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On n-adic Algebra

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Highlights

- **n-adic algebra is a generalization of monadic algebra by extending the number of quantifiers from 1 to n.**
- **The method used in deduction of monadic algebra is generalized to the case of n-adic algebra with respect to terms.**

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ABSTRACT

This paper studies n-adic algebras with terms. Deduction is also done in these algebras.

1. Introduction

n-adic algebra is a generalization of monadic algebra by extending the number of quantifiers from 1 to n. Our main results are given in theorems 12, 13. The method used in (Alzubaidy, 2007) is generalized to the case of n-adic algebra. A Boolean algebra is an algebraic structure $B =< A,\vee,\wedge,' ,0,1 >$ in which $\vee,\wedge$ are binary operations on $A,'$ is a unary operation, while 0 and 1are nullary operations (distinguished elements of A), which satisfy the following conditions: For arbitrary $a, b, c \in A$

i) $a \vee b = b \vee a, a \wedge b = b \wedge a$

ii) $(a \vee b) \vee c = a \vee (b \vee c), (a \wedge b) \wedge c = a \wedge (b \wedge c)$

iii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),\ \ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

iv) $a \vee (a \wedge b) = a,\ \ a \wedge (a \vee b) = a$

v) $a \vee a' = 1, a \wedge a' = 0.$

A partial order $\leq$ on a Boolean algebra is defined by $a \leq b$, if $a \wedge b = a$ or $a \vee b = b$.

A Boolean algebra B is said to be complete if any subset A of B has infimum and supremum in B.

Suppose that B is a complete Boolean algebra. An existential operator on B is a mapping $\exists: B \to B$ such that:

i) $\exists(0) = 0$

ii) $a \leq \exists(a)$ for any $a \in B$

iii) $\exists\big(a \wedge \exists(b)\big) = \exists(a) \wedge \exists(b)$ for any $a, b \in B$

$(B, \exists)$ is called monadic algebra.

2. n-adic algebras

An n-adic algebra $(n \geq 0)(B, \exists_1, \dots, \exists_n)$ consists of a complete Boolean algebra B and n mappings $\exists_i: B \to B; 1 \leq i \leq n$ called existential operators such that:

i) $\exists_i(0) = 0$

ii) $a \leq \exists_i(a)$ for any $a \in B$

iii) $\exists_i\big((a) \wedge \exists_i(b)\big) = \exists_i(a) \wedge \exists_i(b)$ for any $a, b \in B$

iv) $\exists_i\exists_j = \exists_j\exists_i$.

Note that 0-adic algebra is just Boolean algebra B and 1-adic algebra is the monadic algebra $(B, \exists)$. Obviously, the n-adic algebra is a locally finite polyadic algebra in the sense given in (Alzubaidy and Bennour, 2012).

The universal operator $\forall_i: B \to B; 1 \leq i \leq n$ is defined by $\forall_i(a) = (\exists_i(a)')'$, for any $a \in B$.

Proposition 1. (Alzubaidy, 2007)

i) $\forall_i(0) = 0$

ii) $a \leq \forall_i(a)$ for any $a \in B$

iii) $\forall_i\big((a) \wedge \forall_i(b)\big) = \forall_i(a) \wedge \forall_i(b)$ for any $a, b \in B$

iv) $\forall_i\forall_j = \forall_j\forall_i$.

Proposition 2. (Alzubaidy, 2007)

$\exists_i\forall_j = \forall_j\exists_i$ if $i \neq j$.

Suppose Q_i is an existential operator or universal operator. By the previous proposition, we have

Corollary 3.

$Q_1Q_2 \dots Q_n \leq \forall\forall \dots \exists\exists \dots \exists.$

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2.1 Functional n-adic Algebra

Let X be a nonempty set and B a Boolean algebra. Suppose that

$B^{X^n} = \{p|p: X^n \to B \text{ is a function}\}$, where X^n is the Cartesian product of n copies of X.

