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Magnetic field curvature ($\vec{\kappa}$) and poloidal (β_θ) Calculations using single field MHD plasma in Tokamak systems

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Highlights

- **Energy liberated from fusion must balance the energy losses due to radiations.**
- **The power output for a given magnetic field assembly is proportional to the square root of the plasma beta (β^2).**
- **The magnetic field curvature is interpreted in terms of the radius of curvature of the magnetic field lines.**
- **To within a factor of order of unity the plasma beta is the ratio of the square of the sound speed to the square of the Alven wave speed.**

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ABSTRACT

In this paper, the magnetic field curvature ($\vec{\kappa}$) and the poloidal (β_θ), for single fluid magneto-hydrodynamic (MHD) plasma expressions are derived. To simplify the approach, we adhere to plasmas, with a circular cross-section. Thus we use the geometry of the torus, where (φ) is in the toroidal direction and (θ) is in the poloidal direction.

1. Introduction

In a fusion reactor, the high values of the ions temperature (T_i) and density (n_i) must be maintained long enough for the energy liberated by fusion to balance the energy losses due to radiation, conduction, convection and neutron flux. Let τ_E be the time it takes these processes to remove all the energy from the system, then for a given value of $n_i\tau_E$, there is a minimum temperature at which the plasma is said to be ignited, i.e. at which the liberated fusion energy is just adequate to balance all losses.

As D-D plasmas require considerably higher temperatures to achieve ignition, almost all reactor proposals have concentrated on D-T Fusion, which has a minimum at $T_i \backsim 30$ keV (Woods, 2006), where for ignition $n_i\tau_E > 1.5\times10^{20} m^{-3}s$. A slightly lower bound ($n_i\tau_E > 6\times10^{19} m^{-3}s$) known as Lawson's criterion (Lawson, 1957) is obtained if a continuous power supply from outside the system is used to compensate the transport and radiation losses.

The power output for a given magnetic field assembly is proportional to the square of the plasma beta ($2\mu p/B^2$), so for an adequate return on an energy investment in magnetic fields, it has been estimated that β in a reactor exceed at least 10 percent (Wesson, 1987).

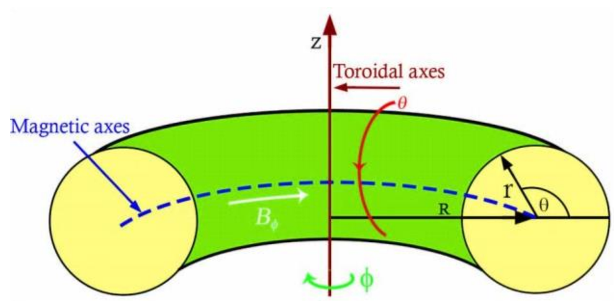

Fig. 1. Primitive toroidal coordinates

The aim of this work is to smooth the way for researchers and engineers in this field by casting the theoretical physical laws into simple quantitative mathematical formulas. To this end, the coordinate system is that of toroidal geometry, which is suitable to most promising fusion reactors mainly tokamaks. A simple torus is shown in Fig. 1 and Fig. 2 depicts the toroidal geometry of a tokamak. Where the toroidal axis vertical by convention; it is encircled by the magnetic axes, a single toroidal field line that generally locates the peak of the plasma current and plasma density profiles. The magnetic axis also identified with the toroidal direction parameter (φ). Similarly closed poloidal curves encircling the magnetic axis, indicate the local poloidal direction (θ) (Harris, 1974).

