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Generalization of b_2-metric spaces and some fixed point theorems

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Highlights

- **Introduce the concept of extended $b_2$metric space.**
- **Establish some fixed-point theorems for self-mapping define on such spaces.**
- **As an application of our results, we will use fixed-point theorem to show that there is a unique solution to some integral equations.**

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ABSTRACT

The primary purpose of this paper is to adduce the concept of extended b_2-metric spaces inspired by the concepts of b_2-metric space and an extended b-metric space. We also create some fixed-point theorems arising in this space.

1. Introduction

The notion of a b-metric space was first introduced by Czerwik in Czerwik, S. (1993); Czerwik, S. (1998) and then many fixed-point results were obtained. Second hand, the notion of a 2-metric space was instituted by Cähler in Cähler, S. (1963), similarly, several fixed-point results were also obtained (Aghajani, A., *et al.*, 2014; Alqahtani, B., *et al.*, 2018; Hicks, T. L.; *et al.*, 1979). Later, Zead Mustafa (Mustafa, Z., *et al.*, 2014) introduced a new type of generalized metric space called b_2-metric space, as a generalization of both 2-metric space and b-metric space. Recently, Kamran *et al.*, (2017) have dealt with an extended b-metric space and obtained unique fixed-point results.

In this paper, we introduce a new type of generalized b_2-metric space , which we call an extended b_2-metric spaces, as a generalization of both b_2-metric space and extended b-metric space. Then we verify some fixed point Theorems.

We provide some notations, definitions and auxiliary facts, which will be need later in this paper. Throughout the manuscript, we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ represents the positive integers.

Definition 1.1 (Samina, B., 2016) Let X be a nonempty set and $T: X \to X$ a self-map. We say that $x \in X$ is a fixed point of T if $T(x) = x$, denote by $Fix(T)$ the set of all fixed points of T.

Definition 1.2 (Kamran *et al.*, 2017) Let X be a nonempty set and $\theta: X \times X \to [1, \infty)$ be a mapping.

A function $d_\theta: X \times X \longrightarrow [0, \infty)$ is an extended b-metric on X if for all $x, y, z \in X$, the following conditions hold:

1) $d_\theta(x, y) = 0$ if and only if $x = y$,
2) $d_\theta(x, y) = d_\theta(y, x)$,
3) $d_\theta(x, y) \le \theta(x, y)[d_\theta(x, y) + d_\theta(y, z)]$.

The pair (X, d_θ) is called an extended b-metric space.

Example 1.1. (Kamran *et al.*, 2017) Let $X = C([a, b], \mathbb{R})$ be the space of all continuous real valued functions defined on $[a, b]$.

Define $\theta : X^2 \longrightarrow [1, \infty)$ by
$$\theta(x, y) = |x(t)| + |y(t)| + 2,$$
and $d_\theta: X^2 \longrightarrow [0, \infty)$ by
$$d_\theta(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|^2.$$

Then d_θ is called an extended b-metric and (X, d_θ) is a complete extended b-metric space.

Definition1.3. (Mustafa *et al.*, 2014) Let X be a nonempty set, $s \ge 1$ be a real number and
$d: X \times X \times X \longrightarrow \mathbb{R}$ be a map satisfying the following conditions:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \ne 0$,
2) If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$,
3) The symmetry: $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$, for all $x, y, z \in X$.
4) The rectangle inequality: $d(x, y, z) \le s[d(x, y, a) + d(y, z, a) + d(z, x, a)]$ for all $x, y, z, a \in X$.

Then d is called a b_2-metric on X and the pair (X, d) is called a b_2-metric space.

Example 1.2. (Mustafa *et al.*, 2014) Let a mapping $d: \mathbb{R}^3 \longrightarrow [0, \infty)$ be defined by
$$d(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}.$$

Then d is a 2-metric space on $\mathbb{R}$. For arbitrary real numbersx, y, z, a. Using convexity of the function $f(x) = x^p$ on $[0, \infty)$ for $p \ge 1$, we obtain that
$$d_p(x, y, z) = [\min\{|x - y|, |y - z|, |z - x|\}]^p,$$

is a b_2-metric on $\mathbb{R}$ with $s \le 3^{p-1}$ and $(\mathbb{R}, d_p)$ is a b_2-metric space.

