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Kripke models for intuitionistic propositional logic

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ABSTRACT

In this paper, we introduce Kripke models for intuitionistic propositional logic showing how these models enable us to understand the intuitionistic point of view to mathematical objects. In addition, we discuss their basic properties and give some applications of them reflecting their importance for intuitionistic propositional logic.

1. Introduction

Mathematical models play a very important role in simplification different systems such that we can describe the given system and predict its behavior. Therefore, we highlight on Kripke models, which were invented by the American philosopher and logician Saul Aaron Kripke in 1964, for intuitionistic propositional logic.

First, we give some principles about intuitionistic logic. Intuitionistic logic is a generalizing of classical logic by deleting the excluded middle and reduction absurdum rules. The best way to understand the idea of intuitionism is through forgetting the classical concept of "truth" and build our judgment about statements without basing on any predefined value of them. We will look at intuitionism as a special case of the principle of constructivism, which says, ''the existence of mathematical objects depends on the existence of a method to construct them and the validity of proofs is derived from these constructions'' (Dalen, 2001). The most important characteristic of intuitionistic propositional logic is that proofs must be constructive in the sense that "they say something quite specific about the additional information which the proof provides" (Dummett, 1974). In addition, the meaning of formulas that include connectives is different comparing with their meanings in classical logic. Brouwer, Heyting and Kolmogorov introduce an interpretation for intuitionistic propositional logic explaining the meanings of these formulas. This interpretation is known as *BHK-interpretation* and it is given as follows:

a) There is no proof for $\bot$.

b) A proof for $A \wedge B$ consists of a proof a of A and a proof b of B.

c) A proof for $A \vee B$ is a pair (b, c) where b determines which disjunct is true, and c is the proof for it.

d) A proof for $\neg A$ is a construction by which we can derive a contradiction from any proof of A . Thus, a proof of $\neg A$ says that A has no proof.

e) A proof for $A \to B$ is a construction that transforms every proof of A into a proof of B.This is in contrast with classical logic where $A \to B$ is false only if A is true and B is false. We cannot use this interpretation in intuitionistic logic because the classical disjunction is used and because of the assumption that we already know the truth values of A and B before determining the truth-value of $A \to B$.

In order to put intuitionistic propositional logic in its form, we define a proof system known as "natural deduction". The system of natural deduction consists of introduction and elimination rules for logical connectives.

Rules for intuitionistic propositional logic

$$\frac{A \in \Gamma}{\Gamma \vdash A}\ (\text{Assumption}) \qquad \frac{\Gamma \vdash A}{\Gamma \cup \{B\} \vdash A}\ (\text{Weaken})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge\,\text{Introduction})$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge\,\text{Elimination1}) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge\,\text{Elimination2})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee\,\text{Introduction1}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee\,\text{Introduction2})$$

$$\frac{\Gamma \cup \{A\} \vdash C \quad \Gamma \cup \{B\} \vdash C \quad \Gamma \vdash A \vee B}{\Gamma \vdash C}\ (\vee\,\text{Elimination})$$

$$\frac{\Gamma \cup \{A\} \vdash B}{\Gamma \vdash A \to B}\ (\to\text{Introduction})$$

$$\frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}\ (\to\text{Elimination}) \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\bot\ \text{Elimination})$$

$$\frac{\Gamma \cup \{A\} \vdash B \quad \Gamma \cup \{A\} \vdash \neg B}{\Gamma \vdash \neg A}\ (\neg\text{Introduction})$$

Now we give some basic concepts that we need in this paper. A lattice $\boldsymbol{L}$ is bounded if it has two elements 0,1 such that $0 \preccurlyeq a$, $a \preccurlyeq$

1 for any $a \in L$. A distributive lattice is a lattice $\boldsymbol{L}$, which satisfies either of the distributive laws,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

For any $a, b, c \in L$ (Burris, S. and Sankappanavar, H., 1981).

A ***Heyting algebra*** $\boldsymbol{H} = \langle H, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a bounded distributive lattice such that for any two elements a and b in H, there is a largest element $a \rightarrow b$, which is called a ***pseudo-complement of a with respect to b***, such that for any $c \in H$ we have

$$c \preccurlyeq a \rightarrow b \text{ if and only if } a \wedge c \preccurlyeq b$$

The operation $\rightarrow$ is called ***Heyting implication*** or simply ***implication*** (Bezhanishvili, N. and Jongh, D.). If V is a variety, we say that an algebra $\boldsymbol{A}$ in V is a ***free algebra over V*** if there is a generating subset X of A and any mapping from X to any algebra $\boldsymbol{B}$ in V can be extended uniquely to a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ (Burris, and Sankappanavar, 1981).