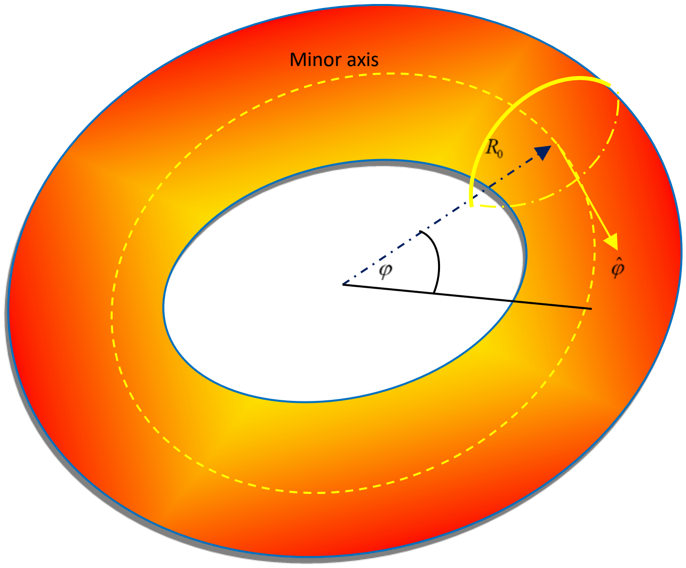

Fig. 2. Top view of the magnetic torus

2. Calculations of the parameters

In deriving the parameter β_0 which is the ratio of the ideal pressure p to the magnetic pressure $\beta^2/2\mu$, we have to adopt a single magnetohydrodynamic (MHD) model, where the timescale $\tau \geq \tau_{ei}$ and $T_e = T_i$, where τ_{ei} is electron-ion collision time and $T_e = T_i$ are the electrons and ions temperatures respectively. When the plasma fluid velocity is negligible, and the magnetic field is steady, the system is in static equilibrium. To obtain the magneto static equations we need the following MHD fluid equations (Shafranov, 1966; Bateman, 1980):

Firstly, the mass conservation equation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{1}$$

Secondly, the momentum conservation equation is

$$\frac{\partial (\rho \vec{v})}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{v} \vec{v} + \vec{P}\right) = \vec{J} \times \vec{B} + \rho \vec{F} \tag{2}$$

Where $\vec{F}$ is the body force and $\vec{P}$ is the pressure tensor. These two equations can be reduced to their intrinsic form by using the identity

$$D \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}, \textit{Hence}$$

$$D\rho + \rho \vec{\nabla} \cdot \vec{v} = 0 \tag{3}$$

$$\rho D \vec{v} + \vec{\nabla} \cdot \vec{P} = \vec{J} \times \vec{B} + \rho \vec{F} \tag{4}$$

If the body force $\vec{F}$, $\partial/\partial t$ and $\vec{v}$ are all zeros. Then the plasma is in static equilibrium and the divergence of the pressure tensor balances the magnetic force density

$$\vec{\nabla} \cdot \vec{P} = \vec{J} \times \vec{B} \tag{5}$$

The pressure tensor in the left-hand side of Eq. 5 consists of two parts (Wesson, 1987):

$$\vec{P} = p\vec{\delta} + \vec{\pi} \tag{6}$$

where p is the ideal pressure with $\vec{\boldsymbol{\delta}}$ the identity unit matrix tensor, the last term $\vec{\pi}$ is a pure viscosity term which can be written as:

$$\vec{\pi} = -\mu' \vec{\nabla} \cdot \vec{v} - 2\mu \vec{\nabla} \vec{v} \tag{7}$$

Where the scalar μ' and μ are known as the bulk and shear viscosity coefficients. However, all the $\vec{\pi}$ tensor terms are velocity dependent so can be dropped altogether. Hence, Eq. 6 reduced to

$$\vec{P} = p\vec{\delta} \tag{8}$$

The electromagnetic force density acting on the plasma in the right-hand side of Eq. 5 can be written in the following form (Wesson, 1987):

$$\vec{J} \times \vec{B} = \vec{\nabla} \cdot \vec{T} - \mu_0 \epsilon_0 \frac{\partial \vec{S}}{\partial x} \tag{9}$$

Where the Pointing energy vector $\vec{S} \equiv (1/\mu_0)\vec{E} \times \vec{B}$ and $\vec{T}$ is the electromagnetic stress tensor, which can be written as:

$$\vec{T} = \frac{1}{\mu_0} \vec{B}\vec{B} + \epsilon_0 \vec{E}\vec{E} - \left(\frac{B^2}{2\mu_0} + \frac{\epsilon_0}{2} E^2\right) \vec{\delta} \tag{10}$$

Writing the following vector identities for later use:

i) $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla} \cdot (\vec{E}\vec{E}) - \vec{E}\vec{\nabla} \cdot \vec{E} - \vec{\nabla}\left(\frac{1}{2}E^2\right)$

ii) $\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla} \cdot (\vec{B}\vec{B}) - \vec{B}\vec{\nabla} \cdot \vec{B} - \vec{\nabla}\left(\frac{1}{2}B^2\right)$ with $\vec{\nabla} \cdot \vec{B} = 0$

iii) $\vec{\nabla} \cdot (\varphi \vec{\delta}) = \vec{\nabla} \varphi$ if φ is a scalar

Dropping the time-dependent pointing energy term the force density now becomes:

$$\vec{J} \times \vec{B} = \vec{\nabla} \cdot \left(\frac{1}{\mu_0} \vec{B}\vec{B} + \epsilon_0 \vec{E}\vec{E}\right) - \vec{\nabla} \cdot \left(\frac{B^2}{2\mu_0} + \frac{\epsilon_0}{2} E^2\right) \vec{\delta} \tag{11}$$

