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Expansion of some well-known functions and classical polynomials in series of the generalized Laguerre polynomials.

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Highlights

- **This article shows how to expand some well-known classical polynomials such as the Hermite, the Legendre polynomials and some well-known functions such as the first kind Bessel functions and gamma function in series of the generalized Laguerre polynomials**
- **The expansion in series of the generalized Laguerre polynomials is very applicable tool in mathematics, physics and engineering**

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ABSTRACT

The generalized Laguerre polynomials form a complete set (orthogonal and normalized) in the space $L^2(0,\infty)$ with respect to a certain weighting function because they are the Eigen functions of a second-order differential operator. Here we shall show how to expand some well-known classical polynomials such as the Legendre and the Hermite polynomials in series of generalized Laguerre polynomials. Since the generalized Laguerre orthogonal polynomials are set of polynomials that are mutually orthogonal to each other with respect to a measure of weighting function that is just the integrand of the gamma function, thus it grants us the guarantee of the ability to expand the first kind of Bessel functions and the gamma function in terms of the generalized Laguerre polynomials. The series expansion in terms of the generalized Laguerre polynomials can be achieved by following various approaches. For instance, the series expansion of the first kind Bessel functions is gained by the generalized hypergeometric function approach, whereas the series expansion of gamma function was obtained directly by the usual way that is by calling the orthogonality and orthonormality properties of the generalized Laguerre polynomials. Another powerful technique to gain the series expansion of the classical polynomials is the generating function approach as it has been followed here to obtain a series expansion of the Legendre and the Hermite polynomials in terms of the generalized Laguerre polynomials. The series expansion in series of the generalized Laguerre polynomials has a variety of applications in mathematics, physics, and engineering.

1. Introduction

The field of orthogonal polynomials (Chihara *et al.*, 2001; Gautsch, 2004; Koornwinder *et al.*, 2010; Mourad, 2005; Sansone, 1991; Szegö, 1975; Vilmos, 2005) has been taken growing interest by many well-known scientists in the twenty century. The most common orthogonal polynomials are the classical polynomials such as the Laguerre, Hermite, Legendre, Chebyshev, Jacobi and Gegenbauer polynomials (Andrew *et al.*, 1999; Bell, 1968; Brychkov, 2008; Rainville, 1960; Sneddon, 1980). The classical polynomials in general and the generalized (or referred to as associated) Laguerre polynomials (Arfken,1985; Bell, 1968; Brychkov, 2008; Lebedev, 1972; Rainville, 1960; Sneddon, 1980; Chihara *et al.*, 1999) in particular are very important in many applications in mathematics, engineering, physics. A comprehensive historical literature review on what has been done on the orthogonal polynomials over the 45 years beginning in 1950 was written by Chihara (Chihara *et al.*, 2001).

The generalized Laguerre polynomials are named after the French mathematician Edmond Laguerre (1834, 1886) and occur in quantum mechanics as the solution of the Schrödinger equation for the Hydrogen atom. The orthogonal polynomials are a set of polynomials that are mutually orthogonal to each other with respect to a measure of weighting function under a certain inner product. Any piecewise continuously differentiable function and square-integrable with respect to a weighting function associated with the classical orthogonal polynomials can be expanded in series of some classes of the classical orthogonal polynomialssuch as the Hermite or the generalized Laguerre polynomials (Uspensky, 1972). In fact, the orthogonal classical orthogonal is the Eigen-functions of a symmetric second-order differential operator, thus such polynomials play an important role in the theory of moments, continued fractions, and spectral theory (Brychkov, 2008).

The generalized Laguerre polynomials are considerably important in the numerical analysis as they can be used for Gaussian quadrature to numerically compute integrals of the form$\int_0^\infty e^{-x} f(x)$ (Gautsch, 2004; Van Assche, 1987). It is worth mentioning that there are some classes of orthogonal polynomials that are orthogonal on some plane regions in the complex plane such as discs or triangles. For instance, Cantero (Cantero *et al.*, 2003) considered orthogonal polynomials for some curves in the complex plane such as the unit circle. Furthermore, Endl (1955)

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generalized the orthogonality of the Hermite and the Laguerre polynomials over some star regions in the complex plane.

By following various approaches (Lebedev, 1972; Rainville, 1960), in this paper we shall show how to expand some well-known classical polynomials such as the Hermite and the Legendre polynomials in series of the generalized Laguerre polynomials. Furthermore, we shall show how to expand some well-known functions such as the first kind Bessel functions and gamma function in series of generalized Laguerre polynomials.

This paper is structured as follows: in section two, we briefly set up some concepts that we need in the paper. These concepts consist of a brief introduction of some needed formulae on double series manipulations, Pochhammer symbol, gamma function, the generalized hypergeometric function, the Hermite polynomials, the first kind Bessel functions, the Legendre polynomials, and finally the associated Laguerre polynomials and its hypergeometric representation. Then the general theory of expanding any function in terms of generalized Laguerre polynomials is introduced in section three. In sections four and five, expansions of the Hermite and the Legendre polynomials in series of the generalized Laguerre polynomials will be obtained respectively. Then expansions of the gamma function and the first kind Bessel functions in series of the generalized Laguerre polynomials will be obtained respectively in sections six and seven. Finally, a conclusion is drawn in section eight.

2. Preliminaries

Here we shall introduce some necessary concepts, which we will need later on, such as a very important tool of dealing with double series.

