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Stability of Periodic solutions by Krasnoselskii fixed point theorem of neutral nonlinear system of dynamical equation with Variable Delays

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Abstract

The fixed point theorem is used in this study to provide stability results for the zero solution of a nonlinear neutral system of differential equations with functional delay.

Key words: Contraction mapping, stability, nonlinear neutral, differential equation, integral equation.

استقرار الحلول الدورية للنظام غير الخطي المحايد للمعادلة الديناميكية ذات المتغيرات المتأخرة

أميرة علي محمد بن فايد & رفيق عبدالفتاح فرج المنصوري & عائشة امبية مفتاح المحمد

الملخص

يتم استخدام نظرية النقطة الثابتة في هذه الدراسة لتوفير نتائج الاستقرار للحل الصفري لنظام محايد غير خطي للمعادلات التفاضلية التي تمتلك الدوال التفاضلية بدلالة الزمن $\{t\}$.

الكلمات المفتاحية : معادلة تكاملية – معادلة تفاضلية – غير خطي محايد – استقرار – دالة انكماش .

1.INTRODUCTION

Over the past two decades, the stability difficulties of the time-delay systems of neutral type have received considerable attention[1,2,3,4,5,6]. Practical examples of such systems include electrical distribution networks containing lossless transmission lines, and population ecology, and vibration of masses attached to an elastic bar [7,8,9].

Lyapunov functions (LF) have been the commonly deployed tool to obtain boundedness, stability, and the existence of periodic solutions of differential equations, differential equations with functional delays, and functional differential equations [10,11,12]. When using LF to investigate differential equations with applicable delays, several difficulties arise if the postponement is unbounded [13,14]. Furthermore, getting the necessary and appropriate conditions is considerably more difficult.

Boundedness, stability, and periodic solutions of differential equations, differential equations with functional delays, and functional differential equations have all been studied using Lyapunov functions (LF) [10,11,12].. However, if the postponement is unbounded, many issues arise when employing LF to explore differential equations with applicable delays [13,14]. Furthermore, obtaining the necessary and appropriate circumstances is significantly more challenging.

For the past ten years, many researchers have looked at specific problems that have posed significant challenges to that theory and proposed answers through various fixed-point theorems. Many of these difficulties can be solved using fixed point theory, including Burton and Furumochi [15,16,17].

Ben Fayed et al [18] showed an investigation into the possibility of finding the existence and uniqueness of periodic solutions of the nonlinear neutral system of differential equations by employing Krasnoselskii's fixed point theorem under slightly more stringent conditions and by applying the solution of the fundamental matrix solution of $y' = A(t)y$.

This paper is motivated by the limitations listed in [19]. We study the stability results of the zero solution method of the nonlinear neutral system of the following differential equations:

$$\frac{d}{dt}x(t) = A(t)x(t) + \frac{d}{dt}Q\left(t, x(t-g(t))\right) + \int_0^t D_1(u,s)\, f_1(t,s,x(s))ds + \int_0^t D_2(u,s)\, f_2(t,s,x(s))ds \qquad (1.1)$$

,with the assumptions of the initial function as in:

$$x(t) = \psi(t),\ t \in [m_0, t_0]$$

where $\psi \in C([m_0,t_0],\mathbb{R}^n), m_0 = \inf\{t - \tau(t): t \geq t_0\}$, and $A(\cdot)$ is nonsingular, $n \times n$ matrix with continuous real-valued functions as its elements, $\tau(t)$ being scalar, continuous, and $\tau(t) \geq \tau^* > 0$. The functions $Q: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $G: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

In the analysis, we use the fundamental matrix solution to convert the system (1.1) into an integral equation. Where we derive a fixed point mapping approach consecutively, we define a suitable complete space prudently with respect to the initial condition so that the mapping is a contraction. Using Banach's contraction mapping principle, we obtain a solution for this mapping, and hence we quantify an asymptotically stable answer for (1.1).

This paper is structured as follows: in section 2, we present some definitions, remarks, and the inversion formulas of the system (1.1). The main results are illustrated in the section 3.