In MHD dynamics, the plasma is quasineutral and any charge in balance will be shielded out and any resulting electric field $\vec{\boldsymbol{E}}$ vanishes. Thus, the force density in Eq. 11 can be given (Shafranov, 1966) as:

$$\vec{J} \times \vec{B} = \vec{\nabla} \cdot \frac{1}{\mu_0} \vec{B}\vec{B} - \vec{\nabla} \cdot \left(\frac{B^2}{2\mu_0}\right) \vec{\delta} \tag{12}$$

Now using Eq. 8 and Eq. 12 we can write Eq. 5 as:

$$\vec{\nabla} \cdot \left(p + \frac{B^2}{2\mu_0}\right) \vec{\delta} = \vec{\nabla} \cdot \frac{1}{\mu_0} \vec{B}\vec{B} \tag{13}$$

$$\vec{\nabla} \cdot \left(p + \frac{B^2}{2\mu_2}\right) \vec{\delta} = \frac{1}{\mu_0} (\vec{\nabla} \cdot \vec{B})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{B} \tag{14}$$

Using $\vec{\nabla} \cdot \vec{\mathrm{B}} = 0$ and identity (iii) in Eq. 14 to get:

$$\vec{\nabla}\left(p + \frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0} (\vec{B} \cdot \vec{\nabla})\vec{B} \tag{15}$$

3. Magnetic field curvature

The term $(\vec{B} \cdot \vec{\nabla})\vec{B}$ in the right hand side of Eq. 15 is related to the field curvature $\vec{\kappa} \equiv (\hat{b} \cdot \vec{\nabla})\hat{b}$ with $\vec{\mathrm{B}} = \mathrm{B}\hat{\mathrm{b}}$ we can write:

$$\mathrm{B}^2 \vec{\kappa} = (\vec{\mathrm{B}} \cdot \vec{\nabla})\vec{\mathrm{B}} \tag{16}$$

The term involving $\vec{\kappa}$ is more readily interpreted in terms of the radius $\vec{\mathbf{r}}$ of curvature of the magnetic field lines and the unit normal $\hat{\mathbf{n}}$

towards the center of curvature. The field line is normal to the curvature, i.e. $\vec{\kappa} \perp \hat{b}$ (Shafranov, 1966). This yield, see Fig. 3

$$\vec{\kappa} = \frac{\hat{n}}{r} = -\frac{\hat{r}}{r} \tag{17}$$

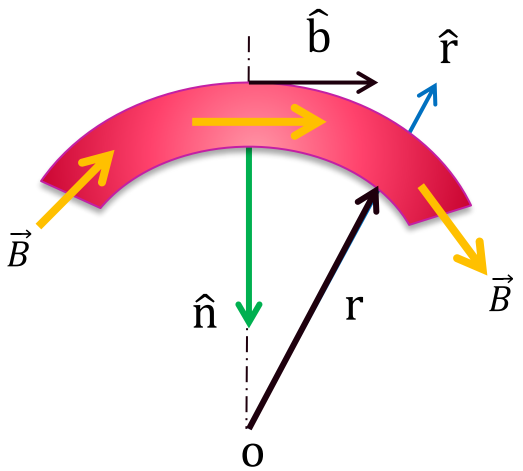

Fig. 3. Field line curvature

The right-hand side of Eq. 15 can be put in the following form:

$$\frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} = \frac{1}{\mu_0}B^2\,\vec{\kappa} + \vec{\nabla}_{\parallel}\left(\frac{B^2}{2\mu_0}\right) \tag{18}$$

Where $\vec{\nabla}_{\parallel}$ is parallel to $\vec{B}$ direction. Comparing Eq.15 and Eq. 18 we deduce that:

$$\vec{\nabla}\left(p + \frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}B^2\vec{\kappa} + \vec{\nabla}_{\parallel}\left(\frac{1}{\mu_0}B^2\right) \tag{19}$$

Using the definition of the field line curvature in Eq. 17 into Eq. 19 to have:

$$\vec{\nabla}\left(p + \frac{B^2}{\mu_0}\right) = -\frac{1}{\mu_0}\frac{B^2}{r}\hat{r} + \vec{\nabla}_{\parallel}\left(\frac{1}{\mu_0}B^2\right) \tag{20}$$

