On n-adic Algebra

Nabila M. Bennour *, Kahtan H. Alzubaidy

Department of Mathematics, University of Benghazi, Libya,

Highlights

- n-adic algebra is a generalization of monadic algebra by extending the number of quantifiers from 1 to n.
- The method used in deduction of monadic algebra is generalized to the case of n-adic algebra with respect to terms.

ARTICLE INFO

Article history:
Received 11 June 2019
Revised 19 December 2019
Accepted 27 December 2019
Available online 30 December 2019

Keywords:
Monadic algebra, n-adic algebra, terms, deduction

Address of correspondence:
E-mail address: n.benour@yahoo.com
N. M. Bennour

1. Introduction

n-adic algebra is a generalization of monadic algebra by extending the number of quantifiers from 1 to n. Our main results are given in theorems 12, 13. The method used in (Alzubaidy, 2007) is generalized to the case of n-adic algebra. A Boolean algebra is an algebraic structure $B =< A, V, \land, \lor, 0, 1>$ in which $\land, \lor$ are binary operations on $A$, $\lnot$ is a unary operation, while 0 and 1 are nullary operations (distinguished elements of $A$), which satisfy the following conditions: For arbitrary $a, b, c \in A$

i) $a \lor b = b \lor a, a \land b = b \land a$

ii) $(a \lor b) \lor c = a \lor (b \lor c), (a \land b) \land c = a \land (b \land c)$

iii) $a \lor (b \land c) = (a \lor b) \land (a \lor c), a \land (b \land c) = (a \land b) \lor (a \land c)$

iv) $a \lor (a \land b) = a, a \land (a \lor b) = a$

v) $a \lor a' = 1, a \land a' = 0$.

A partial order $\leq$ on a Boolean algebra is defined by $a \leq b$, if $a \land b = a$ or $a \lor b = b$.

A Boolean algebra $B$ is said to be complete if any subset $A$ of $B$ has infimum and supremum in $B$.

Suppose that $B$ is a complete Boolean algebra. An existential operator on $B$ is a mapping $\exists: B \rightarrow B$ such that:

i) $\exists(0) = 0$

ii) $a \leq \exists(a)$ for any $a \in B$

iii) $\exists(a \land \exists(b)) = \exists(a \land \exists(b))$ for any $a, b \in B$

$(B, \exists)$ is called monadic algebra.

2. n-adic algebras

An n-adic algebra $(B, \exists_1, ..., \exists_n)$ consists of a complete Boolean algebra $B$ and $n$ mappings $\exists_i: B \rightarrow B; 1 \leq i \leq n$ called existential operators such that:

i) $\exists_i(0) = 0$

ii) $a \leq \exists_i(a)$ for any $a \in B$

iii) $\exists_i((a \land \exists_i(b)) = \exists_i(a) \land \exists_i(b)$ for any $a, b \in B$

iv) $\exists_i \exists_j = \exists_j \exists_i$.

Note that 0-adic algebra is just Boolean algebra $B$ and 1-adic algebra is the monadic algebra $(B, \exists)$. Obviously, the n-adic algebra is a locally finite polyadic algebra in the sense given in (Alzubaidy and Bennour, 2012).

The universal operator $\forall: B \rightarrow B; 1 \leq i \leq n$ is defined by $\forall_i(a) = (\exists_i(a))'$, for any $a \in B$.

Proposition 1. (Alzubaidy, 2007)

i) $\forall_i(0) = 0$

ii) $a \leq \forall_i(a)$ for any $a \in B$

iii) $\forall_i((a \lor \forall_i(b)) = \forall_i(a) \lor \forall_i(b)$ for any $a, b \in B$

iv) $\forall_i \forall_j = \forall_j \forall_i$.

Proposition 2. (Alzubaidy, 2007)

$\exists_i \forall_j = \forall_j \exists_i$ if $i \neq j$.

Suppose $Q_i$ is an existential operator or universal operator. By the previous proposition, we have

Corollary 3.

$Q_1 Q_2 ... Q_n \leq \forall \forall ... \exists \exists ... \exists$. 

© 2019 University of Benghazi. All rights reserved. ISSN 2663-1407; National Library of Libya, Legal number: 390/2018
2.1 Functional n-adic Algebra

Let $X$ be a nonempty set and $B$ a Boolean algebra. Suppose that

$$B^X = \{ p : p(X) \to B \text{ is a function} \},$$

where $X^n$ is the Cartesian product of $n$ copies of $X$.

For $p, q \in X^n$ define $p \land q, p \lor q, p'$ and 0, 1 pointwise as follows:

$$(p \land q)(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) \land q(x_1, \ldots, x_n),$$

$$(p \lor q)(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) \lor q(x_1, \ldots, x_n),$$

$$p'(x_1, \ldots, x_n) = (p(x_1, \ldots, x_n))^\prime = \{0\text{ if } x_i = 1\text{ for }i = 1, \ldots, n \text{ or }1\text{ if } x_i = 0\text{ for }i = 1, \ldots, n \}.$$ For any $t \in X^n$, $t$ is a term of variables appearing explicitly is less than or equal $n$.

