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Some applications of infinite matrices on some sequence spaces.

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Highlights

- **By using different new functional (* -functionals) some inequalities have been proved here like Knopp's core theorems. It is the extension of Knopp's core theorems.**
- **Summability of Fourier series and derived Fourier series has been done earlier by various method, here we have used p-regular matrices for this purpose.**

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ABSTRACT

In this paper, we find some applications of infinite matrices. We prove some results analogues to Knopp's core theorem. Furthermore, we have applied p-regular matrices to find the sum of the derived Fourier series.

1. Introduction

Choudhary (1988) discussed 'An Extension of Knopp's Core Theorem' here the purpose of the author is to generalize Knopp's core theorem. Also, for corresponding work, we refer to Orhan (1990) in which the author discussed 'Sublinear Functionals and Knopp's Core Theorem' here the author is concerned with the inequalities involving certain sublinear functionals on the space of bounded sequences. Such inequalities being analogs of Knopp's core theorem.

2. Preliminaries

Let $\ell_\infty = \{ x = (x_k): sup_k|x_k| < \infty \}$, be the space of all bounded sequences, with $\| x \| = sup_k |x_k|$, and $c = \left\{ x = (x_k): \lim_k x_k = l, l \in \mathbb{C} \right\}$, the space of convergent sequences.

Definition (2.1): An infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ is said to be regular if it transforms convergent sequences into convergent sequences with the same limit. That is, $Ax \in c$, *for all* $x = (x_k)_{k=1}^{\infty} \in c$ with $\lim Ax = \lim x$, where $Ax = (A_n(x))_{n=1}^{\infty}$ *and* $A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k$. The well-known Knopp's core theorem states that: In order that $L(Ax) \leq L(x)$ for every $x \in \ell_\infty$, it is necessary and sufficient that A should be regular and almost positive, where $L(x) = \lim\sup|x|$ *and* $L(Ax) = \lim\sup|Ax|$. This section also deals with necessary preliminaries, which are needed for our result in Sections 3 and 4. We list the following functionals defined on ℓ_∞ (Cooke, 1950; Das, 1987; Devi, 1976). $\ell(x) = lim_k\, inf\, x_k$; $(x) = lim_k\, sup x_k$ $q(x) = lim_k \inf|x_k|$; $Q(x) = lim_k \sup|x_k|$ $w(x) = inf\{ L(x+z) : z \in bs \}$; $\| x \| = sup_k |x_k|$ where bs denote the space of all bounded sequence $x = (x_k)$ such that $sup_n |\sum_{k=0}^{n} x_k| < +\infty$ further, we have

$\ell^*(x) = lim_n\, inf\, sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r$;

$L^*(x) = lim_n sup sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r$; $w^*(x) = inf\{L^*(x+z) : z \in bs\}$. We note the following inequalities (Ahmed, Mursaleen and Khan, 1996; Devi, 1976; Kuttner and Maddox, 1983; Maddox, 1979; Orhan, 1990; Simons, 1969) $\ell \leq w \leq L \leq \| \cdot \|$; $w \leq Q \leq \| \cdot \|$; $L \leq Q$; $\ell \leq q \leq Q$; $\ell \leq \ell^* \leq L^* \leq L$; $w^* \leq L^*$, the functionals which are marked with * are of special interest to us in section 3 and we call them as *-functionals. In fact, these are related to the concept of the Banach limit. It is well-know that the functional $q(x) = inf_{n_1, n_2, \dots, n_r}\ lim_m\ sup \frac{1}{r} \sum_{i=1}^{r} x_{m+n_i}$ is sublinear on ℓ_∞. If $q(x) = -q(-x) = s$, then x is said to be almost convergent to s (Lorentz, 1948). It was also shown in (Das and Mishra, 1981) that $q(x) = L^*(x)$. If f *and* g are any two of the above functionals, we shall write $fA \leq gB$ to denote that, for every $\in \ell_\infty$, Ax *and* Bx are defined and bounded and $fA(x) \leq gB(x)$. A and B are infinite matrices, where $A = (a_{nk})_{n,k=1}^{\infty}$, $B = (b_{nk})_{n,k=1}^{\infty}$. Some preliminaries about Fourier series: Let f be L-integrable and periodic with period 2π, and let the Fourier series of f be

$$\frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \qquad (2.1)$$

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and the derived series

$$\sum_{k=1}^{\infty} k(b_k \cos kx \; - \; a_k \sin kx\,) \tag{2.2}$$

We write

$$\phi_x(t) = \phi(f,t) = \begin{cases} f(x+t) - f(x-t), & 0 < t \le \pi\,, \\ f(x+0) - f(x-0), & t = 0\,, \end{cases}$$

and $h_x(t) = \frac{\phi_x(t)}{4\sin\frac{1}{2}t}$. A partial sum of the derived Fourier series of f is given by

$$s'_k(x) = \frac{1}{\pi}\int_0^{\pi} \phi_x(t)\,[\sum_{m=1}^{k} m\,\sin mt]\,dt,$$

$$= -\frac{1}{\pi}\int_0^{\pi}\phi_x(t)\,\frac{d}{dt}\left[\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\right]dt,$$

$$= I_k \;+\; \frac{2}{\pi}\int_0^{\pi}\sin(\,k+\tfrac{1}{2})\;t\;dh_x\,(t) \tag{2.3}$$

Where $I_k = \frac{1}{\pi}\int_0^{\pi} h_x(t)\,\cos\frac{1}{2}t\,\frac{\sin(\,k+\frac{1}{2}\,)\,t}{\sin\frac{1}{2}t}\,dt$, some preliminaries definitions, and results are used in Section 3.