1. Basic definitions and examples

Definition 2.1.
A ***Kripke model K*** is a partially ordered set $\langle K, \preccurlyeq \rangle$ together with a mapping α from the set of all propositional variables for intuitionistic propositional logic to the power set of K such that for any $x, y \in K$ and any propositional variable p we have

$$\text{If } x \in \alpha(p) \text{ and } x \preccurlyeq y, \text{ then } y \in \alpha(p)$$

This property of α is known as the ***monotonicity property***.

We can extend α to a valuation ρ from the set of all propositional formulas to the power set of K as follows:

$$\rho(p) = \alpha(p), \text{ for any propositional variable } p$$

$$\rho(\perp) = \emptyset$$

$$\rho(A \wedge B) = \rho(A) \cap \rho(B)$$

$$\rho(A \vee B) = \rho(A) \cup \rho(B)$$

$$\rho(A \rightarrow B) = \{x \in K | \{x\} \uparrow \cap \rho(A) \subseteq \rho(B)\}$$

If $\Gamma = \{A_1, \ldots, A_n\}$ is a set of formulas, then the valuation of Γ is the valuation of the conjunction of formulas in Γ, i.e.

$$\rho(\Gamma) = \rho(A_1 \wedge \ldots \wedge A_n) = \rho(A_1) \cap \ldots \cap \rho(A_n)$$

If $\Gamma = \emptyset$, then $\rho(\Gamma) = K$.

Definition 2.2.
Let $\boldsymbol{K}$ be a Kripke model. We define the forcing relation $\vDash$ between elements of K and formulas as follows:

$$x \vDash A \text{ if and only if } x \in \rho(A),$$

for any $x \in K$ and any formula A. In this case, we say that x forces or satisfies A.

Now we can explain how intuitionism is represented by Kripke models as follows: The elements of the set K represent different states of our knowledge such that we can consider any x in K as a fact that we know at a particular time. The partial order $\preccurlyeq$ represents the extending states that we obtain by gaining more knowledge. That is, if we have two states of knowledge x and y such that $x \preccurlyeq y$, this means that we now know x and we may know y in the future. The forcing relation $\vDash$ tells which formulas can be deduced to be true if we know a particular fact (Fitting, 1969).

Definition 2.3.

Let $\boldsymbol{K}$ be a Kripke model. Then

1. A formula A is ***valid at a point x in K*** if $x \vDash A$ and we say that A is ***valid in K***, written as $\boldsymbol{K} \vDash A$, if $x \vDash A$ for all x in K.

2. A set of formulas Γ is said to be ***valid at a point x in K*** if $x \vDash A$ for all $A \in \Gamma$. Moreover, we say that Γ is ***valid in K or the theory of K***, written as $\boldsymbol{K} \vDash \Gamma$, if Γ is valid at each point of $\boldsymbol{K}$.

3. A formula A is a ***Kripke consequence of Γ***, written as $\Gamma \vDash A$, if A is valid in $\boldsymbol{K}$ whenever Γ is valid in $\boldsymbol{K}$. In addition, A is called a ***Kripke valid*** if $\emptyset \vDash A$ or briefly we write $\vDash A$.

Definition 2.4.
A formula A is said to ***be deduced from a set of formulas Γ***, written as $\Gamma \vdash A$, if there is a finite sequence of expressions of the form $\Gamma_1 \vdash A_1, \ldots, \Gamma_{n-1} \vdash A_{n-1}, \Gamma \vdash A$ such that each expression is derived from some previous ones using rules. The natural number n is called the ***height of the derivation***. If $\emptyset \vdash A$ (or simply $\vdash A$), then we say that A is a ***theorem of intuitionistic propositional logic***.

2. Properties of Kripke models

Lemma 3.1.
If $x \in \rho(A)$ and $x \preccurlyeq y$, then $y \in \rho(A)$. In other words, the valuation ρ on a Kripke model $\boldsymbol{K}$ satisfies the monotonicity property.

Proof
We can get the proof by induction on the construction of A.
If $A = p$ for some propositional variable p, then by monotonicity of ρ we have

$$x \in \rho(p) \Longrightarrow y \in \rho(p) \text{ for all } y \succcurlyeq x$$

Suppose that the theorem is true for all formula with less number of connectives than A.