A term $t$ is $n$-ary if the number of variables appearing explicitly is less than or equal $n$.

The term $t$ defines a function $t_a : A^n \to A$ as $t_a(a_1, \ldots, a_n) = t(x_1, \ldots, x_n)$ for $a_1, \ldots, a_n \in A$, where $A$ is an algebra of type $F$.

The set $T(X)$ can be transformed into an algebra (Burris and Sahaappanavar, 1981). The term algebra $T(X)$ of type $F$ over $X$ has as its universe the set $T(X)$ and the fundamental operations satisfies:

$$f_t(x_1, \ldots, x_n) \mapsto (t)(x_1, \ldots, x_n) \text{ for } f \in F_n$$

Now consider the functional $n$-adic algebra $(B^X, \exists_i)$. A term $t$ defines a function $t : B^X \to B^X$ as follows:

$$t_a(x_1, \ldots, x_n) = t(x_1, \ldots, x_n)$$

$$(x_1, \ldots, x_n) \in (B^X, \exists_i)$$

$$(B^X, \forall, \land, \lor, 0, 1)$$

is an Boolean algebra.

Proposition 4. (Alzubaidy, 2007)

$(B^X, \exists_i)$ is a monadic algebra.

Existential and universal operators $\exists_i \forall_i$ on the functional Boolean algebra $B^X$ are defined as follows:

for each $p \in B^X$, $\exists_i(p)(x_1, \ldots, x_n) = \sup_i (p(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in X^n)$

$$(p \land q)(x_1, \ldots, x_n) = \inf_i (p(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in X^n).$$

Proposition 6.

$(B^X, \exists_i)$ is a $n$-adic algebra, $1 \leq i \leq n$.

Proof

i) $\exists_i(0)(x_1, \ldots, x_n) = \sup_i (0(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in X^n) = 0(x_1, \ldots, x_n) = 0$. Then $\exists_i(0) = 0$

ii) $\exists_i(p)(x_1, \ldots, x_n) = \sup_i (p(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in X^n) \supset p(x_1, \ldots, x_n).$ Then $\exists_i(p) \supset p$

iii) $\exists_i(p \land q)(x_1, \ldots, x_n) = \sup_i (p(x_1, \ldots, x_n) \land q(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in X^n.$

iv) $\exists_i(p)(x_1, \ldots, x_n) \land q(x_1, \ldots, x_n) = \sup_i (p(x_1, \ldots, x_n) \land q(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in X^n).$

A more general case is the locally finite polyadic algebra. This is given in (Alzubaidy, 2007).

3. Terms

Let $X$ be a set of variables and $F$ be a type of algebra. The set $T(X)$ of terms of type $F$ over $X$ is the smallest set such that:

i) $X \cup F_0 \subseteq T(X)$

ii) If $t_1, \ldots, t_n \in T(X)$ and $f \in F_n$ then $f(t_1, \ldots, t_n) \in T(X)$ (Burris and Sahaappanavar, 1981).

For $t \in T(X)$ we often write $t$ as $t(x_1, \ldots, x_n)$ to indicate that the variables occurring in $t$ are among $x_1, \ldots, x_n$.

4. Deduction

4.1 n-adic ideals and filters

A subset $U$ of a Boolean algebra $B$ is called a Boolean ideal if

i) $0 \in U$

ii) $a \lor b \in U$ for any $a, b \in U$

iii) If $a \in U$ and $b \leq a$ then $b \in U$.

A subset $F$ of a Boolean algebra $B$ is called a Boolean filter if

i) $1 \in F$

ii) $a \land b \in F$ for any $a, b \in F$

iii) If $a \in F$ and $b \geq a$ then $b \in F$.

A subset $U$ of an $n$-adic algebra $B$ is called an $n$-adic ideal of $B$ if

i) $U$ is a Boolean ideal

ii) $\exists_i(U)(a) \in U, a \in U$ for $1 \leq i \leq n$. 

49
A subset $F$ of an $n$-adic algebra $B$ is called an $n$-adic filter of $B$ if

i) $F$ is a Boolean filter

ii) $\forall_i(F)(a) \in F$, $a \in F$ for $1 \leq i \leq n$.

Thus we have the following two propositions.

**Proposition 8.**

There is a one to one correspondence between ideals and filters.

**Proposition 9.**

The set of all $n$-adic ideals and the set of all $n$-adic filters are closed under the arbitrary intersection.

Let $B$ be an $n$-adic algebra and $\Gamma \subseteq B$. Let $U(\Gamma)$ denote the least $n$-adic ideal containing $\Gamma$ and $F(\Gamma)$ denote the least $n$-adic filter containing $\Gamma$. We say that $U(\Gamma)$ and $F(\Gamma)$ are generating by $\Gamma$.

