Weak solutions for the coupled system of Riemann-Stieltjes integral equations

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Abstract
We present an existence theorem for at least one weak solution for a coupled system of Riemann-Stieltjes integral equations

\[ x(t) = p_1(t) + \lambda_1 \int_0^1 f_1(t, s, x(s), y(s)) \, ds \, g_1(t, s), \quad t \in I = [0, 1], \]
\[ y(t) = p_2(t) + \lambda_2 \int_0^1 f_2(t, s, x(s), y(s)) \, ds \, g_2(t, s), \quad t \in I. \]

in a reflexive Banach space \( E \). As an application, we study the existence of weak solutions \( x, y \in C[I, E] \) for the coupled system of Hammerstein-Stieltjes integral equations in \( E \).

Keywords: Coupled system, weak solution, weakly Riemann-Stieltjes integral, weakly relatively compact, fixed point theorem.

حلول ضعيفة لنظام مزدوّج من المعادلات التكاملية من نوع ريمان استيلجيس

الملخص:
نقدم وجود حل ضعيف للنظام مزدوّج من معادلات التكاملية من نوع ريمان استيلجيس

\[ x(t) = p_1(t) + \lambda_1 \int_0^1 f_1(t, s, x(s), y(s)) \, ds \, g_1(t, s), \quad t \in I = [0, 1], \]
\[ y(t) = p_2(t) + \lambda_2 \int_0^1 f_2(t, s, x(s), y(s)) \, ds \, g_2(t, s), \quad t \in I. \]

في فضاء بناخ الانعكاسي \( E \) تحت شروط معينة. 
وأيضاً إذا كان حل ضعيف واحد على الأقل للنظام المزدوّج من المعادلات التكاملية من نوع هامرستيان - استيلجيس في فضاء بناخ الانعكاسي \( E \).

الكلمات المفتاحية: نظام مزدوّج - حل ضعيف - تكامل ريمان استيلجيس - متراص ضعيف.
1 Introduction and Preliminaries

Consider the Riemann-Stieltjes integral equation

\[ x(t) = p(t) + \int_0^1 f(t, s, x(s)) \, ds \, g(t, s), \quad t \in I = [0, 1] \]  

(1)

where \( g : I \times I \to \mathbb{R} \) is nondecreasing in the second argument on \( I \) and the symbol \( ds \) indicates the integration with respect to \( s \).

Equations of type (1) and some of their generalizations were considered in several papers (see [1]-[4]) and also by some other authors, for example (see [7], [8] and [18]-[20]). In those papers the authors proved that (1) (or more general equations) is solvable in some classes of Banach spaces. Some other properties, were also studied in the mentioned papers. Further facts concerning Stieltjes integrals and their properties (see Banaś[1]).

In this paper, we generalize these results to study the existence of weak solutions \((x, y) \in C[I, E] \times C[I, E]\) for the coupled system of Riemann-Stieltjes integral equations

\[ x(t) = p_1(t) + \lambda_1 \int_0^1 f_1(t, s, x(s), y(s)) \, ds \, g_1(t, s), \quad t \in I = [0, 1], \]

\[ y(t) = p_2(t) + \lambda_2 \int_0^1 f_2(t, s, x(s), y(s)) \, ds \, g_2(t, s), \quad t \in I. \]

(2)

in a reflexive Banach space \( E \).

As an application, we study the existence of weak solutions \( x, y \in C[I, E] \) for the coupled system of Hammerstien-Stieltjes functional integral equations

\[ x(t) = p_1(t) + \lambda_1 \int_0^1 k_1(t, s) h_1(s, x(s), y(s)) \, ds \, g_1(t, s), \quad t \in I = [0, 1]. \]

\[ y(t) = p_2(t) + \lambda_2 \int_0^1 k_2(t, s) h_2(s, x(s), y(s)) \, ds \, g_2(t, s), \quad t \in I. \]

(3)

For the definition, background and properties of the Stieltjes integral we refer to Banaś[1].

However, the coupled system of integral equations have been studied, recently, by some authors (see [13]-[14],[16]).