If $A = B \wedge C$, then

$$
\begin{aligned}
x \in \rho(B \wedge C) &\Leftrightarrow x \vDash B \wedge C \\
&\Leftrightarrow x \vDash B \text{ and } x \vDash C \\
&\Leftrightarrow (\forall y \succcurlyeq x)(y \vDash B \text{ and } y \vDash C) \\
&\Leftrightarrow (\forall y \succcurlyeq x)(y \vDash B \wedge C) \\
&\Leftrightarrow (\forall y \succcurlyeq x)(y \in \rho(B \wedge C))
\end{aligned}
$$

If $A = B \vee C$, then

$$
\begin{aligned}
x \in \rho(B \vee C) &\Leftrightarrow x \vDash B \vee C \\
&\Leftrightarrow x \vDash B \text{ or } x \vDash C \\
&\Leftrightarrow (\forall y \succcurlyeq x)(y \vDash B \text{ or } y \vDash C) \\
&\Leftrightarrow (\forall y \succcurlyeq x)(y \vDash B \vee C) \\
&\Leftrightarrow (\forall y \succcurlyeq x)(y \in \rho(B \vee C))
\end{aligned}
$$

If $A = B \rightarrow C$. Let $x \in \rho(B \rightarrow C) = \rho(A)$ and $y \succcurlyeq x$. To prove that $y \in \rho(B \rightarrow C)$ we take $z \succcurlyeq y$ such that $z \in \rho(B)$.

Since $y \succcurlyeq x, z \succcurlyeq y$ and $\preccurlyeq$ is transitive we have $z \succcurlyeq x$.

Thus $z \in \rho(C)$.

Hence $y \in \rho(B \rightarrow C)$.

Theorem 3.1.
The forcing relation $\vDash$ has the following properties

1. $x \vDash p$ if and only if $x \in \rho(p)$ for any propositional variable p.
2. $x \vDash \perp$ never holds.
3. $x \vDash A \wedge B$ if and only if x satisfies both A and B.
4. $x \vDash A \vee B$ if and only if x satisfies either A or B.
5. $x \vDash A \rightarrow B$ if and only if $(\forall y \succcurlyeq x)(y \vDash A \Rightarrow y \vDash B)$.

Proof

1. It is immediate by the definition of $\vDash$.

2. $x \vDash \perp \Leftrightarrow x \in \rho(\perp) = \emptyset$ which cannot hold since $\emptyset$ does not contain any element.

3. $x \vDash A \wedge B \Leftrightarrow x \in \rho(A \wedge B) = \rho(A) \cap \rho(B)$

$\Leftrightarrow x \in \rho(A)$ and $x \in \rho(B)$

$\Leftrightarrow x \vDash A$ and $x \vDash B$

4. $x \vDash A \vee B \Leftrightarrow x \in \rho(A \vee B) = \rho(A) \cup \rho(B)$

$\Leftrightarrow x \in \rho(A)$ or $x \in \rho(B)$

$\Leftrightarrow x \vDash A$ or $x \vDash B$

5. $x \vDash A \to B \Leftrightarrow x \in \rho(A \to B)$

$\Leftrightarrow \{x\} \uparrow\cap\, \rho(A) \subseteq \rho(B)$

$\Leftrightarrow (\forall y \succcurlyeq x)\big(y \in \rho(A) \Rightarrow y \in \rho(B)\big)$

$\Leftrightarrow (\forall y \succcurlyeq x)(y \vDash A \Rightarrow y \vDash B)$

Corollary 3.1.

1. $x \vDash \neg A$ if and only if $\forall y \succcurlyeq x, y \nvDash A$.
2. $x \vDash \neg\neg A$ if and only if $\forall y \succcurlyeq x, \neg\forall(z \succcurlyeq y)(z \nvDash A)$.

Proof

1. $x \vDash \neg A \Leftrightarrow x \in \rho(\neg A) = \rho(A \to \perp)$

$\Leftrightarrow \{x\} \uparrow\cap\, \rho(A) \subseteq \rho(\perp) = \emptyset$

$\Leftrightarrow \{x\} \uparrow\cap\, \rho(A) = \emptyset$

$\Leftrightarrow \forall y \succcurlyeq x, y \notin \rho(A)$

$\Leftrightarrow \forall y \succcurlyeq x, y \nvDash A$

2. $x \vDash \neg\neg A \Leftrightarrow x \in \rho(\neg\neg A) = \rho(\neg A \to \perp)$

$\Leftrightarrow \{x\} \uparrow\cap\, \rho(\neg A) \subseteq \rho(\perp) = \emptyset$

$\Leftrightarrow \{x\} \uparrow\cap\, \rho(\neg A) = \emptyset$

$\Leftrightarrow \forall y \succcurlyeq x, y \notin \rho(\neg A)$

$\Leftrightarrow \forall y \succcurlyeq x, y \nvDash \neg A$

$\Leftrightarrow \forall y \succcurlyeq x, \neg\forall(z \succcurlyeq y)(z \nvDash A)$ from (1)

Theorem 3.2. (Soundness)

If $\Gamma \vdash A$ in intuitionistic logic, then $\Gamma \vDash A$ for any Kripke model.

Proof

The proof of the theorem is obtained by showing that for any Kripke model $\boldsymbol{K}$ and any $x \in K$, if $x \in \rho(\Gamma)$, then $x \in \rho(A)$ by using induction on the height of the derivation of $\Gamma \vdash A$. We will prove the theorem for the rules of $(\to I)$, $(\to E)$ and $(\neg I)$.

