu_A(x) \wedge \mu_B(y))$$

$$= \min(\mu_A(x), \max_{y \in Y}(\mu_B(y))) = \min_{y \in Y}(\mu_A(x), 1)$$

$$= \mu_A(x).$$

and we have

$$v_{Q_x}(x) = \min_{y \in Y}(v_{A\times B}(x,y)) = \min_{y \in Y}(v_A(x) \wedge v_B(y))$$

$$= \max(v_A(x), \min_{y \in Y}(v_B(y))) = \max_{y \in Y}(v_A(x), 0)$$

$$= v_A(x).$$

As a result, we have $Q_x(A\times B) = A$. Similarly we can proof $Q_y(A\times B) = B$.

$$\mu_{Q_y}(y) = \max_{x \in X}(\mu_{A\times B}(x,y)) = \max_{x \in X}(\mu_A(x) \wedge \mu_B(y))$$

$$= \min_{x \in X}(\max(\mu_A(x),(\mu_B(y))) = \min_{x \in X}(1,\mu_B(y))$$

$$= \mu_B(y).$$

We have

$$\nu_{Q_y}(y) = \min_{x \in X}(\nu_{A\times B}(x,y)) = \min_{x \in X}(\nu_A(x) \wedge \nu_B(y))$$

$$= \max_{x \in X}(\min(\nu_A(x)),\nu_B(y)) = \max_{x \in X}(0,\nu_B(y))$$

$$= \nu_B(y).$$

Thus $Q_y(A\times B) = B$.

We now propose the composition of intuitionistic fuzzy relation; first we need to proof the following lemma to state the definition of the composition of intuitionistic fuzzy relation

Lemma 3.8 $P \circ Q$ is the composition of $P(U,V)$ and $Q(V,W)$ if and only if

$$\mu_{P\circ Q}(x,z) = \max_{y\in V} t(\mu_P(x,y), \mu_Q(y,z))$$

$$\nu_{P\circ Q}(x,z) = \min_{y\in V} s(\nu_P(x,y), \nu_Q(y,z)). \qquad (*)$$

For any $(x,y) \in U \times W$ where t is any t-norm and s is any s-norm.

Proof: We first show that if $P \circ Q$ is the composition according to the definition then (*) is true.

If $P \circ Q$ is the composition, then $(x,z) \in P \circ Q$ implies that there exists $y \in V$ such that $\mu_p(x,y) = 1$ and $\nu_Q(y,z)$ and also $\nu_p(x,y) = 0$ and $\nu_Q(x,z) = 0$, Hence

$$\mu_{PoQ}(x,z) = 0 = \min_{y\in V} s\,[\mu_P(x,y),\ \mu_Q(y,z)]$$

and $\nu_{PoQ}(x,z) = 0 = \min_{y\in v} s(\nu_p(x,y), \nu_Q(y,z))$. That is (*) is true.

If $(x,z) \notin P \circ Q$ then for any $y \in V$ either $\mu_P(x,y) = 0$ or $\mu_Q(y,z) = 0$ and also either $\nu_P(x,y) = 1$ or $\nu_Q(y,z) = 1$.

Hence $\mu_{PoQ}(x,z) = 0 = \min_{y\in V} s\,[\mu_P(x,y),\ \mu_Q(y,z)]$

and $\nu_{PoQ}(x,z) = 1 = \min_{y\in V} s\,[\nu_P(x,y),\ \nu_Q(y,z)]$, therefore(*) is true. For any $(x,z) \in U\times W$, if (*) is true, then $(x,y) \in P \circ Q$ implies $\max_{y\in V} t\,[\mu_P(x,y),\ \mu_Q(y,z)] = 1$, and $\min_{y\in V} s\,[\nu_P(x,y),\ \nu_Q(y,z)] = 0$, which means that there

exists at least one $y \in V$ such that $\mu_P(x,y) = \mu_Q(y,z) = 1$ and $\nu_P(x,y) = \nu_Q(y,z) = 0$.