Definition (2.2): The matrix $A = (a_{nk})$is p-regular if and only if $\lim_n \sum_k a_{nk}\,y^{(k)} = \lim_k y^{(k)},\ \ for\ all\ \ (y^{(k)}) \in c(Y)$, when our sequences are taken in the p-normed space , which we shall take to be $\ell_p(X)$, with $0 < p < 1$. Thus $y = (y_k) \in Y$ means that $y_k \in X$ for all k and $\| y \| = \sum \| y_k \|^p < \infty$, where spaces $c(Y)$ defined by (Maddox, 1988) and $\ell_p(X)$ also defined by (Maddox, 1980).

Theorem 2.1: (Maddox, 1992) The infinite matrix $A = (a_{nk})$ is p-regular if and only if

(i) $\lim_n a_{nk} = 0\ for\ each\ k \ge 1,$

(ii) $\lim_n \sum_{k=1}^{\infty} a_{nk} \;=\; 1$

(iii) $sup_n \sum_k |a_{nk}|^p < \infty,\quad 0 < p < 1.$

Theorem 2.2: (Jordan's Convergence Criterion, 1881) If $f(x)$ is bounded variation in some interval $(a\,,b)$, then its Fourier series converges at every point of the interval. Its sum is $f(x)$ at a point of continuity and $[\,f(x+0) + f(x-0)]/2$ at a point of discontinuity.

Theorem 2.3: (Banach Weak Convergence Theorem, 1932 and Mursaleen, 2014): It states that $\int_0^1 g_n\ dh_x = 0\ (n \to \infty)\ for\ all\ h_x \in BV[0,1]$, If and only if $\| g_n \| < M\ \ for\ all\ \ n\ \ and\ \ g_n \to 0\ as\ n \to \infty$. where BV [0,1] space of all function of bounded variation on [0 , 1].

3. Knopp's Core like theorems

In order to prove our results, we shall need the following lemmas. Throughout this section, the matrix B is taken as the normal matrix.

Lemma A: (Choudhary, 1988), In order that whenever Bx is bounded, $(Ax)_n$ should be defined for fixed n, if and only if

$$c_{nk} = \sum_{j=k}^{\infty} a_{nj}\ b_{jk}^{-1}\ exists\ for\ all\ \ k\,; \tag{3.1}$$

$$\sum_k |c_{nk}| < \infty\,; \tag{3.2}$$

And for any fixed n,

$$\sum_{k=0}^{J}\left|\sum_{j=J+1}^{\infty} a_{nj}\ b_{jk}^{-1}\right| \to 0\ (J \to \infty) \tag{3.3}$$

If these conditions hold then, for bounded Bx

$$(Ax)_n = \sum_k c_{nk}\ y_k = (Cy)_n\,, \quad where \quad y_k = (Bx)_k \tag{3.4}$$

Lemma B: (Orhan, 1990), $L^*\,A \le L^*$ if and only if A is f-regular and

$$\lim_n sup_i \sum_k |\,\tfrac{1}{n+1}\sum_{r=i}^{i+n} a_{rk}\,| = 1, \tag{3.5}$$

We prove the following results:

Theorem 3.1: For any matrix $A = (a_{nk})$, in order that, whenever Bx is bounded, Ax should exist and bounded and satisfy

$$L^*(Ax) \le L^*(Bx), \tag{3.6}$$

It is necessary and sufficient that

$$C = AB^{-1}, exists \tag{3.7}$$

$$C\ \text{ is f-regular}\,; \tag{3.8}$$

$$\lim_n sup_i \sum_k |\,\frac{1}{n+1}\sum_{r=i}^{i+n} c_{rk}\,| = 1, \tag{3.9}$$

for any fixed n ,

$$\sum_{k=0}^{J}\left| \sum_{j=J+1}^{\infty} a_{nj}\ \ b_{jk}^{-1}\,\right| \to 0\ (J \to \infty), \tag{3.10}$$

where Bx is B-transform of $x = (x_k)$. That is

$$Bx = \big(B_n(x)\big)\ where\ \ B_n(x) = \sum_{k=1}^{\infty} b_{nk}\,x_k.$$

Proof. Let the conditions (3.7)-(3.10) hold. Then (3 .7), (3.8) and (3.9) imply that conditions of Lemma A are satisfied and hence, (3.4) holds; moreover Cy is bounded for $y \in \ell_\infty$. Further, by Lemma B, conditions (3.8) and (3.9) together give $L^*(Cy) \le L^*(y)\ \ for\ \ y \in \ell_\infty$. Putting $y = Bx$ we get (3.6). Conversely, suppose that (3.6) holds and assume that $(Ax)_n$ exists for every n whenever $Bx =: y$ is bounded. Then by Lemma A, it follows that the conditions (3 .7) and (3.10) hold, also (3.2) of Lemma A holds for every n. Further, for every $y \in \ell_\infty$, (3.4) holds. Therefore by (3.6) we have $L^*(Cy) \le L^*(y),\ y \in \ell_\infty$ and hence by Lemma B, it follows that (3.8) and (3.9) hold. This completes the proof of the theorem.