2.1. Double series manipulations

Here we shall introduce some elementary operations with double summation which we will need later in the rearrangement of series appearing later in the paper.

Theorem 1: For a convergent power series φ , one has

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\varphi(m,n-m) \tag{1}$$

and

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]}\varphi(m,n-2m) \tag{2}$$

where $[n/2]$ is the greatest integer symbol defined as

$$\left[\frac{n}{2}\right]=\begin{cases} n/2\,, & \text{for n even,}\\ (n-1)/2\,, & \text{for n odd}\end{cases}$$

It should be noted that these identities can be taken in reverse order, that is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\varphi(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi(m,n+m) \tag{3}$$

and

$$\sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]}\varphi(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi(m,n+2m) \tag{4}$$

Also note that a combination of the Identities (1) and (2) yields

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\varphi(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]}\varphi(m,n-m) \tag{5}$$

2.2. Pochhammer. Symbol and gamma function

It is very convenient to introduce the so-called generalized factorial function (Arfken,1985; Rainville, 1960) or Pochhammer symbol $(a)_n$ defined as

$$(a)_n=\prod_{k=1}^{n}(a+k-1),\,(a)_0=1,\quad a\neq 0$$

$$(a)_n=a(a+1)(a+2)\ldots(a+n-1),\qquad n\geq 1$$

Now we introduce the gamma function that is related to the Pochhammer symbol.

Definition 1: For a non-negative real numberα, the gamma function $\Gamma(\alpha)$ is defined by the following Euler integral

$$\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}dx\,,\alpha>0 \tag{6}$$

The gamma function can be written as the sum of two integrals as follows:

$$\Gamma(\alpha)=\int_0^{\delta}x^{\alpha-1}e^{-x}dx+\int_{\delta}^{\infty}x^{\alpha-1}e^{-x}dx\,,\qquad \delta\in[0,\infty)$$

$$\Gamma(\alpha)=\gamma(\alpha,\delta)+\Gamma(\alpha,\delta)\;\;\alpha>0 \tag{7}$$

where $\gamma(\alpha,\delta)$ is called the lower incomplete gamma function and $\Gamma(\alpha,\delta)$ is called the upper incomplete gamma function. For fixed δ, the function $\Gamma(\alpha,\delta)$is analytic for all z, while the function $\gamma(\alpha,\delta)$is not analytic at the values $=0,-1,-2,\ldots$. The factorial function is related to the gamma function by the following relation,

$\Gamma(a+1)=a!$

In addition, the Pochhammer symbol is related to the gamma function by the following relation.

Theorem 2: If a is neither zero nor a negative integer then

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},\qquad a\neq 0,\pm1,\pm2,\ldots,n=0,1,2,\ldots \tag{8}$$

Next, we introduce some beneficial identities that we will need in our derivations in this article.

Lemma 1: Let n be a positive integer and a any real number,

$$(a)_{2n}=2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n \tag{9}$$

Proof: since

$$\begin{aligned}(a)_{2n}&=a(a+1)(a+2)(a+3)\ldots(a+2n-1),\\ &=2^{2n}\left[\frac{a}{2}\left(\frac{a}{2}+\frac{1}{2}\right)\left(\frac{a}{2}+1\right)\left(\frac{a}{2}+\frac{3}{2}\right)\ldots\left(\frac{a}{2}+n-\frac{1}{2}\right)\left(\frac{a}{2}+n-1\right)\right],\\ &=2^{2n}\frac{a}{2}\left(\frac{a}{2}+1\right)\ldots\left(\frac{a}{2}+n-1\right)\left(\frac{a+1}{2}\right)\left(\frac{a+3}{2}\right)\ldots\left(\frac{a}{2}+n-\frac{1}{2}\right)\end{aligned}$$

Thus,

$$(a)_{2n}=2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n$$

This relation can be generalized as in the following lemma.

Lemma 2: If k is a positive integer and $n=0,1,2,\ldots$ then

$$(a)_{kn}=k^{nk}\left(\frac{a}{k}\right)_n\left(\frac{a+1}{k}\right)_n\ldots\left(\frac{a+k-1}{k}\right)_n \tag{10}$$

Lemma 3: For integer numbers $k,\ n$ such that $0\leq k\leq n$, one has the identity

$$\frac{(-n)_k}{n!}=\frac{(-1)^k}{(n-k)!},\qquad 0\leq k\leq n \tag{11}$$

Proof: Since

$$\frac{(-n)_k}{n!} = \frac{(-n)(-n+1)\ldots(-n+k-1)}{n(n-1)\ldots(n-k+1)(n-k)(n-k-1)\ldots 3.2.1},$$

$$= \frac{(-1)^k n(n-1)\ldots(n-k+1)}{n(n-1)\ldots(n-k+1)(n-k)(n-k-1)\ldots 3.2.1}$$

Thus, one obtains the required Relation (11).