2. PRELIMINARIES

For $T > 0$ let $\mho_T$ be the set of all n-vector continuous functions $x(t)$, periodic in t of period T. Then$(\mho_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x(.)\| = |sup_{t\in\mathbb{R}}x(t)| = \left|sup_{t\in[0,T]}x(t)\right|$$

Where $|\cdot|$ denotes the infinity norm for $x \in \mathbb{R}^n$.

In addition, if A is an $n \times n$ real matrix, then we define the norm of A by $|A| = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$. First, we make the following definition.

Let $\psi \in C([m_0, t_0], \mathbb{R}^n)$ be a given continuous bounded initial function. We denote such a solution by $x(t) = x(t, t_0, \psi)$. From the existence theory we can conclude that for each $\psi \in C([m_0, t_0], \mathbb{R}^n)$, there exists a unique solution $x(t) = x(t, t_0, \psi)$ of (1.1) defined on $[t_0, \infty)$. We define $\| \psi \| = \sup\{|\psi(t)| : m_0 \le t \le t_0\}$.

We mention to some definition for fundamental matrix, see also [20]

Definition 2.1. If the matrix *A(t)*is periodic of period *T*, then the linear system

$$x'(t) = A(t)x(t) \qquad (2.1)$$

We said to be noncritical with respect to *T* if it has no periodic solution of period *T* except the trivial solution $x(t) = 0$

Definition 2.2. An Identity real matrix $n \times n$ function $t \to \mathrm{K}(t)$, defined on an open interval J, is called a matrix solution of the homogeneous linear system (2.1) if each of its columns is a (vector) solution.

Definition 2.3. A set of n solutions of the homogeneous linear differential equation (2.1), all defined on the same open interval J, is called a fundamental set of solutions on J if the solutions are linearly independent functions on J.

Definition 2.4. A matrix solution is called a fundamental matrix solution if its columns form a fundamental set of solutions. In addition, a fundamental matrix solution $t \to \mathrm{K}(t)$ is called the principal fundamental matrix solution at $t_0 \in J$ if $\mathrm{K}(t_0) = I$, where I denotes the $n \times n$ identity matrix.

Definition 2.5. The state transition matrix for the homogeneous linear system (2.1) on the open interval J is the family of fundamental matrix solutions $t \to \mathrm{K}(t, r)$ parametrized by $r \in J$ such that $\mathrm{K}(r, r) = I$.

In this paper we assume that, for $t \in \mathbb{R}, x, y, z, w \in \mathbb{R}^n$, the functions $Q(t,x), D_1(u,s), D_2(u,s), f_1(t,x,y)$ and $f_2(t,x,y)$ are globally Lipschitz continuous in x and y, respectively. That, there are positive constants $E_1, E_2, E_3, E_4 E_5, E_6$ and E_7 such that

$$|Q(t,x) - Q(t,y)| \le E_1 \| x - y\| \tag{2.2}$$

$$|f_1(t,x,y) - f_1(t,z,w)| \le E_2\|x - z\| + E_3\|y - w\| \tag{2.3}$$

$$|f_2(t,x,y) - f_2(t,z,w)| \le E_4\|x - z\| + E_5\|y - w\| \tag{2.4}$$

$$\int_0^t |D_1(u,s)| ds \le E_6 < \infty \tag{2.5}$$

and

$$\int_0^t |D_2(u,s)| ds \le E_7 < \infty \tag{2.6}$$

Proposition 2.6 ([20, Proposition 2.14]). If $t \to \mathrm{K}t)$ is a fundamental matrix solution for the system (2.1) on J, then $\mathrm{K}(t,r) := \mathrm{K}(t)\mathrm{K}^{-1}(r)$ is the state transition matrix. Also, the state transition matrix satisfies the Chapman-Kolmogorov identities

$$\mathrm{K}(r,r) = I,\ \mathrm{K}(t,s)\mathrm{K}(s,r) = \mathrm{K}(t,r)$$

Throughout this paper, $\mathrm{K}(t)$ will denote a fundamental matrix solution of the homogeneous (unperturbed) linear problem (2.1). First, we have to transform (1.1) into an equivalent equation that possesses the same basic structure and properties to define a fixed point mapping.