For the rule of $(\to I)$, $A = B \to C$ and we have a derivation of $\Gamma \cup \{B\} \vdash C$. Let $x \in \rho(\Gamma)$.To show that $x \in \rho(\mathrm{A}) = \rho(\mathrm{B} \to \mathrm{C})$ we take $y \in K$ such that $y \succcurlyeq x$ and $y \in \rho(B)$.

By monotonicity, we have $y \in \rho(\Gamma)$.

So $y \in \rho(\Gamma) \cap \rho(B) = \rho(\Gamma \cup \{B\})$. By using the induction hypothesis we get $y \in \rho(C)$.

Thus

$$x \in \rho(\mathrm{B} \to \mathrm{C}) = \rho(\mathrm{A})$$

For the rule of $(\to E)$, we have derivations of $\Gamma \vdash B \to A$, $\Gamma \vdash B$. Let $x \in \rho(\Gamma)$, then the induction hypothesis says that $x \in \rho(B)$ and $x \in \rho(\mathrm{B} \to \mathrm{A})$. So for all $y \succcurlyeq x$ such that $y \in \rho(B)$ we have $y \in \rho(A)$. But $\preccurlyeq$ is reflexive,*i.e.*

$$x \preccurlyeq x, \forall x \in K$$

This implies that $x \in \rho(\mathrm{A})$.

For the $(\neg I)$ rule, $A = \neg C$ and we have derivations of

$\Gamma \cup \{C\} \vdash B$ and $\Gamma \cup \{C\} \vdash \neg B$.

Let $x \in \rho(\Gamma)$.Take $y \in K$ such that $y \succcurlyeq x$ and suppose that $y \in \rho(C)$. By monotonicity we have $y \in \rho(\Gamma)$.

So, $y \in \rho(\Gamma) \cap \rho(C) = \rho(\Gamma \cup \{C\})$. By the induction hypothesis we obtain

$$y \in \rho(B) \text{ and } y \in \rho(\neg B)$$

This is contradiction.

Thus, $y \notin \rho(C)$ which means that $x \in \rho(\neg C) = \rho(A)$.

Theorem 3.3. (Completeness of Kripke models)

If $\Gamma \vDash A$ for any Kripke model, then $\Gamma \vdash A$ in intuitionistic logic **(**SØrsensen, and Urzyczyn, 1998**)**.

3. Applications of Kripke models for intuitionistic propositional logic

i. Counter models

We have seen that a formula A is provable in intuitionistic propositional logic if and only if it is valid in each Kripke model. Thus, to show that a formula A is unprovable in intuitionistic propositional logic we just need a finite Kripke model $\boldsymbol{K}$ such that A is invalid in it, i.e. $\rho(A) \neq K$. Such models are called ***counter models.***

Example 4.1.

1. Counter model for excluded middle and double negation

We use the Kripke model $\boldsymbol{K} = \langle K, \preccurlyeq, \alpha\rangle$ shown in Fig. 1A, where $\alpha(p) = \{b\}$.

$$\rho(\neg p) = \{x \in K | \{x\} \uparrow\cap \{b\} = \emptyset\} = \emptyset$$

$$\rho(p \vee \neg p) = \rho(p) \cup \rho(\neg p) = \{b\} \cup \emptyset = \{b\} \neq K$$

Thus, $p \vee \neg p$ is not provable in intuitionistic logic.

Also,

$$\rho(\neg\neg p) = K$$

$$\rho(\neg\neg p \to p) = \{x \in K | \{x\} \uparrow\cap \rho(\neg\neg p) \subseteq \rho(p)\}$$

$$= \big\{x \in K \big| \{x\} \uparrow\cap K \subseteq \{b\}\big\} = \{b\} \neq K$$

Hence $\neg\neg p \to p$ is not intuitionistically provable.

2. Counter model for Peirce $((\boldsymbol{p} \to \boldsymbol{q}) \to \boldsymbol{p}) \to \boldsymbol{p})$

Let our Kripke model be as shown in Fig. 1B, Where

$$\alpha(p) = \{b, c\}, \alpha(q) = \{c\}$$

Then

$$\rho(p \to q) = \big\{x \in K \big| \{x\} \uparrow\cap \{b, c\} \subseteq \{c\}\big\} = \{c\} \neq K$$

$$\rho\big((p \to q) \to p\big) = \big\{x \in K \big| \{x\} \uparrow\cap \{c\} \subseteq \{b, c\}\big\} = K$$

$$\rho\Big(\big((p \to q) \to p\big) \to p\Big) = \big\{x \in K \big| \{x\} \uparrow\cap K \subseteq \{b, c\}\big\} = \{b, c\} \neq K$$

This implies that $\big((p \to q) \to p\big) \to p$ is unprovable.

