Generalization of $b_2$-metric spaces and some fixed point theorems

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Highlights

- Introduce the concept of extended $b_2$-metric space.
- Establish some fixed-point theorems for self-mapping define on such spaces.
- As an application of our results, we will use fixed-point theorem to show that there is a unique solution to some integral equations.

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1. Introduction

The notion of a $b$-metric space was first introduced by Czerwik in Czerwik, S. (1993); Czerwik, S. (1998) and then many fixed-point results were obtained. Second hand, the notion of a 2-metric space was introduced by Cähler in Cähler, S. (1963), similarly, several fixed-point results were also obtained (Aghanj, A., et al., 2014; Alqahtani, B., et al., 2018; Hicle, T. L.; et al, 1979). Later, Zead Mustafa (Mustafa, Z., et al, 2014) introduced a new type of generalized metric space called $b_2$-metric space, as a generalization of both 2-metric space and $b$-metric space. Recently, Kamran et al., (2017) have dealt with an extended $b$-metric space and obtained unique fixed-point results.

In this paper, we introduce a new type of generalized $b_2$-metric space, which we call an extended $b_2$-metric space, as a generalization of both $b_2$-metric space and extended $b$-metric space. Then we verify some fixed point Theorems.

We provide some notations, definitions and auxiliary facts, which will be used later in this paper. Throughout the manuscript, we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ represents the positive integers.

**Definition 1.1** (Samina, B., 2016) Let $X$ be a nonempty set and $T:X \to X$ a self-map. We say that $x \in X$ is a fixed point of $T$ if $T(x) = x$, denote by $\text{Fix}(T)$ the set of all fixed points of $T$.

**Definition 1.2** (Kamran et al., 2017) Let $X$ be a nonempty set and $\theta:X \times X \to [1, \infty)$ be a mapping.

A function $d_\theta:X \times X \to [0, \infty]$ is an extended $b$-metric on $X$ if for all $x, y, z \in X$, the following conditions hold:

1) $d_\theta(x, y) = 0$ if and only if $x = y$,  
2) $d_\theta(x, y) = d_\theta(y, x)$,  
3) $d_\theta(x, y) \leq \theta(x, y)(d_\theta(x, z) + d_\theta(y, z))$.  

The pair $(X, d_\theta)$ is called an extended $b$-metric space.

**Example 1.1.** (Kamran et al., 2017) Let $X = C([a, b], \mathbb{R})$ be the space of all continuous real valued functions defined on $[a, b]$. Define $\theta : X^2 \to [1, \infty)$ by \[ \theta(x, y) = |x(t)| + |y(t)| + 2, \]
and $d_\theta: X^2 \to [0, \infty)$ by \[ d_\theta(x, y) = \sup_{t \in [a, b]} |x(t) - y(t)|^2. \]

Then $d_\theta$ is called an extended $b$-metric and $(X, d_\theta)$ is a complete extended $b$-metric space.

**Definition 1.3.** (Mustafa et al., 2014) Let $X$ be a nonempty set, $s \geq 1$ be a real number and $d:X \times X \times X \to \mathbb{R}$ be a map satisfying the following conditions:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$,  
2) If at least two of three points $x, y, z$ are the same, then $d(x, y, z) = 0$,  
3) The symmetry: $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$, for all $x, y, z \in X$.
4) The rectangle inequality: $d(x, y, z) \leq s[d(x, y, a) + d(y, z, a) + d(z, x, a)]$ for all $x, y, z, a \in X$.

Then $d$ is called a $b_2$-metric on $X$ and the pair $(X, d)$ is called a $b_2$-metric space.

**Example 1.2.** (Mustafa et al., 2014) Let a mapping $d: \mathbb{R}^3 \to [0, \infty)$ be defined by \[ d(x, y, z) = \min(|x - y|, |y - z|, |z - x|). \]
Then $d$ is a $2$-metric space on $\mathbb{R}$. For arbitrary real numbers $x, y, z, a$. Using convexity of the function $f(x) = x^p$ on $[0, \infty)$ for $p \geq 1$, we obtain that \[ d_p(x, y, z) = \min(|x - y|, |y - z|, |z - x|)^p, \]
is a $b_2$-metric on $\mathbb{R}$ with $s \leq 3p^{-1}$ and $(\mathbb{R},d_p)$ is a $b_2$-metric space.