Theorem 1.1 (Hassan, *et al.*, 2017) Let (X, d) be a complete b_2-metric space with constant $s \ge 1$, such that b_2-metric is a continuous functional. Let $T: X \to X$ be a contraction having contraction constant $k \in [0,1)$ such that $sk < 1$. Then T has a unique fixed point.

2. Main Results

In this section, we adduce a new type of generalized b_2-metric space; we call an extended b_2-metric space. We also provide some fixed-point theorems on such spaces.

Definition 2.1. Let X be a nonempty set and $\theta: X \times X \times X \to [1,\infty)$ be a mapping. A function $d_\theta: X \times X \times X \longrightarrow [0,\infty)$ is an extended b_2-metric on X if for all $a, x, y, z \in X$, the following conditions hold:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d_\theta(x,y,z) \ne 0$,
2) If at least two of three points x, y, z are the same, then $d_\theta(x,y,z) = 0$.
3) The symmetry: $d_\theta(x,y,z) = d_\theta(x,z,y) = d_\theta(y,x,z) = d_\theta(y,z,x) = d_\theta(z,x,y) = d_\theta(z,y,x)$, for all $x, y, z \in X$.
4) The rectangle inequality: $d_\theta(x,y,z) \le \theta(x,y,z)[d_\theta(x,y,a) + d_\theta(y,z,a) + d_\theta(z,x,a)]$ for all $x, y, z, a \in X$.

Then d_θ is called an extended b_2-metric on X and the pair (X, d_θ) is called an extended b_2-metric space.

Remarks 2.1.

1) It is obvious that the class of an extended b_2-metric space is larger than b_2-metric space, because if $\theta(x,y,z) = s$, for $s \ge 1$ then we obtain the definition of a b_2-metric space. Furthermore, for $\theta(x,y,z) = s = 1$, the b_2-metric reduces to a 2-metric.
2) Using condition (1) it readily verified that for all $a \in X$, $d_\theta(x,y,a) = 0$, then $x = y$.

Example 2.1. Let $X = \{1,2,3,4\}$. Define $\theta: X^3 \longrightarrow [1,\infty)$ by

$$\theta(x,y,z) = x + y + z + 1,$$

and $d_\theta: X^3 \longrightarrow [0,\infty)$ by

$$d_\theta(x,y,z) = \begin{cases} 0 & \text{if at least two of } x,y,z \text{ are equal,} \\ 80 & \text{if } x,y,z \in \{1,2,3\}, \\ 300 & \text{if } x,y,z \in \{1,2,4\}, \\ 600 & \text{if } x,y,z \in \{1,3,4\}, \\ 1000 & \text{if } x,y,z \in \{2,3,4\}. \end{cases}$$

Then (X, d_θ) is an extended b_2-metric space.

Proof

In Definition 2.1 conditions 1, 2 and 3 trivially hold. For 4, we have

$d_\theta(1,2,3) = 80$, $\theta(1,2,3)[d_\theta(1,2,4) + d_\theta(1,3,4) + d_\theta(2,3,4)\} = 13300$, $d_\theta(1,3,4) = 600$, $\theta(1,3,4)[d_\theta(1,3,2) + d_\theta(2,3,4) + d_\theta(1,2,4)\} = 12420$, $d_\theta(2,3,4) = 1000$, $\theta(2,3,4)[d_\theta(2,3,1) + d_\theta(2,1,4) + d_\theta(1,3,4)\} = 9800$, $d_\theta(1,2,4) = 300$, $\theta(1,2,4)[d_\theta(1,2,3) + d_\theta(1,3,4) + d_\theta(2,3,4)\} = 13440$.