3. Counter model for $\big(\boldsymbol{p} \to (\boldsymbol{q} \vee \boldsymbol{r})\big) \to \big((\boldsymbol{p} \to \boldsymbol{q}) \vee (\boldsymbol{p} \to \boldsymbol{r})\big)$

Consider the Kripke model $\boldsymbol{K} = \langle K, \preccurlyeq, \alpha\rangle$ which is given in Fig. 1C, where $\alpha(p) = \{a, b\}, \alpha(q) = \{a\}, \alpha(r) = \{b\}$.

We have

$$\rho(p \to (q \vee r)) = \big\{x \in K \big| \{x\} \uparrow\cap \{a, b\} \subseteq \{a, b\}\big\} = K$$

$$\rho(p \to q) = \{a\}$$

$$\rho(p \to r) = \{b\}$$

$$\rho\big((p \to q) \vee (p \to r)\big) = \rho(p \to q) \cup \rho(p \to r) = \{a, b\}$$

$$\rho(p \to (q \vee r)) \to ((p \to q) \vee (p \to r))) = \{a, b\} \neq K$$

Hence $(p \to q \vee r) \to (p \to q) \vee (p \to r)$ is unprovable.

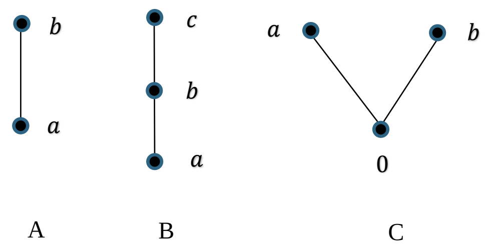

Fig. 1. Counter models for propositional formulas

Remark

Because of the monotonicity of ρ, we can see that if our model $\boldsymbol{K}$ has a smallest element x_0, then a given formula A is valid in $\boldsymbol{K}$ if and only if $x_0 \vDash A$.

Example 4.2.

Consider the Kripke model $\boldsymbol{K} = \langle K, \preccurlyeq, \alpha \rangle$ in Fig.1. C, where

$$\alpha(p) = \{b\}$$

This model has a smallest element 0. Therefore, we will check the validity of $\neg p, \neg\neg p, \neg p \vee \neg\neg p$ using the above remark.

$$0 \notin \rho(\neg p) = \{a\}$$

$$\therefore 0 \nvDash \neg p$$

$$0 \notin \rho(\neg\neg p) = \{b\}$$

$$\therefore 0 \nvDash \neg\neg p$$

$$0 \notin \rho(\neg p \vee \neg\neg p) = \rho(\neg p) \cup \rho(\neg\neg p) = \{a\} \cup \{b\} = \{a, b\}$$

$$\therefore 0 \nvDash \neg p \vee \neg\neg p$$

Hence, these formulas are invalid in the given model.

ii. The proof of the disjunction property for intuitionistic propositional logic

Theorem 4.1.

$\vdash A \vee B$ if and only if $\vdash A$ or $\vdash B$.

Proof

The first direction is trivial by using BHK-interpretation. To prove the another direction we use contradiction through finding two counter models $\boldsymbol{K_1}$ for A and $\boldsymbol{K_2}$ for B and adding a new root x_0 to the union of $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$ such that $A \vee B$ is invalid in x_0 (SØrsensen, M. and Urzyczyn, P., 1998).

iii. Generic Kripke model

We can use Kripke models to construct what is known as a *generic* or *universal model* $\boldsymbol{R_n} = \langle K_n, \preccurlyeq_n, \rho_n \rangle$, $n < \omega$, from which we will form the Heyting algebra of intuitionistic propositional formulas in n variables. We will also see that this Heyting algebra is isomorphic to the free Heyting algebra $\boldsymbol{F_n(H)}$ of n generators which has the universal mapping property for the class of Heyting algebras, so it gives more information about the class. In order to define the generic Kripke model, we construct a chain of Kripke models ordered by inclusion such that each model is obtained from the previous one by adding one new level underneath. Each of these models must be reduced in the sense that there is no two distinct points w, v with the same valuation such that v is the unique cover for w, or such that each strictly dominator of w is a strictly dominator of v. In each case both w and v satisfy the same formulas, so we can omit one of them without any effect on the theory of the model (Bellissima, 1986). To do that we will follow Junker, Darnie`re and Bellissima (Bellissima, 1986; Darnie`re, and Junker, 2008). Bellissima defined the valuation ρ on a Kripke model $\boldsymbol{K} = \langle K, \preccurlyeq, \alpha \rangle$ from the elements of K to the power set of all formulas. In this case, we define the forcing relation $\vDash$ by

$$x \vDash A \iff A \in \rho(x),$$

for all $x \in K$ and any formula A.

Definition 4.1.

