Libyan Journal of Science & Technology 13:2 (2021) 85-90

Faculty of Science - University of Benghazi

Libyan Journal of Science & Technology

journal home page: www.sc.uob.edu.ly/pages/page/77

Non-unique fixed point results in extended b_2-metric spaces.

Abdelhamid S. Elmabrok[a,*], Sameehah R. Alkaleeli[b]

[a]*Department of mathematics, Faculty of Science, University of Benghazi*

[b]*Department of mathematics, Faculty of Science, Al Marj, University of Benghazi*

Highlights

- **Establish some non-unique fixed point theorems for self-mapping on an extended b_2-metric space,**
- **We also give some examples to illustrate the usability of the obtained results.**
- **Generalize some classical results of non-unique fixed-point theorems, in particular, the corresponding results of Ćirić. B. L, (1974), Achari. J, (1976) and Ćirić. B. L., *et al.*, (1998).**

ARTICLE INFO

Article history:

Received 24 April 2021
Revised 12 July 2021
Accepted 23 July 2021

Keywords:

Non-unique fixed point, Extended b_2-metric space, T-orbitally complete, orbitally continuous, α-admissible mappings.

**Address of correspondence:*

E-mail address: abdelhamid.elmabrok@uob.edu.ly

A. S. Elmabrok

ABSTRACT

In this paper, we present some non-unique fixed-point theorems for self-mapping in the setting of an extended b_2-metric space. Our results generalize and widen some results of Alqahtani *et al.* (2018a).

1. Introduction

A non-unique fixed point notion for certain operators was first introduced by Ćirić (1974a), then many results for self-mapping were obtained, see e.g Ćirić (1974b), Achari (1976), Pachpatte (1979), also for non-unique fixed point in 2-metric space see, Paliwal (1987), WAN Mei-ling *et al.* (2017), Yuan Lin and Shuyi Zhang (2017). Later, Alqahtani *et al.* (2018a) have dealt with non- unique fixed-point results on extended b-metric space. Recently, the concept of an extended b_2-metric space is a natural popularization of both b_2-metric space and extended b-metric space. Initially, it has been investigated by Elmabrok and Alkaleeli (2018), then many fixed point result, were obtained. In the present paper, some results of a non-unique fixed-point theorem on the class of extended b-metric space in Alqahtani *et al.* (2018b) are stretch to the class of an extended b_2-metric space. Specifically, it is shown that the self-mappings having non-unique fixed points. Some of our results are the corresponding generalizations of known results Ćirić and Jotic (1998), Liu *et al.* (2006). In the supplement, the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ stand for the sets of real, positive real and positive integers, respectively. Moreover, we will denote to the symbols as $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Preliminaries

We start this section by recall the definition of a generalized b_2-metric space Elmabrok and Alkaleeli (2018), with some results of orbital continues mapping and orbitally complete metric space, see e.g. Samet *et al.* (2012), Popesc (2014) and Alqahtani *et al.* (2018a).

Given a nonempty set X and a self-map $T: X \to X$, then a point $x \in X$, such that $Tx = x$ is called a fixed-point of T.

Definition 2.1 Elmabrok and Alkaleeli (2018).

Let X be a nonempty set and $\theta: X \times X \times X \to [1, \infty)$ be a mapping. A function $d_\theta: X \times X \times X \longrightarrow [0, \infty)$ is an extended b_2-metric on X if for all $a, x, y, z \in X$, the following conditions hold:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d_\theta(x, y, z) \neq 0$,
2) If at least two of three points x, y, z are the same, then $d_\theta(x, y, z) = 0$.
3) The symmetry: $d_\theta(x,y,z) = d_\theta(x,z,y) = d_\theta(y,x,z) = d_\theta(y,z,x) = d_\theta(z,x,y) = d_\theta(z,y,x)$, for all $x, y, z \in X$.
4) The rectangle inequality: $d_\theta(x,y,z) \leq \theta(x,y,z)[d_\theta(x,y,a) + d_\theta(y,z,a) + d_\theta(z,x,a)]$ for all $x, y, z, a \in X$.

Then d_θ is called an extended b_2-metric on X and the pair (X, d_θ) is called an extended b_2-metric space.

Remarks 2.1

1) It is obvious that the class of an extended b_2-metric space is larger than b_2-metric space, because if $\theta(x,y,z) = s$, for $s \geq 1$ then we obtain the definition of a b_2-metric space. Furthermore, for $\theta(x,y,z) = s = 1$, the b_2-metric reduces to a 2-metric.