Due to equilibrium and assumed plasma column symmetry in the torus cross-section, we approximate a short segment of the plasma torus by cylindrical piece, shown in Fig. 4a and Fig. 4b.Thus we can adopt a cylindrical coordinates system for the shown piece. Now the toroidal φ-axis of the torus becomes the z-axial coordinate of the cylindrical piece (Goldstone, 1982). Hence, we need only the r-component of the cylindrical coordinates of Eq. 20:

$$\frac{d}{dr}\left(r^2\left(B_\varphi^2 + B_\theta^2\right)\right) = -2rB_\varphi^2 - 2rB_\theta^2 - 2\mu_0 r^2\frac{dp(r)}{dr} \tag{21}$$

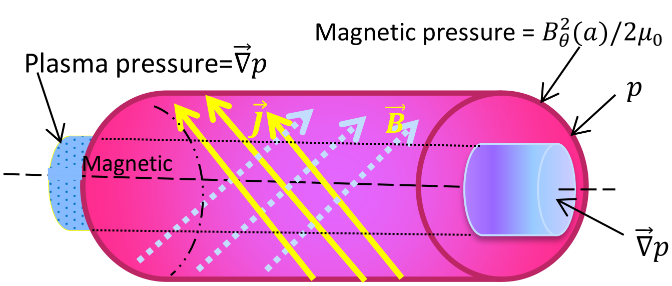

Fig. 4a. Show a short segment of the plasma torus by cylindrical piece

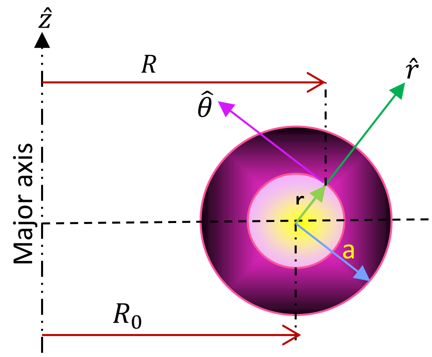

Fig. 4b. Cylindrical and local coordinates

In general, unstated pressure and toroidal field profiles, the solution to Eq. 21 can be written in terms of an average over the volume bounded by the so-called magnetic flux surface. In this work, the fields poloidal B_θ or toroidal B_φ are independent of the variables θ and φ. This amounts to a simple radial averaging, given by the general formula (Kruskel and Kulsrud, 1958).

$$\langle\psi\rangle \equiv \frac{2}{r^2}\int_0^r dr'r'\,\psi(r') \tag{22}$$

After some rearrangement and integration by parts, Eq. 21 yields:

$$B_\varphi^2 + B_\theta^2 - \langle B_\varphi^2(r)\rangle = -2\mu_0(p(r) - \langle p(r)\rangle) \tag{23}$$

To this end, the plasma poloidal β_θ is defined as

$$\beta_\theta(r) = 2\mu_0\frac{\langle p(r)\rangle}{B_\theta^2} \tag{24}$$

Using this definition in Eq. 23 to obtain:

$$\beta_\theta(r) = 1 + \frac{B_\varphi^2(r) - \langle B_\varphi^2(r)\rangle}{B_\theta^2(r)} + 2\mu_0\frac{p(r)}{B_\theta^2(r)} \tag{25}$$

This definition obtained from the plasma MHD theory slightly differs from that usually used in laboratory plasmas (Hutchison, 1987). The expression for $\beta_\theta(r)$ at the plasma boundary $(r = a)$ limiter radius. Thus

$$\beta_\theta(a) = 1 + \frac{B_\varphi^2 - \langle B_\varphi^2\rangle}{B_\theta^2(a)} \tag{26}$$

Where the plasma pressure $p(a) = 0$.

4. Conclusion

At a frequency below relevant collision frequencies; plasmas act like fluids. Almost all plasmas have magnetic fields and they are highly electrically conducting. The magnetic field is then frozen into the fluid and moves with the fluid. A distinction is made between high- β and low-β plasmas. In high- β plasmas; the magnetic field is dragged along the plasma and in low-β plasmas; the plasma is pulled by the magnetic field. Put in another way, for $\beta \gg 1$ the stresses in the plasma are predominantly gas like and are transferred by sound waves with speed

$$C_s = \sqrt{\frac{\gamma p}{\rho}}$$

Where ρ is the plasma mass density and γ is the adiabatic constant. For $\beta \ll 1$ the stresses are predominantly magnetic and are transferred by Alfven waves with speed

$\upsilon_A = \frac{B}{\sqrt{\rho\mu_0}}$

See (Cheng and Chance, 1985). To within a factor of order unity β is the ratio of the square of the sound speed C_s to the square of the Alfven speed υ_A, approximately equal to

$\beta \approx \frac{C_s^2}{\upsilon_A^2} = \frac{\gamma p}{\left(\frac{B}{\mu_0}\right)}$.

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