**Theorem 1.1** (Hassan, et al., 2017) Let $(X,d)$ be a complete $b_2$-metric space with constant $s \geq 1$, such that $b_2$-metric is a continuous functional. Let $T : X \to X$ be a contraction having contraction constant $k \in [0,1)$ such that $sk < 1$. Then $T$ has a unique fixed point.

2. **Main Results**

In this section, we adduce a new type of generalized $b_2$-metric space; we call an extended $b_2$-metric space. We also provide some fixed-point theorems on such spaces.

**Definition 2.1.** Let $X$ be a nonempty set and $\theta : X \times X \times X \to \{0, \infty\}$ be a mapping. A function $d_\theta : X \times X \times X \to [0, \infty)$ is an extended $b_2$-metric on $X$ if for all $x, y, z \in X$, the following conditions hold:

1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d_\theta(x,y,z) \neq 0$.
2. If at least two of three points $x, y, z$ are the same, then $d_\theta(x,y,z) = 0$.
3. The symmetry: $d_\theta(x,y,z) = d_\theta(x,z,y) = d_\theta(y,x,z) = d_\theta(y,z,x) = d_\theta(z,x,y) = d_\theta(z,y,x)$, for all $x, y, z \in X$.
4. The rectangle inequality: $d_\theta(x,y,z) \leq \theta(x,y, z)[d_\theta(x,y,a) + d_\theta(y,z,a) + d_\theta(x,z,a)]$ for all $x, y, z, a \in X$.

Then $d_\theta$ is called an extended $b_2$-metric on $X$ and the pair $(X,d_\theta)$ is called an extended $b_2$-metric space.

**Remarks 2.1.**

1. It is obvious that the class of an extended $b_2$-metric space is larger than $b_2$-metric space, because if $\theta(x,y,z) = s$, for $s \geq 1$ then we obtain the definition of a $b_2$-metric space. Furthermore, for $\theta(x,y,z) = s = 1$, the $b_2$-metric reduces to a $2$-metric.
2. Using condition (1) it readily verified that for all $a \in X, d_\theta(x,y,a) = 0$, then $x = y$.

**Example 2.1.** Let $X = \{1,2,3,4\}$. Define $\theta : X^3 \to [1, \infty)$ by

$$\theta(x,y,z) = \begin{cases} 0 & \text{if at least two of } x, y, z \text{ are equal,} \\ 80 & \text{if } x, y \in \{1,2,3\}, \\ 300 & \text{if } x, y \in \{1,2,4\}, \\ 600 & \text{if } x, y \in \{1,3,4\}, \\ 1000 & \text{if } x, y \in \{2,3,4\}. \end{cases}$$

Then $(X,d_\theta)$ is an extended $b_2$-metric space.

**Proof**

In Definition 2.1 conditions 1, 2 and 3 trivially hold. For 4, we have $d_\theta(x,y,z) \leq \theta(x,y, z)[d_\theta(x,y,a) + d_\theta(y,z,a) + d_\theta(x,z,a)]$.

Then $(X,d_\theta)$ is an extended $b_2$-metric space.

**Definition 2.2.** Let $(X,d_\theta)$ be an extended $b_2$-metric space. Then $d_\theta : X \times X \to [0, \infty)$ by

$$d_\theta(x,y,z) = \begin{cases} \frac{1}{xyz} & \text{if } x,y,z \in \{0,1\} \text{ and } x \neq y \neq z, \\ 0 & \text{if } x,y,z \in \{0,1\} \text{ and at least two of } x, y, z \text{ are equal,} \\ \frac{1}{xy} & \text{if } x,y \in \{0,1\} \text{ and } z = 0. \end{cases}$$

Then $(X,d_\theta)$ is an extended $b_2$-metric space.

**Proof**

Obviously, conditions 1, 2 and 3 in Definition 2.1 hold. For 4, we have the following cases:

1. Let $x,y \in \{0,1\}$, then $d_\theta(x,y) = \theta(x,y,0)[d_\theta(x,y,0) + d_\theta(y,0,0) + d_\theta(x,0,0)] \leq \frac{1}{xy}$.

Then $d_\theta : X \times X \to [0, \infty)$ is a $b_2$-metric.