For $p, q \in X^n$ define $p \wedge q, p \vee q, p'$ and $0, 1$ pointwise as follows:

$(p \wedge q)(x_1, \dots, x_n) = p(x_1, \dots, x_n) \wedge q(x_1, \dots, x_n)$,

$(p \vee q)(x_1, \dots, x_n) = p(x_1, \dots, x_n) \vee q(x_1, \dots, x_n), p'(x_1, \dots, x_n) = (p(x_1, \dots, x_n))', 0(x_1, \dots, x_n) = 0$ and $1(x_1, \dots, x_n) = 1$ for any $(x_1, \dots, x_n) \in X^n$.

Proposition 4. (Alzubaidy, 2007)

$(B^{X^n}, \vee, \wedge, ', 0, 1)$ is an Boolean algebra.

Proposition 5. (Alzubaidy, 2007)

$(B^{X^n}, \exists)$ is a monadic algebra.

Existential and universal operators $\exists_i, \forall_i$ on the functional Boolean algebra B^{X^n} are defined as follows:

for each $p \in B^{X^n}$, $\exists_i(p)(x_1, \dots, x_n) = \sup_i$
$\{p(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\}$ and
$\forall(p)(x_1, \dots, x_n) = \inf_i \{p(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\}$.

Proposition 6.

$(B^{X^n}, \exists_i)$ is a n-adic algebra, $1 \le i \le n$

Proof

i) $\exists_i(0)(x_1, \dots, x_n) = \sup_i \{0(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} = 0(x_1, \dots, x_n) = 0$. Then $\exists_i(0) = 0$

ii) Let $\in B^{X^n}$, $\exists_i(p)(x_1, \dots, x_n) = \sup_i \{p(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} \ge p(x_1, \dots, x_n)$. Then $\exists_i(p) \ge p$

iii) Let $p, q \in B^{X^n}$, $\exists_i(p \wedge \exists_i(q))(x_1, \dots, x_n) = \sup_i \{p \wedge \exists_i(q)(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} = \sup_i \Big\{(p)(x_1, \dots, x_n) \wedge \sup_i (q)(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\Big\} = \sup_i \{p(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} \wedge \sup_i \{q(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} = \exists_i(p)(x_1, \dots, x_n) \wedge \exists_i(q)(x_1, \dots, x_n)$. Then $\exists_i(p \wedge \exists_i(q)) = \exists_i(p) \wedge \exists_i(q)$

iv) $\exists_i(p)(x_1, \dots, x_n))\, \exists_j(p)(x_1, \dots, x_n) = \sup_i \{p(x_1, \dots, x_n)\} \sup_j \{p(x_1, \dots, x_n)\} = \sup_j \{p(x_1, \dots, x_n)\} \sup_i \{p(x_1, \dots, x_n)\} = \exists_j(p)(x_1, \dots, x_n) \exists_i(p)(x_1, \dots, x_n)$. Then $\exists_i \exists_j = \exists_j \exists_i$.

A more general case is the locally finite polyadic algebra. This is given in (Alzubaidy, 2007).

3. Terms

Let X be a set of variables and $\mathcal{F}$ be a type of algebra. The set $T(X)$ of terms of type $\mathcal{F}$ over X is the smallest set such that: i) $X \cup \mathcal{F}_0 \subseteq T(X)$

ii) If $t_1, \dots, t_n \in T(X)$ and $f \in \mathcal{F}_n$ then $f(t_1, \dots, t_n) \in T(X)$ (Burris and Shakappanavar, 1981).

For $t \in T(X)$ we often write t as $t(x_1, \dots, x_n)$ to indicate that the variables occurring in t are among $x_1, \dots, x_n$.

A term t is n-ary if the number of variables appearing explicitly is less than or equal n.

The term t define a function $t_A: A^n \to A$ as $t_A(a_1, \dots, a_n) = t(x_i/a_i)$ for $a_1, \dots, a_n \in A$, where A is an algebra of type $\mathcal{F}$.

The set $T(X)$ can be transformed into an algebra (Burris and Shakappanavar, 1981). The term algebra $T(X)$ of type $\mathcal{F}$ over X has as its universe the set $T(X)$ and the fundamental operations satisfy:

$f^{T(X)}: (t_1, \dots, t_n) \mapsto f(t_1, \dots, t_n)$ for $f \in \mathcal{F}_n$ and $t_i \in T(X), 1 \le t \le n$.