Thus for all $x, y, z, a \in X$

$$d_\theta(x,y,z) \le \theta(x,y,z)[d_\theta(x,y,a) + d_\theta(y,z,a) + d_\theta(z,x,a)].$$

Hence (X, d_θ) is an extended b_2-metric space. Inspired by Example 1.2 (Alqahtani *et al.*, 2018), we have:

Example 2.2. Let $X = [0,1]$. Define $\theta: X \times X \times X \longrightarrow [1,\infty)$ by

$$\theta(x,y,z) = \frac{1+x+y+z}{x+y+z} \quad \text{for all } x,y,z \in X.$$

And $d_\theta: X \times X \times X \longrightarrow [0,\infty)$ by

$$d_\theta(x,y,z) = \begin{cases} \dfrac{1}{xyz} & \text{if } x,y,z \in (0,1] \text{ and } x \ne y \ne z, \\ 0 & \text{if } x,y,z \in [0,1] \text{ and at least two of } x, y, \textit{and } z \textit{ are equal} \\ \dfrac{1}{xy} & \text{if } x,y \in (0,1] \text{ and } z = 0. \end{cases}$$

Then (X, d_θ) is an extended b_2-metric space.

Proof

Obviously, conditions 1, 2 and 3 in Definition 2.1 hold. For 4, we have the following cases:

I. Let $x, y, z \in (0,1]$. For $a \in (0,1]$, we have

$$d_\theta(x,y,z) \le \theta(x,y,z)[d_\theta(x,y,a) + d_\theta(y,z,a) + d_\theta(z,x,a)]$$
$$\Leftrightarrow \frac{1}{xyz} \le \frac{1+x+y+z}{x+y+z} \cdot \frac{x+y+z}{xyza}$$
$$\Leftrightarrow a \le 1 + x + y + z.$$

If $a = 0$, then

$$d_\theta(x,y,z) \le \theta(x,y,z)[d_\theta(x,y,0) + d_\theta(y,z,0) + d_\theta(z,x,0)]$$
$$\Leftrightarrow \frac{1}{xyz} \le \frac{1+x+y+z}{x+y+z} \cdot \frac{x+y+z}{xyz}$$
$$\Leftrightarrow 1 \le 1 + x + y + z.$$

II. Let $x, y \in (0,1]$ and $z = 0$, for $a \in (0,1]$.

$$d_\theta(x,y,0) \le \theta(x,y,0)[d_\theta(x,y,a) + d_\theta(y,0,a) + d_\theta(0,x,a)]$$
$$\Leftrightarrow \frac{1}{xy} \le \frac{1+x+y}{x+y} \cdot \frac{1+x+y}{xya}$$
$$\Leftrightarrow a(x+y) \le (1+x+y)^2.$$

If $a = 0$, then

$$d_\theta(x,y,0) \le \theta(x,y,0)[d_\theta(x,y,0) + d_\theta(y,0,0) + d_\theta(0,x,0)]$$
$$\Leftrightarrow \frac{1}{xy} \le \frac{1+x+y}{x+y} \cdot \frac{1}{xy}$$
$$\Leftrightarrow x + y \le 1 + x + y.$$

III. For $x, y, z, a \in [0,1]$ and at least two of x, y and z are equal. Let $x = y$, then

$$d_\theta(x,x,z) \le \theta(x,x,z)[d_\theta(x,x,a) + d_\theta(x,z,a) + d_\theta(z,x,a)]$$
$$\Leftrightarrow \quad 0 \le \frac{1+2x+z}{2x+z} \cdot \frac{2}{xza}$$
$$\Leftrightarrow \quad 0 \le 2(1+2x+z).$$

Similarly, for $x = z$, $y = z$ and $x = y = z$. In conclusion, for any $x, y, z, a \in X$,

$$d_\theta(x,y,z) \le \theta(x,y,z)[d_\theta(x,y,a) + d_\theta(y,z,a) + d_\theta(z,x,a)].$$

Hence (X, d_θ) is an extended b_2-metric space.