 ISSN: Online 2663-1407, Print 2663-1393; National Library of Libya, Legal number: 390/2018

2) Using condition (1) it readily verified that for all $a \in X, d_\theta(x,y,a) = 0$, then $x = y$.

Example 2.1 Elmabrok and Alkaleeli (2018).

Let $X = [0,1]$. Define $\theta : X \times X \times X \longrightarrow [1,\infty)$ by

$$\theta(x,y,z) = \frac{1+x+y+z}{x+y+z}, \quad \text{for all } x,y,z \in X$$

and $d_\theta : X \times X \times X \longrightarrow [0,\infty)$ by

$$d_\theta(x,y,z) = \begin{cases} \dfrac{1}{xyz} & \text{if } x,y,z \in (0,1] \text{ and } x \neq y \neq z, \\ 0 & \text{if } x,y,z \in [0,1] \text{ and at least two of } x,y, \text{ and } z \text{ are equal}, \\ \dfrac{1}{xy} & \text{if } x,y \in (0,1] \text{ and } z = 0. \end{cases}$$

Then, (X, d_θ) is an extended b_2-metric space.

Definition 2.2 Elmabrok and Alkaleeli (2018).

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an extended b_2-metric space (X, d_θ).

1) A sequence $\{x_n\}$ is convergent to $x \in X$, if for all $a \in X$, there exists $x \in X$, such that $\lim_{n\to\infty} d_\theta(x_n, x, a) = 0$.
2) A sequence $\{x_n\}$ is a Cauchy sequence if and only if $d_\theta(x_n, x_m, a) \to 0$, when $n, m \to \infty$ for all $a \in X$.

Definition 2.3 An extended b_2-metric space (X, d_θ) is called complete if every Cauchy sequence is convergent sequence in X.

Definition 2.4 Ćirić (1974a).

Let X be a non-empty set, $T: X \to X$ and $x_0 \in X$. We construct the sequence $\{x_n\}$ such that, $x_n = Tx_{n-1} = Tx_0$, for all $n \in \mathbb{N}_0$. Then the set $\{x_0, Tx_0, \ldots, T^n x_0, \ldots\}$ is called an orbit of x_0 with respect to T and is defined by $OT(x_0)$.

Definition 2.5 Ćirić (1974b).

Let X be a non-empty set. A self-mapping $T: X \to X$ is said to be orbitally continuous if

$$\lim_{i\to\infty} T^{n_i}(x) = x, \text{ for some } x \in X,$$

implies that

$$\lim_{i\to\infty} T\left(T^{n_i}(x)\right) = Tx.$$

Furthermore, the definition of orbitally continuous on 2-metric space was introduced by Iseki (1975).

Definition 2.6 Samet *et al.* (2012).

Let X be a non-empty set and $\alpha: X \times X \times X \to \mathbb{R}_0^+$ be a mapping.
A mapping $T: X \to X$ is called:

a) An α- admissible, if

$$\alpha(x,y;a) \geq 1 \Rightarrow \alpha(Tx,Ty,a) \geq 1, \tag{2.1}$$

b) An α- orbital admissible, if

$$\alpha(x,Tx;a) \geq 1 \Rightarrow \alpha(Tx,T^2x;a) \geq 1. \tag{2.2}$$

for all $x, y, a \in X$

3. Results

In the sequel, we assume that an extended b_2-metric d_θ is continuous functional.

Definition 3.1 A self –mapping T on an extended b_2-metric space $(X; d_\theta)$ is called orbitally continuous if for all $x, y, a \in X$

$$d_\theta(T^n x, y, a) \to 0, \text{ as } n \to \infty,$$

implies that

$$d_\theta(TT^n x, Ty, a) \to 0, \text{ as } n \to \infty.$$

Definition 3.2 Let T be a self- mapping on a non-empty set X. An extended b_2-metric space $(X; d_\theta)$ is said to be T-orbitally complete, if every Cauchy sequence in $OT(x_0)$ converges in X, where $x_0 \in X$. In the next example, we will show that every complete extended b_2-metric space is T-orbitally complete, but the converse does not hold in general.