A set X is said to be ***upward-closed*** if for any $x \in X$ we have

$$\{x\} \uparrow = \{y \in X | y \succcurlyeq x\} \subseteq X$$

Now we will illustrate the method of constructing $\boldsymbol{R_n}$.
Let $P_n = \{p_i | i < n\}, 0 \leq n < \omega$ be a set of intuitionistic propositional variables, and define a set

$$val_n = \{\beta | \beta \subseteq P_n\} = 2^{P_n}$$

We define each model $\boldsymbol{R_n^d}$ by induction on d as shown in the following steps:

1. To construct the set of states K_n^d, we first define $K_n^{-1} = \emptyset$. Then the elements $w_{\beta,Y}$ of the level $K_n^d \setminus K_n^{d-1}$ satisfy the following conditions:

(i) Y is an upward-closed set in the previous model and must intersect the last level of it. So, for $d = 0$ we have $Y = \emptyset$.

(ii) $\beta \in val_n$ such that $\beta \subseteq \bigcap_{w \in Y} \rho_n^{d-1}(w)$. Thus, if $d = 0$, then the number of elements in K_n^0 is equal to the number of elements in val_n.

(iii) If Y is an upset for some element $w \in K_n^{d-1}$, then we must have $\rho_n^d(w_{\beta,Y}) \subsetneq \rho_n^{d-1}(w)$. Thus, if $\rho_n^{d-1}(w) = \emptyset$, there is no element $w_{\beta,Y}$ such that $\rho_n^d(w_{\beta,Y}) \subset \emptyset$. This means that we cannot add any new element $w_{\beta,Y}$ under w in the new level.

2. The valuation of each element $w_{\beta,Y}$ is defined by

$$\rho_n^d(w_{\beta,Y}) = \beta$$

3. The partial ordering $\preccurlyeq_n^{d-1}$ is extended to $\preccurlyeq_n^d$ as follows:

$$\preccurlyeq_n^d = \preccurlyeq_n^{d-1} \cup \left\{(w_{\beta,Y}, w) | w_{\beta,Y} \in K_n^d \setminus K_n^{d-1} \text{ and } w = w_{\beta,Y} \text{ or } w \in Y\right\}$$

Finally, we define our generic model $R_n$by

$$K_n = \bigcup_{d<\omega} K_n^d, \preccurlyeq_n = \bigcup_{d<\omega} \preccurlyeq_n^d, \rho_n = \bigcup_{d<\omega} \rho_n^d$$

Remark
While we were constructing the elements of each level, we put two conditions

(i) $\beta \subseteq \bigcap_{w \in Y} \rho_n^{d-1}(w)$

(ii) If $Y = \{w\} \uparrow$ for some $w \in K_n^{d-1}$, we must have

$$\rho_n^d(w_{\beta,Y}) \subsetneq \rho_n^{d-1}(w)$$

The first condition ensures that the extension of ρ to the new model remains a valuation. Moreover, if we assume that

$\rho_n^d(w_{\beta,Y}) = \rho_n^{d-1}(w)$ in the second condition, then both w and $w_{\beta,Y}$ have the same valuation and w is the unique cover of $w_{\beta,Y}$. This means that our model will not be reduced.

Example 4.3.

To construct R_1 which is shown in Fig. 2 we define

$$P_1 = \{p\}, val_1 = \{\emptyset, \{p\}\}$$

We show systematically how to construct each model R_1^d by induction on d.

For $d = 0$, the only upward-closed set in $\boldsymbol{R}_1^{-1}$ is $Y = \emptyset$, so

$$K_1^0 = K_1^0 \setminus K_1^{-1} = \{w_{\{p\},\emptyset}, w_{\emptyset,\emptyset}\} = \{w_0, w_1\}$$

Also

$$\preccurlyeq_1^0 = \{(w_0, w_0), (w_1, w_1)\}, \rho_1^0(w_0) = \{p\}, \rho_1^0(w_1) = \emptyset$$

For $d = 1$, the upward-closed subsets in $\boldsymbol{R}_1^0$ are

$$\{w_0\}\uparrow, \{w_1\}\uparrow, K_1^0$$

Since $\rho_1^0(w_1) = \emptyset$ we cannot add any new element under w_1. So, the desired upward-closed subsets of $\boldsymbol{R}_1^0$ are $\{w_0\}\uparrow, K_1^0$.