Definition 2.2. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an extended b_2-metric space (X, d_θ).

1) A sequence $\{x_n\}$ is a Cauchy sequence if and only if $d_\theta(x_n, x_m, a) \to 0$, when $n, m \to \infty$. for all $a \in X$.
2) A sequence $\{x_n\}$ is convergent to $x \in X$, if for all $a \in X$, there exists $x \in X$, such that $\lim_{n\to\infty} d_\theta(x_n, x, a) = 0$.
3) An extended b_2-metric space (X, d_θ) is called complete if every Cauchy sequence is convergent sequence.

Definition 2.3. Let (X, d_θ) be an extended b_2 −metric space. The extended b_2 − metric d_θ is called continuous if

$$d_\theta(x_n, x, a) \to 0 \textit{ and } d_\theta(y_n, y, a) \to 0 \implies d_\theta(x_n, y_n, a) \longrightarrow d_\theta(x, y, a),$$

for all sequence $\{x_n\}, \{y_n\}$ in X and $x, y, a \in X$. Note that, in general a b_2-metric d is not continuous functional and thus so is an extended b_2-metric d_θ.

Example 2.3. Let $X = \mathbb{N} \cup \{\infty\}$ and let $d: X \times X \longrightarrow \mathbb{R}$ be defined by

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right| & \text{if } m \text{ and } n \text{ are even or } mn = \infty, \\ 5 & \text{if } m \text{ and } n \text{ are odd and } m \neq n, \\ 2 & \text{othewise.} \end{cases}$$

Then it is facile to see that, for all $m, n, p \in X$, (X, d) is a b-metric space with $s = 3$ but it is not continuous, (see Hussain *et al.*, (2012).

Now, let $d_\theta(x,y,z) = [\min\{d(x,y), d(x,z), d(y,z)\}]^2$. It is easy to see that d_θ is an extended b_2-metric space, with $\theta(x,y,z) = s = 3$. Now we show that d_θ is not continuous function. Take, $x_n = 2n$, and $y_n = 1$. Then we have $x_n \to \infty, y_n \to 1$. Also,

$$d_\theta(2n, \infty, 3) = [\min\{d(2n,\infty), d(2n,3), d(3,\infty)\}]^2 + $$
$$= \frac{1}{4n^2} \to 0 \text{ as } n \to \infty,$$

and

$$d_\theta(y_n, 1, 3) = 0 \to 0 \quad \text{as } n \to \infty.$$

On other hand,

$$d_\theta(x_n, y_n, 3) = [\min\{d(x_n,3), d(y_n,3), d(x_n,y_n)\}]^2 = 4,$$

and

$$d_\theta(\infty, 1, 3) = [\min\{d(\infty,1), d(1,3), d(\infty,3)\}]^2 = 25.$$

Hence,

$$\lim_{n\to\infty} d_\theta(x_n, y_n, a) \neq d_\theta(x, y, a).$$

Thus, d_θ is not continuous function.

Lemma 2.1 Let (X, d_θ) be an extended b_2- metric space. If d_θ is continuous, then every converges sequence has a unique limit.

Definition 2.4 (Samina B., 2016) Given a mapping $T: X \to X$ and $x_0 \in X$, for all $n \in \mathbb{N}$, the orbit of x_0 with respect to T is defined as the following sequences of points,

$$O(x_0) = \{x_0, Tx_0, \dots, T^n x_0, \dots\}.$$

Theorem 2.2 Let (X, d_θ) be a complete extended b_2- metric space such that d_θ is continuous functional. Let $T: X \to X$ satisfy,

$$d_\theta(Tx, Ty, a) \leq k\, d_\theta(x, y, a), \quad (2.1)$$

for all $x, y, a \in X$, where $k \in [0,1)$ be such that for each $x_0 \in X$, $\lim_{n,m\to\infty} \theta(x_n, x_m, a) < \frac{1}{k}$, where $x_n = T^n x_0, n = 1,2,\dots$. Then T has precisely one fixed point u. Moreover, each $y \in X, T^n y \to u$.