Throughout this paper, unless otherwise stated, \( E \) denotes a reflexive Banach space with norm \( \| \cdot \| \) and dual \( E^* \). Denote by \( C[I, E] \) the Banach space of strongly continuous functions \( x : I \to E \) with sup-norm.
Let $X = C([t,E] \times C([t,E]) = \{(u(t), x(t), y(t)): t \in \mathbb{R}^+ \}, \quad x \in C([t,E], y \in C([t,E], t \in I),$

be a Banach space with the norm defined as

$$\|x,y\|_X = \|x\|_{\mathbb{R}^N} + \|y\|_{\mathbb{R}^M}, \quad \forall (x,y) \in X.$$ 

Now, we shall present some auxiliary results that will be needed in this work. Let $E$ be a Banach space (need not be reflexive) and let $x : [a,b] \to E$, then

(1-) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in [a,b]$ if for every $\varphi \in E^*: \varphi(x(.))$ is continuous (measurable) at $t_0$.

(2-) A function $h : E \to E$ is said to be weakly sequentially continuous if $h$ maps weakly convergent sequences in $E$ to weakly convergent sequences in $E$.

If $x$ is weakly continuous on $I$, then $x$ is strongly measurable and hence weakly measurable (see [17] and [10]). It is evident that in reflexive Banach spaces, if $x$ is weakly continuous function on $[a,b]$ then $x$ is weakly Riemann integrable (see [17]). Since the space of all weakly Riemann-Stieltjes integrable functions is not complete, we will restrict our attention to the existence of weak solutions of the coupled system (2) in the space $C([t,E] \times C([t,E])$.

**Definition 1.** Let $f : I \times E \to E$. Then $f(t,u)$ is said to be weakly-weakly continuous at $(t_0, u_0)$ if given $\varepsilon > 0$, $\varphi \in E^*$ there exists $\delta > 0$ and a weakly open set $U$ containing $u_0$ such that

$$|\varphi(f(t,u) - f(t_0,u_0))| < \varepsilon$$

Whenever

$$|t - t_0| < \delta \text{ and } u \in U.$$

Now, we have the following fixed point theorem, due to O'Regan, in the reflexive Banach space ((see [21])) and some propositions which will be used in the sequel (see [12]).

**Theorem 1.** Let $E$ be a Banach space and let $Q$ be a nonempty, bounded, closed and convex subset of $C([t,E]$ and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in $E$ for each $t \in I$. Then, $F$ has a fixed point in the set $Q$.

**Proposition 1.** In reflexive Banach space, the subset is weakly relatively compact if and only if it is bounded in the norm topology.

**Proposition 2.** Let $E$ be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|y\| = \varphi(y)$. 
2 Main results

In this section we study the existence of weak solutions $u = (x,y) \in X$ for the coupled system of integral equations of Riemann-Stieltjes type (2), under the assumptions:

(i) $p_i \in C[I,E]$, $i = 1,2$.

(ii) $\lambda_i$ is constant in $R$, $i = 1,2$.

(iii) $f_i : I \times I \times D \times D \to E$, where $D \subset E$, satisfy the following conditions:

1. $f(t,\cdot,\cdot,\cdot)$ is continuous function, $\forall s \in I, x, y \in D \subset E$.
2. $f(t,\cdot,\cdot,\cdot,\cdot)$ is weakly-weakly continuous function, $\forall t \in I$.
3. There exist two continuous functions $m_i(t)$, $m_i : I \times I \to I$ and two positive constants $b_i$, such that

$$
\| f_i(t,s,x,y) \| \leq m_i(t,s) + b_i \max\{\|x\|,\|y\|\}, \ i = 1,2.
$$

(iv) The functions $g_i : I \times I \to R$ and the functions $t \to g_i(t,t)$ and $t \to g_i(t,0)$ are continuous on $I$. Put $\mu = \max\{\sup_{t \in I} |g_i(t,t)| + \sup_{t \in I} |g_i(t,0)|, t \in I, i = 1,2\}$.

(v) For all $t_1, t_2 \in I$ such that $t_1 < t_2$ the functions $s \to g_i(t_2,s) - g_i(t_1,s)$ are nondecreasing on $I$.

(vi) $g_i(0,s) = 0$ for any $s \in I$.

**Remark 1.** Observe that assumptions (v) and (vi) imply that the function $s \to g(t,s)$ is nondecreasing on the interval $I$, for any fixed $t \in I$ (Remark 1 in [4]). Indeed, putting $t_2 = t$, $t_1 = 0$ in (v) and keeping in mind (vi), we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in I$, the function $s \to g(t,s)$ is of bounded variation on $I$.