**Definition 2.3.** Let $(X,d_\theta)$ be an extended $b_2$-metric space. The extended $b_2$-metric $d_\theta$ is called continuous if $d_\theta(x_n,x,a) \to 0$ and $d_\theta(y_n,y,a) \to 0$ for all $a \in X$, there exists $x \in X$, such that $d_\theta(x_n,x,a) = 0$.

**Definition 2.3.** Let $(X,d_\theta)$ be an extended $b_2$-metric space. The extended $b_2$-metric $d_\theta$ is called complete if every Cauchy sequence is convergent sequence.

**Example 2.2.** Let $X = \{0,1\}$. Define $\theta : X \times X \times X \to [1, \infty)$ by

$$\theta(x,y,z) = \frac{1 + x + y + z}{x + y + z} \quad \text{for all } x,y,z \in X.$$
for all sequence \( \{x_n\},\{y_n\} \) in \( X \) and \( x, y, a \in X \). Note that, in general a \( b_2 \)-metric \( d \) is not continuous functional and thus so is an extended \( b_2 \)-metric \( d_\theta \).

**Example 2.3.** Let \( X = \mathbb{N} \cup \{\infty\} \) and let \( d : X \times X \to \mathbb{R} \) be defined by

\[
d(m, n) = \begin{cases} 
0 & \text{if } m = n, \\
\frac{1}{m-n} & \text{if } m \text{ and } n \text{ are even or } m = \infty, \\
5 & \text{if } m \text{ and } n \text{ are odd and } m \neq n, \\
2 & \text{otherwise}.
\end{cases}
\]

Then it is facile to see that, for all \( m, n, p \in X, (X, d) \) is a \( b \)-metric space with \( s = 3 \) but it is not continuous, (see Hussain et al., 2012).

Now, let \( d_\theta(x,y,z) = [\min \{d(x,y),d(x,z),d(y,z)\}]^2 \). It is easy to see that \( d_\theta \) is an extended \( b_2 \)-metric space, with \( \theta(x,y,z) = s = 3 \). Now we show that \( d_\theta \) is not continuous function. Take, \( x_n = 2n, \) and \( y_n = 1. \) Then we have \( x_n \to \infty, y_n \to 1. \) Also,

\[
d_\theta(2n,\infty,3) = [\min \{d(2n,\infty),d(2n,3),d(3,\infty)\}]^2 = \frac{1}{4n^2} \to 0 \text{ as } n \to \infty,
\]

and 

\[
d_\theta(y_n,1,3) = 0 \to 0 \text{ as } n \to \infty.
\]

On other hand,

\[
d_\theta(x_n,y_n,3) = [\min \{d(x_n,3),d(y_n,3),d(x_n,y_n)\}]^2 = 4,
\]

and 

\[
d_\theta(\infty,1,3) = [\min \{d(\infty,1),d(1,3),d(\infty,3)\}]^2 = 25.
\]

Hence,

\[
\lim_{n \to \infty} d_\theta(x_n,y_n,a) \neq d_\theta(x,y,a).
\]

Thus, \( d_\theta \) is not continuous function.

**Lemma 2.1** Let \((X,d_\theta)\) be an extended \( b_2 \)-metric space . If \( d_\theta \) is continuous, then every converges sequence has a unique limit.

**Definition 2.4** (Samina B., 2016) Given a mapping \( T : X \to X \) and \( x_0 \in X \), for all \( n \in \mathbb{N} \), the orbit of \( x_0 \) with respect to \( T \) is defined as the following sequences of points,

\[
O(x_0) = \{x_0,Tx_0,\ldots,T^nx_0 \ldots\}.
\]

**Theorem 2.2** Let \((X,d_\theta)\) be a complete \( b_2 \)-metric space such that \( d_\theta \) is continuous functional. Let \( T : X \to X \) satisfy

\[
d_\theta(Tx,Ty,a) \leq k d_\theta(x,y,a), \tag{2.1}
\]

for all \( x, y, a \in X \), where \( k \in (0,1) \) be such that for each \( x_0 \in X, \lim_{n,m \to \infty} \theta(x_0,x_0,m,a) < \frac{1}{k}, \) where \( x_0 = T^n x_0, n = 1, 2, \ldots \). Then \( T \) has precisely one fixed point \( u \). Moreover, each \( y \in X, T^u y \to u. \)