Now consider the functional n-adic algebra $(B^{X^n}, \exists_i)$. A term t defines a function $t_*: B^{X^n} \to B^{X^n}$ as follows:

$t_*(p)(x) = p(t_1(x), \dots, t_n(x))$, for any $x \in X$ and $p \in B^{X^n}$ where $t = (t_1, \dots, t_n)$ as functions.

$End(B^{X^n})$ is the set of an endomorphisms from B^{X^n} into itself.

Proposition 7.

i) $t_* \in End(B^{X^n})$ ii) $\exists_i t_* = t_* \exists_i$

Proof

i) 1) $t_*(p \vee q)(x) = (p \vee q)(t_1(x), \dots, t_n(x)) = p(t_1(x), \dots, t_n(x)) \vee q(t_1(x), \dots, t_n(x)) = t_*(p)(x) \vee t_*(q)(x)$.

Then $t_*(p \vee q) = t_*(p) \vee t_*(q)$.

2) $t_*(p \wedge q)(x) = (p \wedge q)(t_1(x), \dots, t_n(x)) = p(t_1(x), \dots, t_n(x)) \wedge q(t_1(x), \dots, t_n(x)) = t_*(p)(x) \wedge t_*(q)(x)$.

Then $t_*(p \wedge q) = t_*(p) \wedge t_*(q)$.

3) $(t_*(p)(x))' = (p(t_1(x), \dots, t_n(x)))' = p'(t_1(x), \dots, t_n(x)) = t_*(p)'(x)$. Then $(t_*(p))' = t_*(p)'$.

4) $t_*(0)(x) = 0(t_1(x), \dots, t_n(x)) = 0 = 0(x)$. Then $t_*(0) = 0$.

5) $t_*(1)(x) = 1(t_1(x), \dots, t_n(x)) = (t_1(x), \dots, t_n(x)) = t_*(x)$. Then $t_*(1) = t_*$.

ii) $t_* \exists_i(p)(x) = t_* \left(\sup_i \{p(x)\} \right) = \sup_i \{p(t_1(x), \dots, t_n(x))\} = \sup_i \{ t_*(p)(x)\} = \exists_i\, t_*(p)(x)$. Then $\exists t_* = t_* \exists$.

4. Deduction

4.1 n-adic ideals and filters

A subset U of a Boolean algebra B is called a Boolean ideal if

i)$0 \in U$

ii) $a \vee b \in U$ for any $a, b \in U$

iii) If $a \in U$ and $b \le a$ then $b \in U$.

A subset F of a Boolean algebra B is called a Boolean filter if

i) $1 \in F$

ii) $a \wedge b \in F$ for any $a, b \in F$

iii) If $a \in F$ and $b \ge a$ then $b \in F$.

A subset U of an n-adic algebra B is called an n-adic ideal of B if

i) U is a Boolean ideal

ii) $\exists_i(U)(a) \in U, a \in U$ for $1 \le i \le n$.

A subset F of an n-adic algebra B is called an n-adic filter of B if

i) F is a Boolean filter

ii) $\forall_i(F)(a) \in F, a \in F$ for $1 \leq i \leq n$.

Thus we have the following two propositions.

Proposition 8.

There is a one to one correspondence between ideals and filters.

Proposition 9.

The set of all n-adic ideals and the set of all n-adic filters are closed under the arbitrary intersection.

Let B be an n-adic algebra and $\Gamma \subseteq B$. Let $U(\Gamma)$ denote the least n-adic ideal containing Γ and $F(\Gamma)$ denote the least n-adic filter containing Γ. We say that $U(\Gamma)$ and $F(\Gamma)$ are generating by Γ.

Proposition 10.

Let B be an n-adic algebra and $\Gamma \subseteq$ B. Then

i) $U(\Gamma) = \{b \in B: b \leq x_1 \vee x_2 \vee \ldots \vee x_n \; for \; some \; x_1, \ldots, x_n \in \Gamma\} \cup \{0\}$

ii) $F(\Gamma) = \{b \in B: b \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n \; for \; some \; x_1, \ldots, x_n \in \Gamma\} \cup \{1\}$

Proof

i) Let $J = \{b \in B: b \leq x_1 \vee x_2 \vee \ldots \vee x_n \; for \; some \; x_1, \ldots, x_n \in \Gamma\} \cup \{0\}, 0 \in J$. Let $b_1, b_2 \in J$ then $b_1 \leq x_1 \vee x_2 \vee \ldots \vee x_n$ and $b_2 \leq y_1 \vee y_2 \vee \ldots \vee y_m$ for some $x_i, y_i \in \Gamma$.