In a similar fashion to lemma 1, we can prove the following beneficial formula

$$(a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k}, \qquad 0 \le k \le n \tag{12}$$

2.3. The generalized hypergeometric function

In this section, we shall introduce some functions that are used in this paper. Consider the series

$$1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)\;\; n!} z^n \tag{13}$$

where z is a complex variable, α or β and γ are parameters, which can take arbitrary real or complex values provided that $\gamma \neq 0, -1, -2, \ldots$. If we let $\alpha = 1$ and $\beta = \gamma$, then we get the elementary geometric series $\sum_{n=0}^{\infty} z^n$. In terms of the Pochhammer symbol, we can simplify the hypergeometric Series (13) in the following form,

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n$$

We shall denote the convergent hypergeometric Series (13) by the notation $F(\alpha, \beta; \gamma; z)$ that is

$$F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n, \quad |z| < 1, \gamma \neq 0, -1, -2, \ldots \tag{14}$$

The confluent hypergeometric series is defined by

$$\Phi(\alpha;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n n!} z^n, \forall\, z, \gamma \neq 0, -1, -2, \ldots \tag{15}$$

which is convergent for all finite values of z. Thus, the confluent hypergeometric function is analytic for all finite values of z. Next, we show how to rewrite a function in terms of the confluent hypergeometric function

Example 1: Rewrite the function $f(z) = (1-z)^a$ in terms of the hypergeometric function.

We can rewrite the function $f(z)$ in terms of the hypergeometric function as follows,

$$(1-z)^{-a} = 1 + az + \cdots + a(a+1)(a+n-1)\frac{z^n}{n!} + \cdots$$

Thus,

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} = \Phi(a; -; z), \qquad |z| < 1 \tag{16}$$

The Series (14) can be generalized as

$$1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{z^n}{n!} \tag{17}$$

where the denominator parameters b_j are not allowed to be zeros or negative integers. By the ratio test, it can be easily shown that the Series (17) converges for all finite z if $p \le q$ (Rainville, 1960).

We shall denote to the Series (17) by the notation ${}_pF_q(a_1, a_2, \ldots a_p; b_1, b_2, \ldots b_q; z)$ that is

$${}_pF_q(a_1, a_2, \ldots a_p; b_1, b_2, \ldots b_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{z^n}{n!} \tag{18}$$

Or

$${}_pF_q(a_1, a_2, \ldots a_p; b_1, b_2, \ldots b_q; z) = {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots a_p; \\ b_1, b_2, \ldots b_q; \end{matrix}\; z\right] \tag{19}$$

2.4. Hermite polynomials

The Hermite polynomials denoted as $H_n(x)$ are defined by the following generating function

$$e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n, \qquad \forall \text{ finite, real } x, h \tag{20}$$

Now the exponential functions in the left-hand side of equation (20) can be expanded as,

$$e^{2xh-h^2} = \left(\sum_{n=0}^{\infty} \frac{(2xh)^n}{n!}\right)\left(\sum_{j=0}^{\infty} \frac{(-h^2)^j}{j!}\right) \tag{21}$$

Substituting Eq. (21) in Eq. (20) yields

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j (2x)^n h^{n+2j}}{j!\, n!} \tag{22}$$

This relation will be used in obtaining the series expansion of the Hermite polynomials in terms of the generalized Laguerre polynomials as it will be shown in Section 4.

2.5. First kind Bessel functions

Here we present all the needed information about the Bessel function that is will be used in this paper. The Bessel differential equation takes the form
$x^2 y'' + xy' + (x^2 - \nu^2) y = 0$

Solving this equation about the regular singular point at $x = 0$ using the Frobenius method (Arfken, 1985), one obtains the Bessel functions of the first kind of order ν as

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\;\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2k} \tag{23}$$

We now show how to obtain the generalized hypergeometric representation of Bessel functions by using the Identity (8) for the term $\Gamma(k+\nu+1)$ in Eq. (23) to obtain

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(1+\nu)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\;(1+\nu)_k} \left(\frac{x^2}{4}\right)^k$$

Now using the notation of the generalized hypergeometric Function (18), yields

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(1+\nu)}\; {}_0F_1(-; 1+\nu; -\frac{x^2}{4}) \tag{24}$$

This hypergeometric representation of the first kind Bessel functions will be beneficial in expanding the first kind Bessel function in series of the associated Laguerre polynomials as it will be shown in Section 7.

2.6. Legendre polynomials

The Legendre polynomials $P_n(x)$ are defined by the following generating function as

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = (1 - 2xh + h^2)^{-\frac{1}{2}}$$

Using this formula we will derive a very beneficial series expression of the Legendre polynomials that will be used later on. Therefore, by rewriting the function$(1-2xh+h^2)^{-\frac{1}{2}}$ in terms of the hypergeometric function using the Identity (16), one has

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (2x+h)^n}{n!} h^{n+k}$$

Now by using the binomial expansion of the term $(2x+h)^n$, we have

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k \left(\frac{1}{2}\right)_n (2x)^{n-k}}{k!\,(n-k)!} h^{n+k}$$

Rearrange this double series using the Identity (5), one has

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k}}{k!\,(n-2k)!} h^{n} \tag{25}$$

Again rearranging this double series by using the Identity (4) yields,

$$\sum_{n=0}^{\infty} P_n(x)\, h^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k} (2x)^{n}}{k!\ n!} h^{n+2k} \tag{26}$$

This equation will be used in Section 5 to show how to expand the Legendre polynomials in a series of generalized Laguerre polynomials. It should be noted that by equating the coefficients of h^n on both sides of Eq. (25), we have the following useful expression of $P_n(x)$,

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k}}{k!\,(n-2k)!}$$

2.7. Generalized Laguerre polynomials

The generalized Laguerre differential equation takes the form,

$$xy''(x) + (\alpha+1-x)y'(x) + ny(x) = 0 \tag{27}$$

Solving this equation about the regular singular point at $x=0$ using the Frobenius method (Arfken,1985; Andrew *et al.*, 1999), one obtains the generalized Laguerre polynomials$L_n^\alpha(x)$ as,