**Proof**
Assume first that \( x_0 \in X \) be an arbitrary, we define the sequence \( \{x_n\} \) by

\[
x_0, x_1 = T x_0, x_2 = T x_1 = T^2 x_0, \ldots, x_n = T^n x_0 \ldots .
\]

If \( x_n = x_{n+1} \) for some \( n \), there is nothing to prove. Therefore, suppose that

\[
x_n \neq x_{n+1} \text{ for each } n \in \mathbb{N} . \text{ By Eq. (2.1), we have:}
\]

\[
d_\theta(x_n,x_{n+1},a) = d_\theta(T^n x_0, T^{n+1} x_0, a) \leq k^n d_\theta(x_0,x_1,a) \tag{2.2}
\]

Now, we prove that \( \{x_n\} \) is Cauchy sequence in \( X \). For \( m > n \), we have

\[
d_\theta(x_m,x_n,a) \leq d_\theta(x_n,x_m,a) \tag{2.3}
\]

\[
= \theta(x_n,x_m,a) \tag{2.4}
\]

\[
= d_\theta(x_n,x_{n+1},a) \tag{2.5}
\]

Thus for all \( m > n \), inequality Eq. (2.4) implies,

\[
d_\theta(x_m,x_n,a) \leq d_\theta(x_n,x_1,a)(k^{m-n} - s_{n-1}) \tag{2.5}
\]

Therefore, Eq. (2.3), becomes

\[
d_\theta(x_m,x_n,a) \leq \sum_{i=n}^{m-1} k^i \prod_{j=n}^{i} \theta(x_j,x_{j+1},a) \tag{2.4}
\]

Notice the inequality above is dominated by

\[
\sum_{i=n}^{m-1} k^i \prod_{j=n}^{i} \theta(x_j,x_{j+1},a) \leq \sum_{i=1}^{m-n} k^i \prod_{j=n}^{i} \theta(x_j,x_{j+1},a) .
\]

Since \( \lim_{n,m \to \infty} \theta(x_0,x_0,m,a) \to \kappa, \) so that the series \( \sum_{i=1}^{m-n} k^i \prod_{j=n}^{i} \theta(x_j,x_{j+1},a) \), converges by ratio test to some \( s \in (0, \infty) \), for each \( m \in \mathbb{N} \). Let

\[
s = \sum_{i=1}^{\infty} k^i \prod_{j=1}^{i} \theta(x_j,x_{j+1},a),
\]

with partial sum,

\[
s_n = \sum_{i=1}^{n} k^i \prod_{j=1}^{i} \theta(x_j,x_{j+1},a) .
\]

Therefore, Eq. (2.4) implies,

\[
d_\theta(x_m,x_n,a) \leq d_\theta(x_n,x_1,a)(s_{m-n} - s_{n-1}) \tag{2.5}
\]

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Letting $n \to \infty$ in Eq. (2.5), we deduce that, $(x_n)$ is a Cauchy sequence in $X$. Since $X$ is complete, then there exist a point $u \in X$ such that $x_n \to u$. We have,

$$d_\theta(Tu, u, a) \leq \theta(Tu, u, a)[d_\theta(Tu, x_n, u) + d_\theta(u, x_n, a)] = \theta(Tu, u, a)[d_\theta(Tu, T^n x_0, a) + d_\theta(Tu, T^n x_0, u) + d_\theta(u, x_n, a)],$$

$$\leq \theta(Tu, u, a) \left( k d_\theta(u, x_{n-1}, a) + k d_\theta(u, x_{n-1}, u) + d_\theta(u, x_n, a) \right) \leq 0, \quad \text{as} \quad n \to \infty.$$ 

Thus $d_\theta(Tu, u, a) = 0$, for all $a \in X$. Hence $u$ is a fixed point of $T$.

For uniqueness of $u$: Let $u, v$ be distinct fixed points of $T$, then for all $a \in X$, we have

$$d_\theta(u, v, a) = d_\theta(Tu, Tv, a) \leq k d_\theta(u, v, a),$$

This implies that $k \geq 1$, which is a contradiction to $k \in [0, 1)$. Therefore $u$ is a unique fixed point.