Proof

Assume first that $x_0 \in X$ be an arbitrary, we define the sequence $\{x_n\}$ by

$$x_0, \; x_1 = Tx_0, \; x_2 = Tx_1 = T^2 x_0, \dots, \; x_n = T^n x_0 \; \dots.$$

If $x_n = x_{n+1}$ for some n, there is nothing to prove. Therefore, suppose that

$x_n \neq x_{n+1}$ for each $n \in \mathbb{N}_0$. By Eq. (2.1), we have:

$$d_\theta(x_n, x_{n+1}, a) = d_\theta(T^n x_0, T^{n+1} x_0, a) \leq k^n d_\theta(x_0, x_1, a) \quad (2.2)$$

Now, we prove that $\{x_n\}$ is Cauchy sequence in X. For $m > n$, we have

$$d_\theta(x_n, x_m, a) \leq \theta(x_n, x_m, a)[d_\theta(x_n, x_m, x_{n+1}) + d_\theta(x_m, a, x_{n+1}) + d_\theta(a, x_n, x_{n+1})]$$
$$= \theta(x_n, x_m, a)[d_\theta(x_n, x_{n+1}, x_m) + d_\theta(x_{n+1}, x_m, a) + d_\theta(x_n, x_{n+1}, a)]$$
$$= \theta(x_n, x_m, a)[d_\theta(x_n, x_{n+1}, a) + d_\theta(x_n, x_{n+1}, x_m) + d_\theta(x_{n+1}, x_m, a)]$$
$$\leq \theta(x_n, x_m, a)[\, k^n d_\theta(x_0, x_1, a) + k^n d_\theta(x_0, x_1, x_m)\,] + \theta(x_n, x_m, a) d_\theta(x_{n+1}, x_m, a),$$
$$\leq \theta(x_n, x_m, a)\, k^n d_\theta(x_0, x_1, a) + \theta(x_n, x_m, a)\, k^n d_\theta(x_0, x_1, x_m) + \theta(x_n, x_m, a)\theta(x_{n+1}, x_m, a)$$
$$[\, d_\theta(x_{n+2}, x_{n+1}, a) + d_\theta(x_{n+2}, x_{n+1}, x_m) + d_\theta(x_{n+2}, x_m, a)\,],$$
$$\leq [\, \theta(x_n, x_m, a) k^n + \theta(x_n, x_m, a)\theta(x_{n+1}, x_m, a) k^{n+1}\,] d_\theta(x_0, x_1, a)$$
$$+ [\, \theta(x_n, x_m, a) k^n + \theta(x_n, x_m, a)\theta(x_{n+1}, x_m, a) k^{n+1}\,] d_\theta(x_0, x_1, x_m)$$
$$+ \theta(x_n, x_m, a)\theta(x_{n+1}, x_m, a)\, d_\theta(x_{n+2}, x_m, a),$$
$$\leq [\theta(x_n, x_m, a) k^n + \theta(x_n, x_m, a)\theta(x_{n+1}, x_m, a) k^{n+1} + \dots + \theta(x_n, x_m, a)\theta(x_{n+1}, x_m, a) \dots \theta(x_{m-2}, x_m, a)\theta(x_{m-1}, x_m, a) k^{m-1}\,]$$
$$(d_\theta(x_0, x_1, a) + d_\theta(x_0, x_1, x_m)),$$
$$= \left(\sum_{i=n}^{m-1} k^i \prod_{j=n}^{i} \theta(x_j, x_m, a)\right)(\, d_\theta(x_0, x_1, a) + d_\theta(x_0, x_1, x_m)\,). \quad (2.3)$$