$b_1 \vee b_2 \leq x_1 \vee x_2 \vee \ldots \vee x_n \vee y_1 \vee y_2 \vee \ldots \vee y_m$. Therefore $b_1 \vee b_2 \in J$.

If $a \leq b \leq x_1 \vee x_2 \vee \ldots \vee x_n$, then $a \in J$. Then J is a Boolean ideal containing Γ.

If $a \in J$, then $a \leq x_1 \vee x_2 \vee \ldots \vee x_n$, $\exists_i(J)(a) \leq \exists_i(J)(x_1 \vee x_2 \vee \ldots \vee x_n) = \exists_i(J)(x_1) \vee \ldots \vee \exists_i(J)(x_n)$. Then $\exists_i(J)(a) \subseteq U(\Gamma)$.

ii) A similar argument leads to (ii).

A filter F of an n-adic algebra is called ultrafilter if F is maximal with respect to the property that $0 \notin F$.

Ultrafilters satisfy the properties of the following proposition.

Proposition 11. (Burris and Shakappanavar, 1981)

Let F be a filter of n-adic algebra, then

i) F is an ultrafilter of B iff for any $a \in F$ exactly one of a, a' belong to F.

ii) F is an ultrafilter of B iff $0 \in F$ and $a \vee b \in F$ iff $a \in F$ or $b \in F$ for any $a, b \in F$.

iii) If $a \in B - F$, then there is an ultrafilter L such that $F \subseteq L$ and $a \notin L$.

Let $\Gamma \subseteq B$, the ultrafilter containing $F(\Gamma)$ is denoted by $UF(\Gamma)$.

A mapping $\mu: B_1 \rightarrow B_2$ between two n-adic algebras is called n-adic homomorphism if

i) μ is a Boolean homomorphism,

ii) $\mu\exists_i = \exists_i \mu$.

Obviously, $\mu\forall_i = \forall_i \mu$.

For $\Gamma \subseteq B, b \in B$, we define the deduction $\Gamma \vdash$ b iff $b \in UF(\Gamma)$.

Theorem 12.

Let $\Gamma \subseteq B^{X^n}$

i) $\Gamma \vdash p \wedge q$ iff $\Gamma \vdash p$ and $\Gamma \vdash q$

ii) $\Gamma \vdash p \vee q$ iff $\Gamma \vdash p$ or $\Gamma \vdash q$

iii) $\Gamma \vdash p$ iff $\Gamma \nvdash p'$

iv) $\Gamma \vdash p$ iff $\Gamma \vdash \forall_i p, 1 \leq i \leq n$

v) $F(\Gamma) \vdash p(x_0)$ iff $F(\Gamma) \vdash \exists_i p(x), 1 \leq i \leq n$.

Proof

(i), (ii) follow from the definition of filter and (Alzubaidy, 2007, p.2)

iii) follows from proposition 10.

iv) This is by $\forall_i p(x) \leq p(x)$ and $\forall_i(F) \subseteq F$.

v) This is by the definition $\exists_i p(x) = sup\{p(x)\}, x = (x_1, \ldots, x_n)$.

Note that $t_*(UF(\Gamma)) = UF(t_*(\Gamma))$. Define $\Gamma \vdash pt$ if $pt \in F(t_*(\Gamma))$. Then theorem 12 can be generalized with respect to terms as follows:

Theorem 13.

i) $\Gamma \vdash pt \wedge qt$ iff $\Gamma \vdash pt$ and $\Gamma \vdash qt$

ii) $\Gamma \vdash pt \vee qt$ iff $\Gamma \vdash pt$ or $\Gamma \vdash qt$

iii) $\Gamma \vdash pt$ iff $\Gamma \nvdash p't$

iv) $\Gamma \vdash p$ iff $\Gamma \vdash \forall_i pt, 1 \leq i \leq n$

v) $F(\Gamma) \vdash pt(x_0)$ iff $F(\Gamma) \vdash \exists_i pt(x), 1 \leq i \leq n$.

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