**Definition 2.** By a weak solution for the coupled system (2), we mean the pair of functions $(x,y) \in C[I,E] \to C[I,E]$ such that

$$
\varphi(x(t)) = \varphi(p_1(t)) + \lambda_1 \int_0^1 \varphi(f_1(t,s,x(s),y(s))) d_1 g_1(t,s),
$$

$$
\varphi(y(t)) = \varphi(p_2(t)) + \lambda_2 \int_0^1 \varphi(f_2(t,s,x(s),y(s))) d_2 g_2(t,s), \quad t,s \in I,
$$

for all $\varphi \in E^*$. 
Theorem 2. Under assumptions (i)-(vi), the coupled system of Riemann-Stieltjes functional integral equation (2) has at least one weak solution $(x, y) \in C[I, E]$. 

Proof. Define an operator $A$ by

$$A(x, y) = (A_1 x, A_2 y)$$

where

$$A_1 x(t) = p_1(t) + \lambda_1 \int_0^1 f_1(t, s, x(s), y(s)) d_z g_1(t, s),$$

$$A_2 y(t) = p_2(t) + \lambda_2 \int_0^1 f_2(t, s, x(s), y(s)) d_z g_2(t, s), \quad t, s \in I.$$ 

For every $x \in C[I, E]$, $f_i(t, .., x(s), y(s))$ are weakly continuous on $I$, then $\phi(f_i(t, .., x(s), y(s)))$ are continuous for every $\phi \in E^*$. Hence, in view of bounded variational of $g_i$, it follows, $f_i(t, s, x(s), y(s))$ are weakly Riemann-Stieltjes integrable on $I$ with respect to $s \rightarrow g_i(t, s)$. Thus $A_1$ make sense.

Define the sets $Q_1$ and $Q_2$ by

$$Q_1 = \{ x \in C[I, E] : \| x \| \leq r_1 \}, \quad r_1 = \| p_1 \| + | \lambda_1 | M \mu.$$ And

$$Q_2 = \{ y \in C[I, E] : \| y \| \leq r_2 \}, \quad r_2 = \| p_2 \| + | \lambda_2 | M \mu.$$

Now, define the set $Q$ by

$$Q = \{ u = (x, y) \in X : \| u \| \leq r_1 + r_2 \}.$$

The remainder of the proof will be given in four steps.

Firstly: The operator $A$ maps $X$ into $X$.

We can prove that $A_1 : C[I, E] \rightarrow C[I, E]$ and for $t_1, t_2 \in I$, $t_1 < t_2$ (without loss of generality, we may assume that $A_1 x(t_1) - A_1 x(t_2) \neq 0$) and there exists $\phi \in E^*$, such that

$$\| A_1 x(t_2) - A_1 x(t_1) \| \leq \phi(A_1 x(t_2) - A_1 x(t_1))$$

$$\leq | \phi(p_1(t_2) - p_1(t_1)) | + | \lambda_1 | \int_0^1 \phi \left( f_1(t_2, s, x(s), y(s)) \right) d_z g_1(t_2, s)$$

$$- \lambda_1 \int_0^1 \phi \left( f_1(t_1, s, x(s), y(s)) \right) d_z g_1(t_1, s) \right|$$

$$\leq | p_2(t_2) - p_2(t_1) | + | \lambda_1 | \int_0^1 \phi \left( f_2(t_2, s, x(s), y(s)) \right) d_z g_2(t_2, s)$$

$$- \int_0^1 \phi \left( f_2(t_1, s, x(s), y(s)) \right) d_z g_2(t_1, s) \right| + | \lambda_1 | \int_0^1 \phi \left( f_1(t_1, s, x(s), y(s)) \right) d_z g_1(t_2, s)$$
Similarly we can show that

\[ ||A_2x(t_2) - A_1x(t_1)|| \leq ||p_2(t_2) - p_1(t_1)|| \]

\[ + ||A_1||\int_0^1 f_1(t_2, s, x(s), y(s)) - f_2(t_2, s, x(s), y(s))||g_1(t_2, 1) - g_1(t_2, 0)|| \]

\[ + (M + br_2)||\int_0^1 g_1(t_2, s) - g_1(t_1, s)|| \]