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{\Gamma(\alpha+n+1)(-1)^k}{\Gamma(\alpha+k+1)(n-k)!\,k!} x^k \tag{28}$$

In addition, the generalized Laguerre polynomials $L_n^\alpha(x)$ are defined by the following formula of Rodrigues type as,

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^{x}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}), n = 0,1,2,\ldots \tag{29}$$

The generating function of the generalized Laguerre polynomials $L_n^\alpha(x)$ are defined by the following formula,

$$\frac{e^{-xh/(1-h)}}{(1-h)^{\alpha+1}} = \sum_{n=0}^{\infty} L_n^\alpha(x) h^n, \qquad |h|<1 \tag{30}$$

Letting $\alpha=0$ in Eq. (30) leads to the generating function of the simple Laguerre polynomials $L_n(x)$ as,

$$\frac{e^{-xh/(1-h)}}{(1-h)} = \sum_{n=0}^{\infty} L_n(x) h^n, \qquad |h|<1 \tag{31}$$

2.8. The confluent hypergeometric representation of the generalized Laguerre polynomials

Here we introduce the confluent hypergeometric representation of the generalized Laguerre polynomials. To achieve that, we implement the Identity (11) in the series of the generalized Laguerre polynomials (28), thus one has

$$L_n^\alpha(z) = \sum_{k=0}^{n} \frac{\Gamma(\alpha+n+1)(-n)_k(-1)^k}{\Gamma(\alpha+k+1)n!\ k!} z^k$$

Now using the property of gamma Function (8) leads to

$$\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)} = \frac{\Gamma(\alpha+n+1)/\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)/\Gamma(\alpha+1)} = \frac{(\alpha+1)_n}{(\alpha+1)_k}$$

Thus, one has

$$L_n^\alpha(z) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n} \frac{(-n)_k z^k}{(\alpha+1)_k k!}$$

Using the notation of the confluent hypergeometric Function (15), yields the confluent hypergeometric representation of the generalized Laguerre polynomials as,

$$L_n^\alpha(z) = \frac{(\alpha+1)_n}{n!}\Phi(-n, \alpha+1; z) \tag{32}$$

Next, we prove a very important property of the generalized Laguerre polynomials.

2.9. Orthogonality property of the generalized Laguerre polynomials

Now we prove one of the most important properties of the generalized Laguerre polynomials, which are the orthogonlaity and the orthonormality with respect to the weight function $w(x) = x^\alpha e^{-x}$ on the half-line $x \in (0,\infty)$ (Andrew *et al.*, 1999). We shall start the proof by using the generating function of the generalized Laguerre polynomials (30) as

$$\sum_{n=0}^{\infty} t^n L_n^\alpha(x) = \frac{1}{(1-t)^{\alpha+1}} e^{-xt/(1-t)}, \qquad |t|<1 \tag{33}$$

$$\sum_{m=0}^{\infty} s^m\ L_m^\alpha(x) = \frac{1}{(1-s)^{\alpha+1}} e^{-xs/(1-s)}, \qquad |s|<1 \tag{34}$$

Multiplying Eq. (33) by Eq. (34), then doing the integration on the half-line $x\in[0,\infty)$ with the weight function $w(x) = x^\alpha e^{-x}$, yields

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} s^m t^n \int_0^\infty x^\alpha\, e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx$$
$$= \frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}} \int_0^\infty e^{-\frac{(1-st)}{(1-t)(1-s)}x}\, dx$$

To do the integral on the right-hand side of the last equation, we make the substitution

$$y = \frac{(1-st)}{(1-t)(1-s)} x$$

and recall the Euler definition of Gamma function (6) to obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} s^m t^n \int_0^\infty x^\alpha\, e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(\alpha+1)}{(1-st)^{\alpha+1}} \qquad \alpha > -1.$$

Now since

$$(1-st)^{\alpha+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}(st)^n(n+\alpha)!}{n!\,\alpha!}$$

where we have used the binomial coefficient in the form

$$\binom{\alpha+1}{n} = (-1)^n \binom{n+\alpha+1-1}{n}$$

Thus, one has

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} s^m t^n \int_0^{\infty} x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+1)}{n!}(st)^n$$

Equating the coefficients of $(st)^n$ on both sides of this equation, yields

$$\int_0^{\infty} x^\alpha\, e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{m,n}\ , \qquad \alpha > -1 \qquad (35)$$

where $\delta_{m,n}$ is the Kronker delta symbol defined as

$$\delta_{m,n} = \begin{cases} 0, & \text{for } m \neq n, \\ 1, & \text{for } m = n \end{cases}$$

This is the orthogonality and orthonormality properties of the generalized Laguerre polynomials (eigenfunctions of the second-order Differential operator (27)).

Thus the normalized and mutually orthogonal $L_n^\alpha(x)$ with respect to the measure weighting function $w(x) = x^\alpha e^{-x}$on the half-line $x \in (0,\infty)$ for$\alpha > -1 \left\{\sqrt{\frac{n!}{\Gamma(\alpha+n+1)}} L_n^\alpha(x), n \in \mathbb{N}\right\}$form a complete set in the Hilbert space $L^2(0,\infty)$.

3. Expansion of functions in series of the generalized Laguerre polynomials

It is useful to introduce the following important definitions.

Definition 2: The real function $f(x)$ is piecewise continuous on the interval $[a,b]$ if:

1- $f(x)$is continuous on $[a,b]$ except at a finite number of points $\{x_1, x_2, \dots x_n\}$.