$$K_1^1 \setminus K_1^0 = \{w_{\emptyset,\{w_0\}\uparrow}, w_{\emptyset,K_1^0}\} = \{w_2, w_3\}$$

$$K_1^1 = K_1^0 \cup \{w_2, w_3\} = \{w_0, w_1, w_2, w_3\}$$

$$\preccurlyeq_1^1 = \preccurlyeq_1^0 \cup \{(w_2, w_2), (w_3, w_3), (w_2, w_0), (w_3, w_0), (w_3, w_1)\}$$

$$\rho_1^1(w_2) = \rho_1^1(w_3) = \emptyset$$

For $d = 2$, the upward-closed subsets in $\boldsymbol{R}_1^1$ that intersect the last level of the model are

$$\{w_2\}\uparrow, \{w_3\}\uparrow, \{w_1\}\uparrow\cup\{w_2\}\uparrow, K_1^1$$

There is no any element to add under $\{w_2\}\uparrow, \{w_3\}\uparrow$ because their valuation is $\emptyset$. Thus the required upward-closed subsets in $\boldsymbol{R}_1^1$ are

$$\{w_1\}\uparrow\cup\{w_2\}\uparrow, K_1^1$$

In addition

$$K_1^2 \setminus K_1^1 = \{w_{\emptyset,\{w_1\}\uparrow\cup\{w_2\}\uparrow}, w_{\emptyset,K_1^1}\} = \{w_4, w_5\}$$

$$K_1^2 = K_1^1 \cup \{w_4, w_5\} = \{w_0, w_1, w_2, w_3, w_4, w_5\}$$

$$\preccurlyeq_1^2 = \preccurlyeq_1^1 \cup \{(w_4, w_4), (w_5, w_5), (w_4, w_0), (w_5, w_0), (w_4, w_1), (w_5, w_1), (w_4, w_2), (w_5, w_2), (w_5, w_3)\}$$

$$\rho_1^2(w_4) = \rho_1^2(w_5) = \emptyset$$

Continuing like this we obtain

$$K_1 = \bigcup_{d<\omega} K_1^d, \preccurlyeq_1 = \bigcup_{d<\omega} \preccurlyeq_1^d, \rho_1 = \bigcup_{d<\omega} \rho_1^d$$

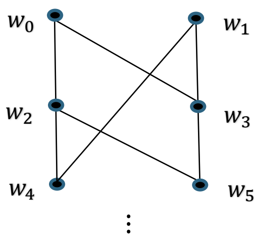

Fig. 2. The generic Kripke model R_1

Definition 4.2.1

For $n < \omega$, we define the ***Heyting algebra of intuitionistic propositional formulas in n variables*** $\boldsymbol{H_n} = \langle H_n, \vee, \wedge, \to, 0, 1\rangle$ as follows:

$$H_n = \{X | X \text{ is upward} - \text{closed subset in } \boldsymbol{R_n}\}$$

$$X \vee Y = X \cup Y$$

$$X \wedge Y = X \cap Y$$

$$X \Longrightarrow Y = \{x \in K_n | \{x\}\uparrow \cap X \subseteq Y\}$$

$$0 = \emptyset, 1 = K_n$$

for any upward-closed subsets X, Y of K_n.

Example 4.4.

After we have constructed $\boldsymbol{R_1}$, we will show how we can get the algebra of intuitionistic propositional variables in one variable p.

To do that we find the upward-closed subsets in $\boldsymbol{R_1}$

$\{w_0\}\uparrow = \{w_0\}, \{w_1\}\uparrow = \{w_1\}, \{w_2\}\uparrow = \{w_0, w_2\}$,

$\{w_3\}\uparrow = \{w_0, w_1, w_3\}$,

$\{w_0\}\uparrow\cup\{w_1\}\uparrow = \{w_0, w_1\}, \{w_1\}\uparrow\cup\{w_2\}\uparrow = \{w_0, w_1, w_2\}$,

$\{w_2\}\uparrow\cup\{w_3\}\uparrow = \{w_0, w_1, w_2, w_3\}$,`

$\{w_4\}\uparrow = \{w_0, w_1, w_2, w_4\}, \{w_5\}\uparrow = \{w_0, w_1, w_2, w_3, w_5\}$,

$\{w_4\}\uparrow\cup\{w_5\}\uparrow = \{w_0, w_1, w_2, w_3, w_4, w_5\}$…

The resulting Heyting algebra is shown in Fig. 3.

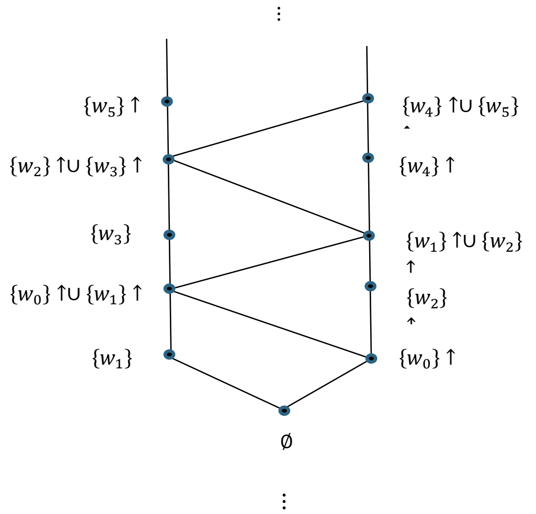

Fig. 3. The algebra of intuitionistic propositional variables in one variable H_1

Theorem 4.2.