Also, we have:

$$d_\theta(x_0, x_1, x_m) \leq \theta(x_0, x_1, x_m)[d_\theta(x_0, x_1, x_{m-1}) + d_\theta(x_{m-1}, x_m, x_0) + d_\theta(x_{m-1}, x_m, x_1)]$$
$$\leq \theta(x_0, x_1, x_m)[d_\theta(x_0, x_1, x_{m-1}) + k^{m-1} d_\theta(x_0, x_1, x_0) + k^{m-2} d_\theta(x_1, x_2, x_1)\,],$$
$$= \theta(x_0, x_1, x_m) d_\theta(x_0, x_1, x_{m-1})$$
$$\leq \theta(x_0, x_1, x_m)\theta(x_0, x_1, x_{m-1}) d_\theta(x_0, x_1, x_{m-2})$$
$$\leq \dots$$
$$\leq \theta(x_0, x_1, x_m)\, \theta(x_0, x_1, x_{m-1}) \dots \theta(x_0, x_1, x_2) d_\theta(x_0, x_1, x_1) = 0.$$

Hence, $d_\theta(x_0, x_1, x_m) = 0$.

Therefore, Eq. (2.3), becomes

$$d_\theta(x_n, x_m, a) \leq \left(\sum_{i=n}^{m-1} k^i \prod_{j=n}^{i} \theta(x_j, x_m, a)\right) d_\theta(x_0, x_1, a). \quad (2.4)$$

Notice the inequality above is dominated by

$$\sum_{i=n}^{m-1} k^i \prod_{j=n}^{i} \theta(x_j, x_m, a) \leq \sum_{i=1}^{m-1} k^i \prod_{j=1}^{i} \theta(x_j, x_m, a).$$

Since $\lim_{n,m\to\infty} \theta(x_n, x_m, a) < \frac{1}{k}$, so that the series, $\sum_{i=1}^{\infty} a_i$ where, $a_i = k^i \prod_{j=1}^{i} \theta(x_j, x_m, a)$, converges by ratio test to some $s \in (0, \infty)$, for each $m \in \mathbb{N}$. Let

$$s = \sum_{i=1}^{\infty} k^i \prod_{j=1}^{i} \theta(x_j, x_m, a),$$

with partial sum,

$$s_n = \sum_{i=1}^{n} k^i \prod_{j=1}^{i} \theta(x_j, x_m, a).$$

Thus for all, $m > n$, inequality Eq. (2.4) implies,

$$d_\theta(x_n, x_m, a) \leq d_\theta(x_0, x_1, a)[s_{m-1} - s_{n-1}\,] \quad (2.5)$$

Letting $n \to \infty$ in Eq. (2.5), we deduce that, $\{x_n\}$ is a Cauchy sequence in X. Since, X is complete, then there exist a point $u \in X$ such that $x_n \to u$. We have,

$d_\theta(Tu,u,a) \le \theta(Tu,u,a)[d_\theta(Tu,x_n,a) + d_\theta(Tu,x_n,u) + d_\theta(u,x_n,a)]$, $= \theta(Tu,u,a)[d_\theta(Tu,T^n x_0,a) + d_\theta(Tu,T^n x_0,u) + d_\theta(u,x_n,a)]$,

$\le \theta(Tu,u,a)\,[kd_\theta(u,x_{n-1},a) + k\,d_\theta(u,x_{n-1},u) + d_\theta(u,x_n,a)]$

≤ 0 , as $n \to \infty$.

Thus $d_\theta(Tu,u,a) = 0$, for all $a \in X$. Hence u is a fixed point of T.

For uniqueness of u: Let u, v be distinct fixed points of T, then for all $a \in X$, we have

$$d_\theta(u,v,a) = d_\theta(Tu,Tv,a) \le kd_\theta(u,v,a)\,,$$

This implies that $k \ge 1$, which is a contradiction to $k \in [0,1)$. Therefore u is a unique fixed point.