2- The two limits $\lim_{x\to x_i^-} f(x) = f(x_i^-)$, and $\lim_{x\to x_i^+} f(x) = f(x_i^+)$

are existed except at the two endpoints $x = a, b$ where only the limits $f(b^-), f(a^+)$ are existed.

Definition 3: The real function $f(x)$ is piecewise smooth on the interval$[a,b]$if the function itself and its first derivative are both piecewise continuous on $[a,b]$.

The real function $f(x)$ defined in the half-line $0 \le x < \infty$ can be expanded in series of the generalized Laguerre polynomials in the following form

$$f(x) = \sum_{n=0}^{\infty} a_n L_n^\alpha(x),\ \ 0 < x < \infty, \qquad \alpha > -1 \qquad (36)$$

provided that the function $f(x)$ meets certain conditions that are mentioned in the following theorem.

Theorem 3: If the real function $f(x)$ defined through the half-line $0 < x < \infty$ is piecewise smooth in every finite subinterval $[x_1, x_2]$ where $0 < x_1 < x_2 < \infty$ and well-behaved at the two endpoints $x_1 = 0$, $x_2 = \infty$. Then, the Expansion (36) converges to the function $f(x)$ at each continuity point of $f(x)$ and converges to $\frac{[f(x^+)+f(x^-)]}{2}$ at each discontinuity point of $f(x)$ in the Hilbert space $L^2(0,\infty)$ if and only if $f(x)$ is square-integrable, that is, the norm should satisfy

$$\|f\|_{L^2}^2 = \int_0^{\infty} e^{-x} x^\alpha f^2(x)\, dx < \infty$$

The unknown coefficient a_n in the Expansion (36) can be computed using the orthogonality and orthonormality properties of the generalized Laguerre polynomials (35). That is by multiplying the Expansion (36) by the function $x^\alpha e^{-x} L_m^\alpha(x)$ and then integrating on the interval $0 < x < \infty$ to obtain the coefficients a_n as

$$a_n = \langle f(x), L_n^\alpha(x)\rangle \ = \frac{n!}{\Gamma(\alpha+n+1)} \int_0^{\infty} e^{-x}\, x^\alpha f(x) L_n^\alpha(x) dx \qquad (37)$$

where the angled brackets ⟨ , ⟩ denote the inner product.

Example 2: Expand the function $f(x) = x^\beta$ in terms of the generalized Laguerre polynomials

The function$f(x)$ can be expanded as

$$x^\beta = \sum_{k=0}^{\infty} a_k L_k^\alpha(x),\ \ 0 < x < \infty, \qquad \alpha > -1$$

The coefficients$a_k$can be computed using Eq. (37) as

$$a_k = \langle x^\beta, L_k^\alpha(x)\rangle = \frac{k!}{\Gamma(\alpha+k+1)} \int_0^{\infty} e^{-x}\, x^{\alpha+\beta} L_k^\alpha(x) dx$$

Now replace the function $L_k^\alpha(x)$ by its equivalence using the Rodrigues formula (29) to obtain,

$$a_k = \frac{k!}{\Gamma(\alpha+k+1)} \int_0^{\infty} x^\beta \frac{d^k}{dx^k}(e^{-x} x^{k+\alpha}) dx.$$

Doing this integral by parts k times, one has

$$a_k = \frac{(-1)^n \beta(\beta-1)\dots(\beta-k+1)}{\Gamma(\alpha+k+1)} \int_0^{\infty} e^{-x} x^{\alpha+\beta}\, dx$$

Calling the definition of Gamma function (6), multiplying, and dividing by the factor $\Gamma(\beta-k+1)$ obtain the required expansion as,

$$x^\beta = \sum_{k=0}^{\infty} \frac{(-1)^k \beta!\ \Gamma(\alpha+\beta+1)}{\Gamma(\alpha+k+1)\Gamma(\beta-k+1)} L_k^\alpha(x) \qquad (38)$$

By letting $\beta = n = 0,1,2,\dots$is a positive integer in the Expansion (38), one has

$$x^n = \sum_{k=0}^{\infty} \frac{(-1)^k n!\ \Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)\Gamma(n-k+1)} L_k^\alpha(x)$$

To simplify the coefficients in this expansion we use the property of Gamma function (8) to obtain

$$\frac{\Gamma(\alpha+\mathrm{n}+1)}{\Gamma(\alpha+\mathrm{k}+1)} = \frac{\Gamma(\alpha+\mathrm{n}+1)/\Gamma(\alpha+1)}{\Gamma(\alpha+\mathrm{k}+1)/\Gamma(\alpha+1)} = \frac{(\alpha+1)_n}{(\alpha+1)_k}$$

$$\Gamma(n-k+1) = (n-k)!, \qquad k \le n$$

Thus, one has

$$x^n = \sum_{k=0}^{n} \frac{(-1)^k n!\ (\alpha+1)_n}{(n-k)!\,(\alpha+1)_k} L_k^\alpha(x) \qquad (39)$$

This expansion will be very beneficial in obtaining the series expansion of the Hermite polynomials in terms of the generalized Laguerre polynomials as shown in the next section.