Now, we obtain

\[ A(x, y)(t_2) - A(x, y)(t_1) = (A_2x(t_2), A_2y(t_2)) - (A_1x(t_2), A_1y(t_1)) \]

\[ = (A_2x(t_2) - A_1x(t_1), A_2y(t_2) - A_1y(t_1)) \]

and

\[ ||A(x, y)(t_2) - A(x, y)(t_1)|| \leq ||A_2x(t_2) - A_2x(t_1)|| + ||A_2y(t_2) - A_2y(t_1)|| \]

\[ \leq ||p_2(t_2) - p_1(t_1)|| \]

\[ + ||A_2||\int_0^1 f_1(t_2, s, x(s), y(s)) - f_2(t_2, s, x(s), y(s))||g_1(t_2, 1) - g_1(t_2, 0)|| \]

\[ + (M + br_2)||\int_0^1 g_1(t_2, s) - g_1(t_1, s)|| \]

\[ + ||A_2||\int_0^1 f_1(t_2, s, x(s), y(s)) - f_2(t_2, s, x(s), y(s))||g_1(t_2, 1) - g_1(t_2, 0)|| \]

\[ + (M + br_2)||\int_0^1 g_1(t_2, s) - g_1(t_1, s)|| \]
then from the continuity of the functions \( g_i \), assumption (iv) we deduce that \( A \) maps \( X \) into \( X \).

Secondly: The operator \( A \) maps \( Q \) into \( Q \).

Let \( y \in Q_1 \) and \( x \in Q_2 \). Without loss of generality we may assume that

\[ A_1 x \neq 0, \quad A_2 y(t) \neq 0, \quad t \in I. \]

By proposition 2, we have

\[
\| A_1 x(t) \| = \varphi(A_1 x(t)) \\
\leq \| \varphi(p_1(t)) \| + \| \varphi(\int_0^1 \int_0^1 (t, s, x(s), y(s)) \, ds \, g_1(t, s)) \| \\
\leq \| a \| + \| \lambda_1 \| \| \varphi(\int_0^1 (t, s, x(s), y(s)) \, ds \, g_1(t, z)) \| \\
\leq \| p_1 \| + \| \lambda_1 \| \| \int_0^1 (t, s, x(s), y(s)) \, ds \, g_1(t, z) \| \\
\leq \| p_1 \| + \| \lambda_1 \| \| \int_0^1 (m_1(t, s) + b_1 \max(\|x\|, \|y\|)) \, ds \, g_1(t, z) \| \\
\leq \| p_1 \| + (M + b_1 \lambda_1) \| \lambda_1 \| \int_0^1 g_1(t, z) \\
\leq \| p_1 \| + (M + b_1 \lambda_1) \| \lambda_1 \| \sup_{t \in I} |g_1(t, 1) - g_1(t, 0)| \\
\leq \| p_1 \| + (M + b_1 \lambda_1) \| \lambda_1 \| \mu.
\]

Then

\[
\| A_1 x(t) \| \leq \| p_1 \| + (M + b_1 \lambda_1) \| \lambda_1 \| \mu = r_1
\]

Similarly we can prove that

\[
\| A_2 y(t) \| \leq \| p_2 \| + (M + b_2 \lambda_2) \| \lambda_2 \| \mu = r_2
\]

Therefore, for any \( u \in Q \)

\[
\| A u(t) \| = \| A(x, y)(t) \| = \| (A_1 x(t), A_2 y(t)) \| \\
\leq \| A_1 x(t) \| + \| A_2 y(t) \| \\
\leq \| p_1 \| + (M + b_1 \lambda_1) \| \lambda_1 \| \mu + \| p_2 \| + (M + b_2 \lambda_2) \| \lambda_2 \| \mu = r_1 + r_2
\]

i.e., \( \forall u \in Q : A u \in Q \Rightarrow AQ \subseteq Q \). Thus \( A : Q \rightarrow Q \).

Thirdly: \( AQ(t) \) is relatively weakly compact in \( E \).

Note that \( Q \) is nonempty, uniformly bounded and strongly equi-continuous subset of \( X \), by the uniform boundedness of \( AQ \). Thus, according to propositions 1, \( AQ \) is relatively weakly compact.

Finally: The operator \( A \) is weakly sequentially continuous.