Lemma 2.7

$x(\mathrm{t})$ is a solution of the equation (1.1) if and only if

$$x(t) = Q(t, x(t-g(t))) + \mathrm{K}(t,t_0)\left[x(t_0) - Q\left(t_0, x(t_0 - g(t_0))\right)\right] + \int_{t_0}^{t} \mathrm{K}(t,s)[A(s)Q(s,x(s-g(s))) + \int_0^s D_1(u,v)\, f_1(s,v,x(v))dv + \int_0^s D_2(u,v)\, f_2(s,v,x(v))dv]ds$$

Proof. Let x be a solution of (1.1) and $K(t)$ is a fundamental system of solutions of (2.1). Rewrite the equation (1.1) as

$$\frac{d}{dt}\left[x(t) - Q\left(t, x(t-g(t))\right)\right]$$
$$= A(t)x(t) + \int_0^t D_1(u,s)\, f_1(t,s,x(s))ds + \int_0^t D_2(u,s)\, f_2(t,s,x(s))ds$$

Define a new function z by $z(t) = K^{-1}(t)[x(t) - Q(t, x(t-g(t)))]$. We have

$$\frac{d}{dt}z(t) = \frac{d}{dt}K^{-1}(t)\left[x(t) - Q\left(t, x(t-g(t))\right)\right]$$
$$+K^{-1}(t)\frac{d}{dt}[x(t) - Q(t, x(t-g(t)))]$$

By the Proposition 2.6, it follows that

$$\frac{d}{dt}K^{-1}(t) = -K^{-1}(t)A(t)$$

Then

$$\frac{d}{dt}[x(t) - Q(t, x(t-g(t)))] = A(t)[x(t) - Q(t, x(t-g(t)))] + K(t)\frac{d}{dt}z(t)$$

Thus,

$$A(t)x(t) + \int_0^t D_1(u,s)\, f_1(t,s,x(s))ds + \int_0^t D_2(u,s)\, f_2(t,s,x(s))ds$$
$$= A(t)[x(t) - Q(t, x(t-g(t)))] + K(t)\frac{d}{dt}z(t)$$

and

$$\frac{d}{dt}z(t) = K^{-1}(t)[A(t)Q(t, x(t-g(t))) + \int_0^t D_1(u,s)\, f_1(t,s,x(s))ds$$
$$+ \int_0^t D_2(u,s)\, f_2(t,s,x(s))]ds \qquad (2.7)$$

Also notice that $z(t_0) = K^{-1}(t_0)\left[x(t_0) - Q\left(t_0, x\left(t_0 - g(t_0)\right)\right)\right]$.

An integration of the equation (2.2) from t_0 to t yields

$$z(t) - z(t_0) = \int_{t_0}^{t} K^{-1}(s)[A(s)Q(s, x(s - g(s))) + \int_0^s D_1(u,v)\, f_1\left(s, v, x(v)\right) \mathrm{d}v + \int_0^s D_2(u,v)\, f_2\left(s, v, x(v)\right) dv] ds$$

Or, in other words,

$$K^{-1}(t)\left[x(t) - Q\left(t, x\left(t - g(t)\right)\right)\right] = K^{-1}(t_0)\left[x(t_0) - Q\left(t_0, x\left(t_0 - g(t_0)\right)\right)\right] + \int_{t_0}^{t} K^{-1}(s) \left[A(s)Q\left(s, x\left(s - g(s)\right)\right) + \int_0^s D_1(u,v)\, f_1\left(s, v, x(v)\right) \mathrm{d}v + \int_0^s D_2(u,v)\, f_2\left(s, v, x(v)\right) dv\right] ds \quad 2.8$$

For studying the behavior (2.5), (2.8) can be expressed by

$$x(t) = Q(t, x(t - g(t))) + K(t, t_0)\left(x(t_0) - Q\left(t_0, x\left(t_0 - g(t_0)\right)\right)\right) + \int_{t_0}^{t} K(t,s)[A(s)Q(s, x(s - g(s))) + \int_0^s D_1(u,v)\, f_1\left(s, v, x(v)\right) \mathrm{d}v + \int_0^s D_2(u,v)\, f_2\left(s, v, x(v)\right) dv] ds.$$

The converse implication is easily obtained and the proof is complete.