**Definition 2.5** (Samina B., 2016) Let $T: X \to X$. A function $G: X \to \mathbb{R}$ is said to be $T$-orbitally lower semi-continuous at $t \in X$ if the sequence $\{x_n\} \subset O(x_0)$ is such that, $x_n \to t$, we have $G(t) \leq \lim inf G(x_n)$.

**Theorem 2.3** Let $(X, d_\theta)$ be a complete extended $b_2^\infty$ metric space such that $d_\theta$ is continuous functional. Let $T: X \to X$ and there exists $x_0 \in X$, such that

$$d_\theta(Ty, T^2y, a) \leq k d_\theta(y, Ty, a),$$

for each $y \in O(x_0)$ (2.6) where $k \in [0, 1)$ be such that for each $x \in X$,

$$\lim_{n, m \to \infty} \theta(x_n, x_m, a) < \frac{1}{k}, \quad \text{and} \quad T^n x_0 \to u \quad \text{as} \quad n \to \infty.$$ 

Furthermore $u$ is a fixed point of $T$ if and only if $G(u) = d_\theta(u, Tu, a) \leq T$-orbitally lower semi-continuous at $u$.

**Proof**

We choose any $x_0 \in X$ be an arbitrary, define the iterative sequence $(x_n)$ by

$$x_0, x_1 = T x_0, x_2 = T^2 x_0, ..., x_n = T^n x_0.$$ 

Now for $y = T x_0$ by successively applying inequality Eq. (2.6) we obtain

$$d_\theta(T^n x_0, T^{n+1} x_0, a) = d_\theta(x_n, x_{n+1}, a) \leq k^n d_\theta(x_0, x_1, a).$$

Following the same procedure as in proof of Theorem 2.2, we infer that $(x_n)$ is a Cauchy sequence in $X$, since $X$ is complete, then there exist a point $u \in X$ such that $x_n = T^n x_0 \to u$. Assume that $G$ is defined as in Definition 2.5, then for all $a \in X$, we have

$$d_\theta(u, Tu, a) \leq \lim_{n \to \infty} \inf d_\theta(T^n x_0, T^{n+1} x_0, a) \leq \lim_{n \to \infty} \inf k^n d_\theta(x_0, x_1, a) = 0,$$

thus $Tu = u$, and hence $u$ is a fixed point of $T$.

Conversely, let $u = Tu \in X$ and $x_n \in O(x_0)$ with $x_n \to u$. Then,

$$G(u) = d_\theta(u, Tu, a) = 0 \leq \lim_{n \to \infty} \inf G(x_n) = d_\theta(T^n x_0, T^{n+1} x_0, a).$$

**Example 2.4.** Let $X = [0, 1]$. Define $\theta : X^3 \to [1, \infty)$

$$\theta(x, y, z) = 2 + x + y + z,$$

and $d_\theta: X^3 \to [0, \infty)$ by

$$d_\theta(x, y, z) = \min(\max(|x - y|, |x - z|, |y - z|)).$$

Clearly, $\theta(x, y, a) < 3$, therefore $(X, d_\theta)$ is complete extended $b_2^\infty$-metric space.

Define $T: X \to X$ by,

$$T x = \frac{x}{5}.$$ 

Therefore, $d_\theta(Tx, Ty, a) \leq k d_\theta(x, y, a)$.
Hence all the conditions of Theorem 2.2 are satisfied and the mapping $T$ has one fixed point. Thus, we conclude that the integral equation Eq. (3.1) has a unique solution in $X$.

4. Conclusion

In this paper, we introduce a new type of generalized $b_2$-metric space; we call an extended $b_2$-metric space. We also provide some fixed-point theorems on these spaces. This provides a background for extended $b_2$-metric spaces technique in the fixed-point theory. Several consequences can be observed from the main results. For example, taking $\theta(x,y,z)$ with $s \geq 1$ implies corresponding fixed-point results in the context of $b_2$-metric space. Furthermore, for $\theta(x,y,z) = 1$, the $b_2$-metric reduces to a 2-metric. In addition, we can indicate several directions from our results for further work, which go through fixed-point theory. As a new work, it will be interesting to extend known fixed-point results on an extended $b$-metric space and $b_2$-metric space to our results on an extended $b_2$-metric.

References