4. Expansion of the Hermite polynomials in series of the generalized Laguerre polynomials

The core idea of obtaining the expansion of the Hermite polynomials in series of the generalized Laguerre polynomials is to replace the expansion of x^n in the generating function of Hermite polynomials by its series expansion of the generalized Laguerre polynomials and then doing some lengthy and tedious calculations to approach the desired limit as shown here (Rainville, 1960). Therefore, we start by implementing the Expansion (39) into the equivalent form of the generating function of Hermite polynomials given by Eq. (22) to achieve

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{k+j} 2^n (1+\alpha)_n L_k^\alpha(x)}{j!\,(n-k)!\,(1+\alpha)_k} h^{n+2j}$$

Rearrange this triple series using the Identity (3), one has

$$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}h^n=\sum_{n,k,j=0}^{\infty}\frac{(-1)^{k+j}2^{n+k}(1+\alpha)_{n+k}L_k^{\alpha}(x)}{j!\,n!\,(1+\alpha)_k}h^{n+k+2j}$$

Rearrange this triple series using the Identity (2), one has

$$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}h^n$$
$$=\sum_{n,k=0}^{\infty}\sum_{j=0}^{[n/2]}\frac{(-1)^{k+j}2^{n+k-2j}(1+\alpha)_{n+k-2j}L_k^{\alpha}(x)}{j!\,(n-2j)!\,(1+\alpha)_k}h^{n+k} \tag{40}$$

Now we try to simplify the coefficients of this series, so using the Relation (12) leads to

$$(1+\alpha)_{n+k-2j}=\frac{(-1)^{2j}(1+\alpha)_{2j}}{(-\alpha-n-k)_{2j}} \tag{41}$$

Also, using the Identity (9) yields,

$$(-\alpha-n-k)_{2j}\ =2^{2j}\left(\frac{-\alpha-n-k}{2}\right)_j\left(\frac{-\alpha-n-k+1}{2}\right)_j \tag{42}$$

In addition, using the Identity (11) leads to

$$\frac{1}{(n-2j)!}=\frac{(-1)^{2j}(-n)_{2j}}{n!},\qquad 0\le 2j\le n \tag{43}$$

Again, using the Identity (9) to obtain,

$$(-n)_{2j}=2^{2j}\left(\frac{-n}{2}\right)_j\left(\frac{-n+1}{2}\right)_j \tag{44}$$

Substituting all the Relations (41), (42), (43), and (44) in the Eq. (40) and using the definition of the generalized hypergeometric function (19) leads to,

$$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}h^n$$
$$=\sum_{n,k=0}^{\infty}{}_2F_2\left[\begin{matrix}-\frac{n}{2},\frac{1-n}{2};\\ -\frac{(\alpha+n+k)}{2},-\frac{(\alpha+n+k-1)}{2};\end{matrix}\ -\frac{1}{4}\right]\frac{(-1)^k\,2^{n+k}(1+\alpha)_{n+k}L_k^{\alpha}(x)}{n!\,(1+\alpha)_k}h^{n+k} \tag{45}$$

Now to be able to equate the power of h^n on both sides of the Series (45), we need to use the Identity (1) and finally equating the coefficients of h^n on both sides. Then reuse the Relation (11) to achieve the following expansion of the Hermite polynomials in series of the generalized Laguerre polynomials as,

$$\sum_{k=0}^{n}{}_2F_2\left[\begin{matrix}\frac{-(n-k)}{2},-\frac{(n-k-1)}{2};\\ -\frac{(\alpha+n)}{2},-\frac{(\alpha+n-1)}{2};\end{matrix}\ -\frac{1}{4}\right]\times\frac{(-n)_k}{(1+\alpha)_k}L_k^{\alpha}(x) \tag{46}$$

Letting $\alpha=0$ in Eq. (55) leads to the expansion of the Legendre polynomials in series of the simple Laguerre polynomials $L_n(x)$, that is

$$H_n(x)$$
$$=2^n n!\sum_{k=0}^{n}{}_2F_3\left[\begin{matrix}\frac{-(n-k)}{2},-\frac{(n-k-1)}{2};\\ -\frac{n}{2},-\frac{(n-1)}{2};\end{matrix}\ -\frac{1}{4}\right]\frac{(-n)_kL_k(x)}{k!},$$

$n>1,(1)_n=n!,(1)_k=k!$

5. Expansion of the Legendre polynomials in series of the generalized Laguerre polynomials

Here in this section, we follow the same steps as the previous section to obtain the expansion of the Legendre polynomials in series of generalized Laguerre polynomials. That is we replace the expansion of x^n in the generating function or (its equivalence) of Legendre polynomials by its series expansion of the generalized Laguerre polynomials (Rainville, 1960) and then doing some lengthy and tedious operations as shown below. Therefore, we start by implementing the expansion of x^n given by Eq. (39) into the equivalent form of the generating function of Legendre polynomials given by Eq. (26) to achieve

$$\sum_{n=0}^{\infty}P_n(x)\,h^n=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k+j}2^n\left(\frac{1}{2}\right)_{n+j}(1+\alpha)_nL_k^{\alpha}(x)}{j!\,(n-k)!\,(1+\alpha)_k}h^{n+2j}$$

Rearrange this triple series using the Identity (3), one has

$$\sum_{n=0}^{\infty}P_n(x)\,h^n$$
$$=\sum_{n,k,j=0}^{\infty}\frac{(-1)^{k+j}2^{n+k}\left(\frac{1}{2}\right)_{n+k+j}(1+\alpha)_{n+k}L_k^{\alpha}(x)}{j!\,n!\,(1+\alpha)_k}h^{n+k+2j}$$