**Proposition 10.**

Let $B$ be an $n$-adic algebra and $\Gamma \subseteq B$. Then

i) $U(\Gamma) = \{b \in B : b \leq x_1 \lor x_2 \lor \ldots \lor x_n \text{ for some } x_1, \ldots, x_n \in \Gamma \} \cup \{0\}$

ii) $F(\Gamma) = \{b \in B : b \geq x_1 \land x_2 \land \ldots \land x_n \text{ for some } x_1, \ldots, x_n \in \Gamma \} \cup \{1\}$

**Proof**

i) Let $J = \{b \in B : b \leq x_1 \lor x_2 \lor \ldots \lor x_n \text{ for some } x_1, \ldots, x_n \in \Gamma \} \cup \{0\}, 0 \in J$. Let $b_1, b_2 \in J$ then $b_1 \leq x_1 \lor x_2 \lor \ldots \lor x_n$ and $b_2 \leq y_1 \lor y_2 \lor \ldots \lor y_m$ for some $x_i, y_i \in \Gamma$.

$b_1 \lor b_2 \leq x_1 \lor x_2 \lor \ldots \lor x_n \lor y_1 \lor y_2 \lor \ldots \lor y_m$. Therefore $b_1 \lor b_2 \in J$.

If $a \leq b \leq x_1 \lor x_2 \lor \ldots \lor x_n$, then $a \in J$. Then $J$ is a Boolean ideal containing $\Gamma$.

If $a \in J$, then $a \leq x_1 \lor x_2 \lor \ldots \lor x_n$, $\exists_i(J)(a) \leq \exists_i(J)(x_1 \lor x_2 \lor \ldots \lor x_n) = \exists_i(J)(x_1) \lor \ldots \lor \exists_i(J)(x_n)$. Then $\exists_i(J)(a) \subseteq U(\Gamma)$.

ii) A similar argument leads to (ii).

A filter $F$ of an $n$-adic algebra is called ultrafilter if $F$ is maximal with respect to the property that $0 \notin F$.

Ultrafilters satisfy the properties of the following proposition.

**Proposition 11.** (Burris and Shakappanavar, 1981)

Let $F$ be a filter of $n$-adic algebra, then

i) $F$ is an ultrafilter of $B$ iff for any $a \in F$ exactly one of $a, a'$ belong to $F$.

ii) $F$ is an ultrafilter of $B$ iff $0 \notin F$ and $a \lor b \in F$ iff $a \in F$ or $b \in F$ for any $a, b \in F$.

iii) If $a \in F \setminus B$, then there is an ultrafilter $L$ such that $F \subseteq L$ and $a \in L$.

Let $\Gamma \subseteq B$, the ultrafilter containing $F(\Gamma)$ is denoted by $UF(\Gamma)$.

A mapping $\mu : B_1 \rightarrow B_2$ between two $n$-adic algebras is called $n$-adic homomorphism if

i) $\mu$ is a Boolean homomorphism,

ii) $\mu \exists_i = \exists_i \mu$.

Obviously, $\mu \forall_i = \forall_i \mu$.

For $\Gamma \subseteq B, b \in B$, we define the deduction $\Gamma \vdash b$ iff $b \in UF(\Gamma)$.

**Theorem 12.**

Let $\Gamma \subseteq B^*$

i) $\Gamma \vdash p \land q$ iff $\Gamma \vdash p$ and $\Gamma \vdash q$

ii) $\Gamma \vdash p \lor q$ iff $\Gamma \vdash p$ or $\Gamma \vdash q$

iii) $\Gamma \vdash p$ iff $\Gamma \not\vdash p'$

iv) $\Gamma \vdash p$ iff $\forall_i(p, 1 \leq i \leq n)$

v) $F(\Gamma) \vdash p(x)$ iff $F(\Gamma) \vdash \exists_i p(x), 1 \leq i \leq n$.

**Proof**

(i), (ii) follow from the definition of filter and (Alzubaidy, 2007, p.2)

iii) follows from proposition 10.

iv) This is by $\forall_i p(x) \leq p(x)$ and $\forall_i F \subseteq F$.

v) This is by the definition $\exists_i p(x) = \sup(p(x)). x = (x_1, \ldots, x_n)$.

Note that $t_i(UF(\Gamma)) = UF(t_i(\Gamma))$. Define $\Gamma \vdash pt$ if $pt \in F(t_i(\Gamma))$. Then theorem 12 can be generalized with respect to terms as follows:

**Theorem 13.**

i) $\Gamma \vdash pt \land qt$ iff $\Gamma \vdash pt$ and $\Gamma \vdash qt$

ii) $\Gamma \vdash pt \lor qt$ iff $\Gamma \vdash pt$ or $\Gamma \vdash qt$

iii) $\Gamma \vdash pt$ iff $\Gamma \not\vdash p't$

iv) $\Gamma \vdash p$ iff $\forall_i pt, 1 \leq i \leq n$

v) $F(\Gamma) \vdash pt(x)$ iff $F(\Gamma) \vdash \exists_i pt(x), 1 \leq i \leq n$.

**5. References**