For all $n < \omega$, the theory of the universal model and the set of intuitionistic tautologies are equal (Bellissima, 1986).

Theorem 4.3.

The Heyting algebra $\boldsymbol{H_n}$ is isomorphic to the free Heyting algebra $\boldsymbol{F_H(n)}$ on n generators.

Proof

Define a map $f: \boldsymbol{F_H(n)} \to \boldsymbol{H_n}$ by $f(\varphi) = \{w \in K_n | w \vDash \varphi\}$. Then

By definition of $\vDash$ we can see easily that the image of a contradiction statement is $\emptyset$ and the image of a tautology statement is K_n.

$$\begin{aligned} f(\varphi \vee \psi) &= \{w \in K_n | w \vDash \varphi \vee \psi\} \\ &= \{w \in K_n | w \vDash \varphi \text{ or } w \vDash \psi\} \\ &= \{w \in K_n | w \vDash \varphi\} \cup \{w \in K_n | w \vDash \psi\} \\ &= f(\varphi) \cup f(\psi) \end{aligned}$$

$$\begin{aligned} f(\varphi \wedge \psi) &= \{w \in K_n | w \vDash \varphi \wedge \psi\} \\ &= \{w \in K_n | w \vDash \varphi \text{ and } w \vDash \psi\} \\ &= \{w \in K_n | w \vDash \varphi\} \cap \{w \in K_n | w \vDash \psi\} \\ &= f(\varphi) \cap f(\psi) \end{aligned}$$

$$\begin{aligned} f(\varphi \to \psi) &= \{w \in K_n | w \vDash \varphi \to \psi\} \\ &= \{w \in K_n | (\forall v \succcurlyeq w)(v \vDash \varphi \text{ implies } v \vDash \psi\} \\ &= \{w \in K_n | (\forall v \succcurlyeq w)(v \in f(\varphi) \text{ implies } v \in f(\psi))\} \\ &= \{w \in K_n | \{w\}\uparrow \cap f(\varphi) \subseteq f(\psi)\} = f(\varphi) \Longrightarrow f(\psi) \end{aligned}$$

Every upward-closed set in $\boldsymbol{H_n}$ satisfy the tautology statement. So f is onto.

$$\begin{aligned} \ker f &= \{(\varphi, \psi) \in \boldsymbol{F_H^2(n)} | f(\varphi) = f(\psi)\} \\ &= \{(\varphi, \psi) \in \boldsymbol{F_H^2(n)} | w \vDash \varphi \Leftrightarrow w \vDash \psi \text{ for any } w \in K_n\} \end{aligned}$$

From theorem 4.2. we can see that ker f is the diagonal set. Hence f is one-to-one.

Example 4.5.

Consider the free Heyting algebra of one generator p

$$F_H(1) = \{p, \neg p, \bot, p \vee \neg p, p \to \neg p, \neg p \to p, \dots\}$$

If we define a map $f: \boldsymbol{F_H}(\mathbf{1}) \to \boldsymbol{H_1}$ by $f(\varphi) = \{w \in K_1 : w \vDash \varphi\}$

Then we have

$$f(\bot) = \emptyset$$
$$f(p) = \{w_0\} = \{w_0\}\uparrow$$
$$f(\neg p) = f(p \to \bot) = f(p) \Rightarrow f(\bot) = \{w_0\}\uparrow \Rightarrow \emptyset = \{w_1\}\uparrow$$
$$f(p \vee \neg p) = f(p) \cup f(\neg p) = \{w_0\}\uparrow \cup \{w_1\}\uparrow$$
$$f(\neg p \to p) = f(\neg p) \Rightarrow f(p) = \{w_1\}\uparrow \Rightarrow \{w_0\}\uparrow = \{w_2\}\uparrow$$
$$f(p \to \neg p) = f(p) \Rightarrow f(\neg p) = \{w_0\}\uparrow \Rightarrow \{w_1\}\uparrow = \{w_1\}\uparrow$$

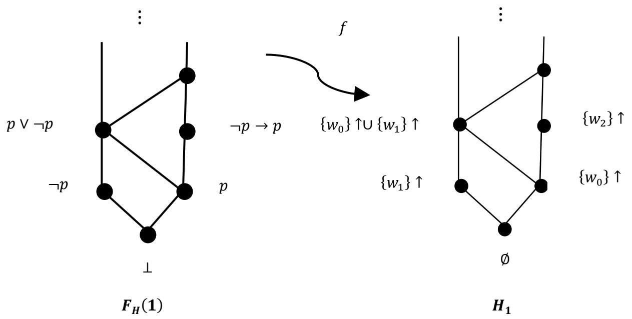

Fig. 4. The isomorphism between $F_H(1)$ and H_1

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