Definition2.5 (Samina B., 2016) Let $T: X \to X$. A function $G: X \to \mathbb{R}$ is said to be T-orbitally lower semi-continuous at $t \in X$ if the sequence $\{x_n\} \subset \mathcal{O}(x_0)$ is such that, $x_n \to t$, we have $G(t) \le \liminf_{n\to\infty} G(x_n)$.

Theorem 2.3 Let (X, d_θ) be a complete extended b_2- metric space such that d_θ is continuous functional. Let $T: X \to X$ and there exists $x_0 \in X$, such that

$$d_\theta(Ty,T^2y,a) \le k\,d_\theta(y,Ty,a), \text{ for each } \quad y \in \mathcal{O}(x_0) \qquad (2.6)$$

where $k \in [\,0,1\,)$ be such that for each $x_0 \in X$,

$\lim_{n,m\to\infty} \theta(x_n,x_m,a) < \frac{1}{k}$, and $x_n = T^n x_0, n = 1,2,\dots$. Then $T^n x_0 \to u\ (as\ n \to \infty)$. Furthermore u is a fixed point of T if and only if $G(u) = d_\theta(u,Tu,a)$ is T-orbitally lower semi-continuous at u.

Proof

We choose any $x_0 \in X$ be an arbitrary, define the iterative sequence $\{x_n\}$ by

$$x_0,\ x_1 = Tx_0,\ x_2 = Tx_1 = T^2x_0, \dots,\ x_n = T^n x_0\ \dots$$

Now for $y = Tx_0$ by successively applying inequality Eq. (2.6) we obtain

$$d_\theta(T^n x_0, T^{n+1}x_0, a) = d_\theta(x_n,x_{n+1},a) \le k^n d_\theta(x_0,x_1,a).$$

Following the same procedure as in proof of Theorem 2.2, we infer that $\{x_n\}$ is a Cauchy sequence in X. since X is complete , then there exist a point $u \in X$ such that $x_n = T^n x_0 \to u$. Assume that G is defined as in Definition 2.5 , then for all $a \in X$,we have

$$d_\theta(u,Tu,a) \le \liminf_{n\to\infty}\ d_\theta(T^n x_0, T^{n+1}x_0, a)$$

$$\le \liminf_{n\to\infty}\ k^n d_\theta(x_0,x_1,a) = 0,$$

thus $Tu = u$, and hence u is a fixed point of T .

Conversely, let $u = Tu$ and $x_n \in \mathcal{O}(x_0)$ with $x_n \to u$. Then,

$$G(u) = d_\theta(u,Tu,a) = 0 \le \liminf_{n\to\infty}\ G(x_n) = d_\theta(T^n x_0, T^{n+1}x_0, a).$$

Example 2.4. Let $X = \left[0, \frac{1}{4}\right]$. Define $\theta : X^3 \longrightarrow [1,\infty)$ by

$$\theta(x,y,z) = 2 + x + y + z\,,$$

and $d_\theta: X^3 \longrightarrow [0,\infty)$ by

$$d_\theta\,(x,y,z) = [\min\{|x-y|, |x-z|, |y-z|\}]^2.$$

Clearly, $\theta(x,y,a) < 3$, therefore (X, d_θ) is complete extended b_2-metric space .

Define $T: X \to X$ by,

$$Tx = \frac{x}{5}.$$

Then, we have

$$d_\theta(Tx,Ty,a) \le \frac{1}{4}\ d_\theta(x,y,a) \ = k\ d_\theta(x,y,a).$$

Now for each $x \in X$, $T^n x = \frac{x}{5^n}$. Thus we obtain

$$\lim_{m,n\to\infty} \theta(T^n x, T^m x, a) < 4 = \frac{1}{k}\,.$$

Therefore all condition of Theorem 2.2 are satisfied, hence T has a unique fixed point.

3. An application to integral equations

There are too many applications of fixed point theorem in mathematics, Specifically, the most widely quoted is an application to integral equations. Inspired by Kamran, et al., (2017), we will use Theorem 2.2 to show that the integral equation

$$x(t) = g(t) + \int_I M(t,s,x(s))ds\,, \quad t,s \in I, \quad I = [0,1], \qquad (3.1)$$

has a unique solution.