Example 3.3 Let $X = \mathbb{R}$ and $d: X^2 \to \mathbb{R}_0^+$ be defined by $d(x;y) = |e^x - e^y|$. We define an extended b_2-metric d_θ on X^3 by $d_\theta(x,y,z) = [\min\{d(x,y), d(y,z), d(z,x)\}]^2$ with $\theta(x,y,z) = |x| + |y| + |z| + 1$. It is easy to see that (X, d_θ) is an extended b_2-metric space.
Now we show that (X, d_θ) is T-orbitally complete extended b_2-metric space but not complete extended b_2-metric space. Take, $x_n = -n$, then we have:

$$\begin{aligned} d_\theta(-n,-m,a) &= [\min\{d(-n,-m), d(-n,a), d(-m,a)\}]^2, \\ &= [\min\{|e^{-n} - e^{-m}|, |e^{-n} - e^a|, |e^{-m} - e^a|\}]^2, \\ &\leq [|e^{-n} - e^{-m}|]^2, \text{ as } n, m \to \infty. \end{aligned}$$

Then the sequence $\{x_n\}$ is Cauchy, but it is not convergent.

If $x_n \to x$, for some $x \in X$, then

$$\begin{aligned} d_\theta(-n,x,a) &= [\min\{d(-n,x), d(-n,a), d(x,a)\}]^2, \\ &= [\min\{|e^{-n} - e^x|, |e^{-n} - e^a|, |e^x - e^a|\}]^2, \\ &\leq [|e^{-n} - e^x|]^2, \text{ as } n \to \infty, \text{ which gives that, } e^x = 0, \end{aligned}$$

a contradiction. Now, for $x_0 \in X$, we define a self-mapping

$T: X \to X$ by $Tx = x_0$. Then $(X; d_\theta)$ is T-orbitally complete extended b_2-metric space.

Lemma 3.4 Let (X, d_θ) be an extended b_2-metric space. If there exists, $k \in [0,1)$, such that the sequence $\{x_n\}$ for an arbitrary $x_0 \in X$ satisfies:

$$\lim_{n,m\to\infty} \theta(x_n, x_m, a) < \frac{1}{k} \tag{3.1}$$

and

$$0 < d_\theta(x_n, x_{n+1}, a) \leq k\, d_\theta(x_{n-1}, x_n, a) \tag{3.2}$$

For any $n \in \mathbb{N}$ and $a \in X$. Then the sequence $\{x_n\}$ is Cauchy in X.

Proof

We construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that, for an arbitrary $x_0 \in X$, satisfies (3.1) and by hiring In (3.2), recursively, we deduce that

$$0 < d_\theta(x_n, x_{n+1}, a) \leq k^n d_\theta(x_0, x_1, a) \tag{3.3}$$

Now, for $\in \mathbb{N}$, with $m > n$, using the rectangle inequality and In (3.3), we have:

$$\begin{aligned} d_\theta(x_n, x_m, a) &\leq \theta(x_n, x_m, a)[d_\theta(x_n, x_m, x_{n+1}) + d_\theta(x_m, a, x_{n+1}) \\ &\quad + d_\theta(a, x_n, x_{n+1})], \\ &= \theta(x_n, x_m, a)[d_\theta(x_n, x_{n+1}, x_m) + d_\theta(x_{n+1}, x_m, a) \\ &\quad + d_\theta(x_n, x_{n+1}, a)], \\ &= \theta(x_n, x_m, a)[d_\theta(x_n, x_{n+1}, a) + d_\theta(x_n, x_{n+1}, x_m) \\ &\quad + d_\theta(x_{n+1}, x_m, a)], \\ &\leq \theta(x_n, x_m, a)[\, k^n d_\theta(x_0, x_1, a) + k^n d_\theta(x_0, x_1, x_m)\,] \\ &\quad + \theta(x_n, x_m, a) d_\theta(x_{n+1}, x_m, a), \\ &\leq \theta(x_n, x_m, a)\, k^n d_\theta(x_0, x_1, a) + \theta(x_n, x_m, a) \\ &\quad k^n d_\theta(x_0, x_1, x_m) + \theta(x_n, x_m, a) \\ &\quad \theta(x_{n+1}, x_m, a)\,[\, d_\theta(x_{n+2}, x_{n+1}, a) + \\ &\quad d_\theta(x_{n+2}, x_{n+1}, x_m) + d_\theta(x_{n+2}, x_m, a)\,], \\ &\leq [\, \theta(x_n, x_m, a) k^n + \\ &\quad \theta(x_n, x_m, a)\theta(x_{n+1}, x_m, a) k^{n+1}\,] \\ &\quad d_\theta(x_0, x_1, a) \\ &\quad + [\, \theta(x_n, x_m, a) + \theta(x_n, x_m, a)\theta(x_{n+1}, x_m, a) k^{n+1}\,] \end{aligned}$$