Again, rearrange this triple series using the Identity (2) yields,

$$\sum_{n=0}^{\infty}P_n(x)\,h^n$$
$$=\sum_{n,k=0}^{\infty}\sum_{j=0}^{[n/2]}\frac{(-1)^{k+j}2^{n+k-2j}\left(\frac{1}{2}\right)_{n+k-j}(1+\alpha)_{n+k-2j}L_k^{\alpha}(x)}{j!\,(n-2j)!\,(1+\alpha)_k}h^{n+k} \tag{47}$$

Now we try to simplify the coefficients of this series, so use the Relation (12) to obtain,

$$\left(\frac{1}{2}\right)_{n+k-j}=\frac{(-1)^j\left(\frac{1}{2}\right)_{n+k}}{\left(\frac{1}{2}-n-k\right)_j} \tag{48}$$

Substituting all the Relations (41), (42), (43), (44) and (48) in the Eq. (47) and using the definition of the generalized hypergeometric Function (19) leads to

$$\sum_{n=0}^{\infty}P_n(x)\,h^n$$
$$=\sum_{n,k=0}^{\infty}{}_2F_3\left[\begin{matrix}-\frac{n}{2},\frac{1-n}{2};\\ \frac{1}{2}-n-k,-\frac{(\alpha+n+k)}{2},-\frac{(\alpha+n+k-1)}{2};\end{matrix}\ \frac{1}{4}\right]$$
$$\times\frac{(-1)^k\,2^{n+k}(1+\alpha)_{n+k}L_k^{\alpha}(x)}{n!\,(1+\alpha)_k}h^{n+k}. \tag{49}$$

Now to be able to equate the power of h^non both sides of the Series (49), we need to use the Identity (1) and finally equating the coefficients of h^non both sides. Then reuse the Relation (11) to achieve the following expansion of the Legendre polynomials in series of the associated Laguerre polynomials as

$$\sum_{k=0}^{n}{}_2F_3\left[\begin{matrix}\frac{-(n-k)}{2},-\frac{(n-k-1)}{2};\\ \frac{1}{2}-n,-\frac{(\alpha+n)}{2},-\frac{(\alpha+n-1)}{2};\end{matrix}\ \frac{1}{4}\right]\times\frac{(-n)_k}{(1+\alpha)_k} \tag{50}$$

Letting $\alpha=0$ in Eq. (55) leads to the expansion of the Legendre polynomials in series of the simple Laguerre polynomials $L_n(x)$, that is

$$P_n(x)=2^n\left(\frac{1}{2}\right)_n\sum_{k=0}^{n}{}_2F_3\left[\begin{matrix}\frac{-(n-k)}{2},-\frac{(n-k-1)}{2};\\ \frac{1}{2}-n,-\frac{n}{2},-\frac{(n-1)}{2};\end{matrix}\ \frac{1}{4}\right]\frac{(-n)_k}{k!}L_k(x),$$
$$n>1,(1)_n=n!,(1)_k=k!$$

6. Expansion of the gamma function in series of the generalized Laguerre polynomials

Since the generalized Laguerre polynomials are mutually orthogonal with respect to the weight function $w(x) = x^{\alpha}e^{-x}$ on the half-line $x \in [0, \infty)$ for $\alpha > -1$ which is just the integrand of the integral definition of Gamma function (6). In addition, the gamma function is analytic except at $x = 0, -1, -2, \dots$, $x = \infty$ where $\Gamma(x)$has an essential singular point at $x = \infty$. Therefore; one could expand the gamma function in terms of the generalized Laguerre polynomials in the interval $(0, \infty)$ as shown here. We start by expanding the function $f(x) = e^{-\beta x}$ as

$$e^{-\beta x} = \sum_{n=0}^{\infty} a_n L_n^{\alpha}(x)$$

with the condition $\beta > -\frac{1}{2}$to guarantee that the conditions in Theorem 3 are hold. The coefficients a_n can be computed easily similar to Example 2, thus

$$a_n = \langle e^{-\beta x}, L_n^{\alpha}(x)\rangle = \frac{\beta^n}{(1+\beta)^{n+\alpha+1}}, \quad \beta > -\frac{1}{2},\ 0 < x < \infty \tag{51}$$

Thus, one has

$$e^{-\beta x} = (\beta+1)^{-(\alpha+1)} \sum_{n=0}^{\infty} \frac{\beta^n}{(1+\beta)^{n+\alpha+1}} L_n^{\alpha}(x) \tag{52}$$

The next step is to multiply the Expansion (52) by the factor $(1+\beta)^{\alpha-1}$ and then integrate with respect to β from $\beta = 0$ to $\beta = \infty$ to obtain

$$\int_0^{\infty} e^{-\beta x}(1+\beta)^{\alpha-1} d\beta = \sum_{n=0}^{\infty} L_n^{\alpha}(x) \int_0^{\infty} \left(\frac{\beta}{1+\beta}\right)^n \frac{d\beta}{(1+\beta)^2} = \sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{n+1} \tag{53}$$

The integral on the left hand side of Eq. (53) can be computed by making the substitution $u = \beta x$ to obtain,

$$\int_0^{\infty} e^{-\beta x}(1+\beta)^{\alpha-1} d\beta = x^{-\alpha} \int_0^{\infty} e^{-u}(u+x)^{\alpha-1} du$$