If $x: [t_0, \infty) \to \mathbb{R}^n$ is a given solution of (1.1), then discussing the behavion of another solution y of this equation relative to the solution x, i.e. discussing the behavior of the difference $y - x$ is equivalent to studying the behavior of the solution $z = y - x$ of the equation

$$z'(t) = A(t)[y(t) - x(t)] + \frac{d}{dt}\left[Q\left(t, z(t-g(t)) + x(t-g(t))\right) - Q\left(t, x(t-g(t))\right)\right] + \int_0^t D_1(u,s)\, f_1(t,s,z(s)+x(s))ds + \int_0^t D_2(u,s)\, f_2(t,s,z(s)+x(s))ds - \int_0^t D_1(u,s)\, f_1(t,s,x(s))ds + \int_0^t D_2(u,s)\, f_2(t,s,x(s))ds$$

relative to the trivial solution $z \equiv 0$. Thus we may, without loss in generality, assume that (1.1) has the trivial solution as a reference solution, i.e.

$$Q(t,0) = \int_0^t D_1(u,s)\, f_1(t,s,0)ds = \int_0^t D_2(u,s)\, f_2(t,s,0)ds \equiv 0$$

an assumption we shall henceforth make.

3. Conclusion

Our aim here is to give a necessary and sufficient condition for asymptotic stability of the zero solution of (1.1). Stability definitions may be found in [15], for example. By the Lemma 2.7, let a mapping $\mathcal{H}$ given by $(\mathcal{H}\varphi)(t) = \psi(t)$ for $t \in [m_0, t_0]$ and for $t \geq t_0$

$$(\mathcal{H}\varphi)(t) = Q\left(t, \varphi(t-g(t))\right) + \mathrm{K}(t,t_0)\left[\psi(t_0) - Q\left(t_0, \psi(t_0 - g(t_0))\right)\right] + \int_{t_0}^t \mathrm{K}(t,s) \left[A(s)Q\left(s, x(s-g(s))\right) + \int_0^s D_1(u,v)\, f_1(s,v,x(v))dv + \int_0^s D_2(u,v)\, f_2(s,v,x(v))dv\right]ds \quad (3.1)$$

and define the space $\mho_T$ by

$$\mho_T = \{\varphi: \mathbb{R} \to \mathbb{R}^n, \varphi(t) = \psi(t) \text{ if } m_0 \leq t \leq t_0, \varphi(t) \to 0 \text{ as } t \to \infty \;\; \varphi \in C \text{ is bounded}\}$$

Then, $(\mho_T, \|\cdot\|)$ is a complete metric space where $\|\cdot\|$ is the supremum norm.

Theorem 3.1. Assume (2.2) - (2.6) hold. Further assume that

$$\mathrm{K}(t) \to 0 \text{ as } t \to \infty \quad (3.2)$$
$$t - g(t) \to \infty \text{ as } t \to \infty \quad (3.3)$$

$$Q(t,0) \to 0 \text{ as } t \to \infty \quad (3.4)$$

$$f_1(t,0,0) \to 0 \text{ as } t \to \infty \quad (3.5)$$

$$f_2(t,0,0) \to 0 \text{ as } t \to \infty \quad (3.6)$$

and there is $\alpha > 0$ such that

$$E_1 + \int_{t_0}^{t} |\mathrm{K}(t,s)|(E_6[E_2+E_3] + E_7[E_4+E_5] + |A|E_1)ds \le \alpha < 1,\ t \ge t_0$$

hold. Then every solution $x(t,t_0,\psi)$ of (1.1) with small continuous initial function ψ, is bounded and asymptotically stable. Moreover, the zero solution is stable at t_0.

Proof.