For $(x, z) \notin P \circ Q$ we have from (*) that

$\max_{y\in V} t\,[\mu_P(x,y),\ \mu_Q(y,z)] = 0,$

$\min_{y\in V} s\,[\nu_P(x,y),\ \nu_Q(y,z)] = 1,$

Which means that there is no $y \in v$ such that $\mu_P(x,y) = \mu_Q(y,z) = 1$ and also $\nu_P(x,y) = \nu_Q(y,z) = 0$. Therefore (*) implies that $P \circ Q$ is the composition.

Now we generalize the concept of composition of intuitionistic fuzzy relation by using Lemma 3.8

Definition3.9 The composition of intuitionistic of fuzzy relations $P(U,V)$ and $Q(V,W)$, denoted by $P \circ Q$, is defined as an intuitionistic fuzzy relation in $U \times W$ whose membership function and non-membership function are

$$\mu_{P\circ Q}(x,z) = \max_{y\in V} t(\mu_P(x,y), \mu_Q(y,z)) \,,\ \nu_{P\circ Q}(x,z) = \min_{y\in V} s(\nu_P(x,y), \nu_Q(y,z))$$

For any $(x,z) \in U \times W$ where t is any t-norm.

Similar to the fuzzy case, we give two cases of the composition of intuitionistic fuzzy relations which we call the max-min composition and the max-product composition which are defined as follows.

- The max-min composition of intuitionistic fuzzy relations $P(U,V)$ and $Q(V,W)$is an intuitionistic fuzzy relations $P \circ Q$ in $U\times W$ defined by the membership function and non-membership function

$$\mu_{P\circ Q}(x,z) = \max_{y\in v} \min\left[\ \mu_P(x,y), \mu_Q(y,z)\right],\ \nu_{P\circ Q}(x,z) = \min_{y\in v} \max\left[\ \nu_P(x,y), \nu_Q(y,z)\right]$$

where $(x,z) \in U \times W$.

- The max-product composition of intuitionistic fuzzy relation $P(U,V)$ and $Q(V,W)$

is an intuitionistic fuzzy relation $P \circ Q$in U×W defined by the membership function and non-member ship function

$$\mu_{PoQ}(x,z)=\max_{y\in v}\left[\mu_P(x,y)\,\mu_Q(y,z)\right],\ \nu_{PoQ}(x,z)=\min_{y\in v}\left[\nu_P(x,y)\,\nu_Q(y,z)\right]$$

Example 3.10 Consider two intuitionistic fuzzy relations defined by the relation matrices.

$$\mu_{Q_1}=\begin{pmatrix}1 & 0 & 0.7\\ 0.3 & 0.2 & 0\\ 0 & 0.5 & 1\end{pmatrix}\qquad \nu_{Q_1}=\begin{pmatrix}0 & 1 & 0.2\\ 0.1 & 0 & 1\\ 0 & 0.2 & 0\end{pmatrix}$$

$$\mu_{Q_2}=\begin{pmatrix}0.6 & 0.6 & 0\\ 0 & 0.6 & 0.1\\ 0 & 0.1 & 0\end{pmatrix}\qquad \nu_{Q_2}=\begin{pmatrix}0.1 & 0 & 1\\ 0 & 0.2 & 0\\ 0.2 & 0.3 & 1\end{pmatrix}$$

We compute the max-product composition.

$$\mu_{PoQ}(x,z)=\max_{y\in v}\left[\mu_P(x,y)\,\mu_Q(y,z)\right]\ \nu_{PoQ}(x,z)=\min_{y\in v}\left[\nu_P(x,y)\,\nu_Q(y,z)\right]$$

$$\mu_{Q_1oQ_2}=\begin{pmatrix}1 & 0 & 0.7\\ 0.3 & 0.2 & 0\\ 0 & 0.5 & 1\end{pmatrix} o \begin{pmatrix}0.6 & 0.6 & 0\\ 0 & 0.6 & 0.1\\ 0 & 0.1 & 0\end{pmatrix}$$

$$=\begin{pmatrix}\vee((1)(0.6),(0)(0),(0.7)(0)) & \vee((1)(0.6),(0)(0.6),(0.7)(0.1)) & \vee((1)(0),(0)(0.1),(0.7)(0))\\ \vee((0.3)(0.6),(0.2)(0),(0)(0)) & \vee((0.3)(0.6),(0.2)(0.6),(0)(0.1)) & \vee((0.3)(0),(0.2)(0.1),(0)(0))\\ \vee((0)(0.6),(0.5)(0),(1)(0)) & \vee((0)(0.6),(0.5)(0.6),(1)(0.1)) & \vee((0)(0),(0.5)(0.1)(0),(0.1)(0))\end{pmatrix}$$