Now let $u + x = t$, thus

$$\int_0^{\infty} e^{-\beta x}(1+\beta)^{\alpha-1} d\beta = x^{-\alpha}e^{x} \int_x^{\infty} e^{-t} t^{\alpha-1} dt = x^{-\alpha}e^{x}\Gamma(x,\alpha)$$

where $\Gamma(x,\alpha)$ is the upper incomplete gamma function defined by Eq. (7). Now substituting the result of this integral back in Eq. (53) yields,

$$\Gamma(x,\alpha) = x^{\alpha}e^{-x} \sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{n+1}, \quad 0 < x < \infty, \alpha > -1 \tag{54}$$

Letting $\alpha = 0$ in Eq. (54) leads to the expansion of the gamma function in series of the simple Laguerre polynomials $L_n(x)$, that is

$$\Gamma(x) = e^{-x} \sum_{n=0}^{\infty} \frac{L_n(x)}{n+1}, \quad 0 < x < \infty \tag{55}$$

To prevent the repetition by following a similar fashion to the above derivation one could obtain the expansion of the lower incomplete gamma function $\gamma(x,\alpha)$ defined by Eq. (7) in series of the generalized Laguerre polynomials as

$$\gamma(x,\alpha) = x^{\alpha} \sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{2^{n+\alpha}(n+\alpha)}, \quad 0 < x < \infty, \alpha > 0 \tag{56}$$

where the condition $\alpha > 0$ is essential for the definition of the lower incomplete gamma function $\gamma(x,\alpha)$.

7. Expanding the first kind of Bessel functions in series of the generalized Laguerre polynomials

Here in this section, we follow various approaches to obtain the series expansion of the first kind of Bessel functions in series of the generalized Laguerre polynomials. This approach relies on calling the hypergeometric representation of the expanded function. Therefore, using the Relation (11) in the confluent hypergeometric representation of the generalized Laguerre polynomials (32) yields

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{(-1)^k(\alpha+1)_n x^k}{k!\,(n-k)!\,(\alpha+1)_k}$$

Taking the summation of both sides of this equation and multiplying by$h^n$to obtain

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{(\alpha+1)_n} h^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k x^k h^n}{k!\,(n-k)!\,(\alpha+1)_k}$$

Rearrange this double series by using the Identity (3) with replacing $n \to n+k$ to obtain

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{(\alpha+1)_n} h^n = \left(\sum_{n=0}^{\infty} \frac{h^n}{n!}\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k x^k h^k}{k!\,(\alpha+1)_k}\right)$$

Using the notation of the generalized hypergeometric Function (19), yields

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{(\alpha+1)_n} h^n = e^h\ {}_0F_1(-; 1+\alpha; -xh) \tag{57}$$

Now from the hypergeometric representation of the first kind of Bessel functions given by Eq. (24), one has

$${}_0F_1(-; 1+\alpha; -xh) = \Gamma(1+\alpha)(xh)^{-\frac{\alpha}{2}} J_{\alpha}\left(2\sqrt{xh}\right) \tag{58}$$

Finally, implementing Eq. (58) back into Eq. (57) yields

$$J_{\alpha}\left(2\sqrt{xh}\right) = \frac{e^{-h}(xh)^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{(\alpha+1)_n} h^n \tag{59}$$

This is the expansion of the first kind of Bessel functions in terms of the generalized Laguerre polynomials $L_n^{\alpha}(x)$.

Letting $\alpha = 0$ in Eq. (59) leads to the expansion of the zeroth-order first kind Bessel functions in series of the simple Laguerre polynomials$L_n(x)$, that is

$$J_0\left(2\sqrt{xh}\right) = e^{-h} \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} h^n, (1)_n = n! \tag{60}$$

8. Discussion and conclusion

To conclude we have presented the series expansions of some well-known classical polynomials such as the Legendre and the Hermite polynomials in terms of the generalized and simple Laguerre polynomials. Furthermore, because of the fact that the generalized Laguerre orthogonal polynomials are mutually orthogonal with respect to a weighting function, which is just the integrand of the gamma function, thus we also expand the first kind of Bessel functions and the gamma function in series of the generalized and simple Laguerre polynomials. To sum up, in this paper, the series expansions in terms of the generalized Laguerre polynomials have been achieved by following various approaches. For instance, respectively the series expansion of the Legendre and the Hermite polynomials in terms of the generalized Laguerre Polynomials (46) and (50) are gained through the generating function approach. It should be noted that by a similar fashion of the derivations in Sections 4 and 5 we can gain the opposite results. That is one could obtain the series expansion of the generalized Laguerre polynomials in series of the Hermite or Legendre polynomials respectively.

To achieve this goal we replace the expansion of x^n in series of Hermite or Legendre polynomials in the generating function or (its equivalence) of generalized Laguerre polynomials and then doing similar lengthy operations as shown in Sections 4 and 5 respectively. On the other hand, the series expansions of the first kind Bessel Functions (59) and (60) are gained by the generalized hypergeometric function approach, whereas the Series expansions of gamma function (54), (55), and (56) are obtained directly by the usual way that is by calling the orthogonality and ortho-normality properties of the generalized and simple Laguerre polynomials (35). One could claim that by expanding some functions in terms of the generalized Laguerre polynomials, one could treat the former as a special case of the latter polynomials. As future work, a series expansion in terms of the generalized Laguerre polynomials can be obtained to other well-known classical polynomials, such as Chebychev, Jacobi, Gegenbauer polynomials, etc. In turn, the series expansion in series of the generalized Laguerre polynomials should bring a variety of applications for such polynomials in mathematics, physics, and engineering.

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