Let the mapping $\mathcal{H}$ defined by (3.1). Since Q, D_1, D_2, f_1 and f_2 are continuous, it is easy to show that $\mathcal{H}$ is. Let ψ be a small given continuous initial function with

$\parallel \psi \parallel < \delta\ (\delta > 0)$. Since $\varphi \in \mho_T$ then there exist a positive constant M, such that

$\parallel \varphi \parallel \le M$, this and the conditions (3.2)- (3.6) imply

$$|(\mathcal{H}\varphi)(t)| \le \left|Q\left(t,\varphi(t-g(t))\right)\right| + |\mathrm{K}(t,t_0)|\left[|\psi(t_0)| + \left|Q\left(t_0,\psi(t_0-g(t_0))\right)\right|\right] + \int_{t_0}^{t} |\mathrm{K}(t,s)| \left[|A(s)|\left|Q\left(s,x(s-g(s))\right)\right| + \left|\int_0^s D_1(u,v) f_1(s,v,x(v))dv + \int_0^s D_2(u,v) f_2(s,v,x(v))dv\right|\right] ds$$

$$\le E_1M + |\mathrm{K}|\delta(1+E_1) + M\int_{t_0}^{t} |\mathrm{K}(t,s)|(E_6[E_2+E_3] + E_7[E_4+E_5] + |A|E_1)ds$$

$$\le |\mathrm{K}|\delta(1+E_1) + \alpha M$$

which implies $\mathcal{H}\varphi$ is bounded, for the right δ. Next we show that $(\mathcal{H}\varphi)(t) \to 0$ as $t \to \infty$. The first term on the right side of (3.1) tends to zero, by condition (3.2). Also, the second term on the right side tends to zero, because of (3.3), (3.4) and the fact that $\varphi \in \mho_T$. Let $\epsilon > 0$ be given, then there exists a $t_1 > t_0$ such that for $t > t_1$, $|\varphi(t - g(t))| < \epsilon$. By the condition (3.2), there exists a $t_2 > t_1$ such that for $t > t_2$ implies that

$$|\mathrm{K}(t, t_2)| < \frac{\epsilon}{\alpha M}$$

Thus for $t > t_2$, we have

$$\begin{aligned}
&\int_{t_0}^{t} |\mathrm{K}(t,s)|([E_2 + E_3]|\varphi(s)| + [E_4 + E_5]|\varphi(s)| + |A|E_1|\varphi(s - g(s))|)ds \\
&\le M \int_{t_0}^{t_1} |\mathrm{K}(t,s)|([E_2 + E_3] + [E_4 + E_5] + |A|E_1)ds \\
&+ \epsilon \int_{t_1}^{t} |\mathrm{K}(t,s)|([E_2 + E_3] + [E_4 + E_5] + |A|E_1)ds \\
&\le M|\mathrm{K}(t,t_2)| \int_{t_0}^{t_1} |\mathrm{K}(t_2,s)|([E_2 + E_3] + [E_4 + E_5] + |A|E_1)ds + \alpha\epsilon \\
&\le \alpha M|\mathrm{K}(t,t_2)| + \alpha\epsilon < \alpha\epsilon + \epsilon
\end{aligned}$$

Hence, $(\mathcal{H}\varphi)(t) \to 0$ as $t \to \infty$. It is natural now to prove that $\mathcal{H}$ is contraction under the supremum norm. Let, $\varphi_1, \varphi_2 \in \mho_T$. Then

$$\begin{aligned}
&|(\mathcal{H}\varphi_1)(t) - (\mathcal{H}\varphi_2)(t)| \\
&\le |Q(t, \varphi_1(t - g(t))) - Q(t, \varphi_2(t - g(t)))| \\
&+ \int_{t_0}^{t} |\mathrm{K}(t,s)|(E_6[E_2 + E_3]\|\varphi_1 - \varphi_2\| + E_7[E_4 + E_5]\|\varphi_1 - \varphi_2\| + E_1|A|\|\varphi_1 - \varphi_2\|)ds \\
&\le \alpha\|\varphi_1 - \varphi_2\|
\end{aligned}$$

Hence, the contraction mapping principle implies, $\mathcal{H}$ has a unique fixed point in $\mho_T$ which solves (1.1), bounded and asymptotically stable. The stability of the zero solution of (1.1) follows simply by replacing M by ϵ.

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