Let, $X = C(I,\mathbb{R})$ be the space of all continuous real valued functions defined on I. Note that X is a complete extended b_2-metric space by considering $d_\theta: X^3 \to [0,\infty)$,

$$d_\theta(x,y,z) = \left\{\max_{t\in I} \min\{|x(t)-y(t)|, |x(t)-z(t)|, |y(t)-z(t)|\}\right\}^2 \qquad (3.2)$$

with $\theta: X^3 \to [1,\infty)$ by $\theta(x,y,z) = |\,x(t)| + |\,y(t)| + |\,z\,(t)\,| + 3$.

Let $T: X \to X$, the operator given by:

$$Tx(t) = g(t) + \int_I M(t,s,x(s))ds\,, \quad t,s \in I. \qquad (3.3)$$

Assume that the following conditions are satisfied:

i. $g: I \to \mathbb{R}$, and $M: I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous functions,
ii. for $t,s \in I$ and $x,y,a \in X$,

$$|M(t,s,x(s)) - M(t,s,y(s))| \le$$

$$\frac{1}{2}\left[\max_{t\in I}\ \min\{|x(s)-y(s)|, |x(s)-a(s)|, |y(s)-a(s)|\}\right].$$

Theorem 3.1. Under assumptions i and ii the integral equation Eq. (3.1) has a unique solution in X.

Proof:

Define the extended b_2-metric $d_\theta: X \times X \times X \to [0,\infty)$ as above by Eq. (3.2). Then (X, d_θ) is a complete extended b_2-metric space. In addition, we define the operator$T: X \to X$, as given by Eq. (3.3). By using assumptions, we obtain that:

$$d_\theta(Tx,Ty,a) =$$

$$\left[\max_{t\in I}\ \min\{|Tx(t)-Ty(t)|, \quad |Tx(t)-a(t)|, |Ty(t)-a(t)|\}\right]^2,$$

$$\le [|\,Tx(t) - Ty(t)\ |]^2,$$

$$= \left[\left|\int_I \Big(M(t,s,x(s))\Big)ds + g(t) - \int_I \Big(M(t,s,y(s))\Big)ds - g(t)\right|\right]^2,$$

$$\le \left[\int_I |M(t,s,x(s)) - M(t,s,y(s))|ds\right],^2$$

$$\le \left[\int_I \frac{1}{2}\left[\max_{t\in I}\ \min\{|x(t)-y(t)|, |x(t)-a(t)|, |y(t)-a(t)|\}\right] ds\right]^2,$$

$$= \frac{1}{4}\left[\max_{t\in I}\ \min\{|x(t)-y(t)|, |x(t)-a(t)|, |y(t)-a(t)|\}\right]^2 \left[\int_I ds\right]^2$$

$$\le \frac{1}{4}\ d_\theta(x,y,a)$$

Therefore, $d_\theta(Tx,Ty,a) \le k\ d_\theta(x,y,a)$.

Hence all the conditions of Theorem 2.2 are satisfied and the mapping T has one fixed point. Thus, we conclude that the integral equation Eq. (3.1) has a unique solution in X.

4. Conclusion

In this paper, we introduce a new type of generalized b_2-metric space; we call an extended b_2-metric space. We also provide some fixed-point theorems on these spaces. This provides a background for extended b_2-metric spaces technique in the fixed-point theory. Several consequences can be observed from the main results. For example, taking $\theta(x, y, z)\ s$, with $s \geq 1$ implies corresponding fixed-point results in the context of b_2-metric space. Furthermore, for $\theta(x, y, z\ ) = 1$, the b_2-metric reduces to a 2-metric. In addition, we can indicate several directions from our results for further work, which go through fixed-point theory. As a new work, it will be interesting to extend known fixed-point results on an extended b-metric space and b_2-metric space to our results on an extended b_2-metric.

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