Let \( \{ y_n(t) \} \) and \( \{ x_n(t) \} \) be sequence in \( C[I,E] \) weakly convergent to \( y(t) \) and \( x(t) \) respectively \((\forall t \in I)\), since \( f_1(t, s, \ldots) \) and \( f_2(t, s, \ldots) \) are weakly continuous. Then \( f_1(t, s, x_n(s), y_n(s)) \) and
$f_2(t,s,x_n(s),y_n(s))$ converge weakly to $f_1(t,s,x(s),y(s))$ and $f_2(t,s,x(s),y(s))$ respectively. Furthermore, $(\forall \varphi \in E^*) \varphi f_1(t,s,x_n(s),y_n(s))$ and $\varphi f_2(t,s,x(s),y(s))$ converge strongly to $\varphi f_1(t,s,x(s),y(s))$ and $\varphi f_2(t,s,x(s),y(s))$ respectively. Applying Lebesgue dominated convergence theorem, we get

$$\varphi \left( \lambda_1 \int_0^1 f_1(t,s,x_n(s),y_n(s)) d_x g_1(t,s) \right) = \lambda_1 \int_0^1 \varphi \left( f_1(t,s,x_n(s),y_n(s)) \right) d_x g_1(t,s), \quad \forall \varphi \in E^*, \ t,s \in I.$$ 

And

$$\varphi \left( \lambda_2 \int_0^1 f_2(t,s,x_n(s),y_n(s)) d_x g_2(t,s) \right) = \lambda_2 \int_0^1 \varphi \left( f_2(t,s,x_n(s),y_n(s)) \right) d_x g_2(t,s), \quad \forall \varphi \in E^*, \ t,s \in I.$$

Thus, $A$ is weakly sequentially continuous on $Q$.

Since all conditions of Theorem 1 are satisfied, then the operator $A$ has at least one fixed point $(x,y) = u \in Q$ and the coupled system of Urysohn-Stieltjes functional integral equations (2) has at least one weak solution.

### 3 Hammerstien-Stieltjes coupled system

This section, as an application, deals with the existence of weak continuous solution for the coupled system of Hammerstien-Stieltjes functional integral equations (3).

Consider the following assumption:

(iii)* Let $h_i, l \times E \times E \rightarrow E$ and $k_i: l \times l \rightarrow \mathbb{R}_+$ assume that $h_i, k_i$ satisfy the following assumptions:

1. $h_i(s,x(s),y(s))$ are weakly-weakly continuous functions, $i = 1,2$.
2. There exists a continuous functions $m_i(t)$ and constants $b_i > 0$ such that
   $$\|h_i(t,x,y)\| \leq m_i(t) + b_i \max\{\|x\|,\|y\|\},$$
   for $t,s \in I$, $x,y \in E$. Moreover, we put $M^* = \max\{m_i(t): t \in I\}$, $M^* > 0$, $i = 1,2$.
3. $k_i(t,s)$ is continuous function such that $K = \sup_{t} |k_i(t,s)|$ is positive constant.

Definition 3. By a weak solution for the coupled system (3), we mean the pair of functions $(x,y) \in C[I,E] \times C[I,E]$ such that

$$\varphi(x(t)) = \varphi_1(t) + \lambda_1 \int_0^1 k_1(t,s)h_1(s,x(s),y(s))d_s g_1(t,s), \quad t \in I.$$
New for the existence of a weak solutions of (3), we have the following theorem.

**Theorem 4.** Let the assumptions (i),(ii),(iv)-(v) and (iii)* be satisfied. Then the coupled system of Hammerstien-Stieltjes functional integral equations (3) has at least one weak solution \((x,y) \in X\).

**Proof.** Let

\[ f_i(t,s,x(s),y(s)) = k_i(t,s)h_i(s,x(s),y(s)). \]

Then from the assumption (iii)*, we find that the assumptions of Theorem 3 are satisfied and result follows.

**Example:** Consider the functions \(g_i : I \times I \to \mathbb{R}\) defined by the formula

\[ g_1(t,s) = t^2 + ts, \quad t \in I, \]
\[ g_2(t,s) = t^2(2s + 1), \quad t \in I. \]

It can be easily seen that the functions \(g_1(t,s)\) and \(g_2(t,s)\) satisfies assumptions (iv)-(vi) given in Theorem 2. In this case, the coupled system of the Urysohn-Stieltjes integral equations (2) having the form

\[ x(t) = p_1(t) + \lambda_1 \int_0^1 t f_1(t,s,x(s),y(s))ds, \quad t \in I, \]
\[ y(t) = p_2(t) + \lambda_2 \int_0^1 2t^2 f_2(t,s,x(s),y(s))ds, \quad t \in I. \]

Therefore, the coupled system (4) has at least one weak solution \((x,y) \in X\), if the functions \(p_i, \lambda_i\) and \(f_i\) satisfy the assumptions (i),(ii) and (iii).
References