Corollary 3.2: For a row-finite matrix A, $L^*(Ax) \le L^*(Bx),\ for\ all\ \ x \in \ell_\infty$ if and only if (3.8) and (3.9) hold.

Remark: For a row-finite matrix A, the expression inside the modulus in (3.10) is 0 for sufficiently large J (and all k). Thus (3.10) is necessarily satisfied and we get Corollary (3.2).

Theorem 3.3: In order that, whenever Bx is bounded, Ax should exists and satisfy

$$L^*(Ax) \le w^*(Bx), \tag{3.11}$$

if and only if conditions (3.7)-(3.10) of Theorem 3.1 hold.

Proof. Sufficiency follows on the same lines as in the proof of Theorem 3.1. For necessity, let Ax be defined whenever $y \coloneqq Bx$ be bounded. Using Lemma A, we get (3.4), i.e. $Ax = Cy$.

Now $w^*(Bx) \coloneqq \inf\ \{L^*\,(Bx+z) : z \in bs\ \} \le L^*\,(Bx+z)$. Taking $z = 0$, we have $w^*(Bx) \le L^*(Bx)$. By (3.11), it follows that $L^*(Ax) \le L^*(Bx)$ which is (3 .6) of Theorem 3.1 and hence necessity follows. This completes the proof of the Theorem. The following result is a consequence of the above theorem.

Corollary 3.4: (Orhan, 1990), For a row-finite matrix A, $L^*(Ax) \le w^*(Bx);\ x \in bs$ If and only if (3.8) and (3 .9) of Theorem 3.1 hold. By taking $B = I$ (identity matrix) in Corollary (3.4) we have the required result.

Theorem 3.5: (Orhan, 1990), Let w^*be sublinear functional defined by $w^*(x)$. Then for a row finite matrix A, $L^*(Ax) \le w^*(x)\ \ for\ all\ \ x \in bs$ if and only if A is almost positive and f-regular.

4. A-summability of a class of derived Fourier series

In this section, we find necessary and sufficient condition for A-summability of the sequence $(s_k'(x))$.

Theorem 4.1: Let $A = (a_{nk})$ be a p-regular matrix. Then, for each $x \in [-\pi, \pi]$ for which $h_x(t) \in BV\,[0,\pi]$, the sequence $(s_k'(x))$ is A-summable to $h_x(0_+)$ if and only if

$$\sum_k a_{nk} \sin(k+\tfrac{1}{2})t \to 0, \quad n \to +\infty, \quad for\ every\ t \in [0, \pi], \qquad (4.1)$$

Proof. Since $h_x(t) \in BV\,[0,\pi]$ and $h_x(t) \to h_x(0_+)$ as $t \to \infty$, $h_x(t)\cos\frac{1}{2}t$ has the same properties. Therefore, by Jordan's Convergence Criterion for Fourier series $I_k \to h_x(0_+)$ as $k \to \infty$. Now by (2.3), we have

$\sum_{k=1}^{\infty} a_{nk}\, s_k'(x) = \sum_k a_{nk}\, I_K + \frac{2}{\pi}\int_0^{\pi}[\,\sum_k a_{nk} \sin(k+\frac{1}{2})\,t\,]\,dh_x(t) = J_1 + J_2$, say. Since A is p-regular, condition (ii) of Theorem 2.1 implies that $J_1 \to h_x(0_+)$ as $n \to \infty$. Therefore, we have to show that condition (4.1) holds if and only if $J_2 \to 0$ as $n \to \infty$. Now, in particular, if we choose the sequence $g_n = \{\sum_k a_{nk}\sin(k+\frac{1}{2})\,t\,\}$ $for\ any$ $t \in \Re$, then $\| g_n \| = \| \sum_k a_{nk} \sin(\,k+\frac{1}{2})\,t\,\| \le sup_n \sum_k |\,a_{nk}\,|^p < \infty$, by condition (iii) of Theorem 2.1; and $g_n \to 0$, by condition (i) of Theorem 2.1. Hence by Banach Weak Convergence Theorem,

$\lim_n \int_0^1 \sum_k a_{nk} \sin(k+\frac{1}{2})\,t\,dh_x(t) = 0$. If and only if condition (4.1) holds, i.e. $(s_k'(x))$ is A-summable to $h_x(0_+)$ if and only if condition (4.1) holds. This completes the proof of the Theorem.

5. Conclusion

In Section 3, our results also generalized results due (Choudhary, 1988)

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