$$=\begin{pmatrix}0.6 & 0.6 & 0\\ 0.19 & 0.18 & 0.02\\ 0 & 0.3 & 0.05\end{pmatrix}$$

$$\nu_{Q_1oQ_2}=\begin{pmatrix}0 & 1 & 0.2\\ 0.1 & 0 & 1\\ 4 & 0.2 & 0.1\end{pmatrix} o \begin{pmatrix}0.1 & 0 & 1\\ 0 & 0.2 & 0.2\\ 0.3 & 0.3 & 1\end{pmatrix}$$

$$=\begin{pmatrix}\wedge[(0)(0.1),(1)(0),(0.2)(0.3)] & \wedge[(0)(0),(1)(0.2),(0.2)(0.3)] & \wedge[(0)(1),(1)(0.2),(0.2)(1)]\\ \wedge[(0.1)(0.1),0,(1)(0.3)] & \wedge[(0)(0),(0)(0.2),(1)(0.3)] & \wedge[\ (0.1)(1),(0.2)1\ ]\\ \wedge[(1)(0.1),(0.2)(0),(0)(0.3)] & \wedge[\ 0,(0.2)(0.2),(0)(0.3)\ ] & \wedge[(1)(1),(0.2)(0.2),(0.1)(1)]\end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.04 \end{pmatrix}$$

Thus $Q_1 \circ Q_2$ has $\mu_{Q_1 \circ Q_2} = \begin{pmatrix} 0.6 & 0.6 & 0 \\ 0.18 & 0.18 & 0.02 \\ 0 & 0.3 & 0.05 \end{pmatrix}$ $\nu_{Q_1 \circ Q_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.04 \end{pmatrix}$

We now propose the composition of intuitionistic fuzzy relation is assistive by the following theorem.

Theorem3.11 Let Q be an intuitionistic fuzzy relation in $X \times Y$, and P an intuitionistic fuzzy relation in $Y \times Z$ and R an intuitionistic fuzzy relation in $Z \times U$. Then$(Q \circ P) \circ R = Q \circ (P \circ R)$where the composition between the relations is the max-min composition.

Proof i) $\mu_{(Q \circ p) \circ R}(x,t) = \max\min[\mu_{Q \circ p}(x,z), \mu_R(z,t)]$

$$= \max \ \min[\max\min(\mu_Q(x,y), \mu_p(y,z)), \mu_R(z,t)$$

$$= \max \ \max[\min(\mu_Q(x,y), \min \mu_p(y,z), \mu_R(z,t)))]$$

$$= \max\min[(\mu_Q(x,y), \max\min(\mu_p(y,z), \mu_R(z,t)))]$$

$$= \max\min[(\mu_Q(x,y), (\mu_{p \circ R)}(y,t),))]$$

$$= \mu_{Q \circ (P \circ R)}(x,t).$$

The equality $V_{Q \circ (P \circ R)}(x,t)$is computed as follows

$$\nu_{(Q \circ p) \circ R}(x,t) = \min\max[\nu_{Q \circ p}(x,z), \nu_R(z,t)]$$

$$= \min \ (\max[\min \max(\nu_Q(x,y), \nu_p(y,z)), \nu_R(z,t))])$$

$$= \min \ (\min[\max(\nu_Q(x,y), \max(\nu_p(y,z), \nu_R(z,t)])$$

$$= \min \ (\max[\nu_Q(x,y), \min \ \max(\nu_p(y,z)), \nu_R(z,t)])$$

$$= \min\max[\nu_Q(x,y), (\nu_{P \circ R}(y,t),))]$$

$= \nu_{Q \circ (P \circ R)}(x,t)$. Therefore $(Q \circ P) \circ R = Q \circ (P \circ R)$, where $(x,z) \in U \times W$.

4.Conclusion

In this paper, we have defined the Cartesian product of intuitionistic fuzzy sets and introduce the concepts of intuitionistic fuzzy relation and projection, cylindric extension and the composition of intuitionistic fuzzy relations.

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