$$d_\theta(x_0,x_1,x_m)$$
$$+\,\theta(x_n,x_m,a)\theta(x_{n+1},x_m,a)\,d_\theta(x_{n+2},x_m,a),$$
$$\leq [\theta(x_n,x_m,a)k^n+\theta(x_n,x_m,a)\theta(x_{n+1},x_m,a)k^{n+1}+\cdots$$
$$+\quad \theta(x_n,x_m,a)\theta(x_{n+1},x_m,a)\ldots\theta(x_{m-2},x_m,a)$$
$$\theta(x_{m-1},x_m,a)k^{m-1}]\,(d_\theta(x_0,x_1,a)\,+\,d_\theta(x_0,x_1,x_m)\,),$$
$$=\textstyle\sum_{i=n}^{m-1}k^i\prod_{j=n}^{i}\theta(x_j,x_m,a)\,[d_\theta(x_0,x_1,a)+d_\theta(x_0,x_1,x_m)]. \quad (3.4)$$

Also, we have:

$$\begin{aligned} d_\theta(x_0,x_1,x_m) &\leq \theta(x_0,x_1,x_m)\quad [d_\theta(x_0,x_1,x_{m-1})\,+ \\ &\quad d_\theta(x_{m-1},x_m,x_0)+d_\theta(x_{m-1},x_m,x_1)], \\ &\leq\ \theta(x_0,x_1,x_m)\ [d_\theta(x_0,x_1,x_{m-1})\ \ + \\ &\quad k^{m-1}d_\theta(x_0,x_1,x_0)+\ k^{m-2}d_\theta(x_1,x_2,x_1)\,]\,, \\ &= \theta(x_0,x_1,x_m)d_\theta(x_0,x_1,x_{m-1}), \\ &\leq \theta(x_0,x_1,x_m)\theta(x_0,x_1,x_{m-1})d_\theta(x_0,x_1,x_{m-2}), \\ &\ \ \vdots \\ &\leq\ \theta(x_0,x_1,x_m)\ \theta(x_0,x_1,x_{m-1})\ldots \\ &\quad \theta(x_0,x_1,x_2)d_\theta(x_0,x_1,x_1), \\ &= 0. \end{aligned}$$

Hence, $d_\theta(x_0,x_1,x_m)=0$. Therefore, In. (3.3), becomes

$$d_\theta(x_n,x_m,a)\leq\left(\sum_{i=n}^{m-1}k^i\prod_{j=n}^{i}\theta(x_j,x_m,a)\right)d_\theta(x_0,x_1,a). \qquad (3.5)$$

Since, $\lim_{n,m\to\infty}\theta(x_n,x_m,a)<\frac{1}{k}$, so that the series , $\sum_{i=1}^{\infty}a_i$ where, $a_i=k^i\prod_{j=1}^{i}\theta(x_j,x_m,a)$, converges by ratio test to some $s\in(0,\infty)$, for each $m\in\mathbb{N}$. Let

$$s=\sum_{i=1}^{\infty}k^i\prod_{j=1}^{i}\theta(x_j,x_m,a)\,,$$

with partial sum

$$s_n=\sum_{i=1}^{n}k^i\prod_{j=1}^{i}\theta(x_j,x_m,a)\,.$$

Thus, for all $m>n$, In. (3.4) implies,

$$d_\theta(x_n,x_m,a)\leq\ d_\theta(x_0,x_1,a)[s_{m-1}-s_{n-1}\,] \qquad (3.6)$$

Letting, $n\to\infty$ in In. (3.6), we deduce that, $\{x_n\}$ is a Cauchy sequence in X.

Lemma 3.5

Let $T:\ X\ \to X$ be an α-admissible mapping and $x_n=Tx_{n-1}, n\in\mathbb{N}$. If there exists $x_0\in X$, such that $\alpha(x_0,Tx_0,a)\geq 1$, then we have $\alpha(x_{n-1},x_n,a)\geq 1$, for all $n\in\mathbb{N}_0$ and $a\ \in\ X$.

Proof Suppose that, there exist $x_0\in X$, such that $\alpha(x_0,Tx_0,a)\geq 1$, and let the constructive sequence $\{x_n\}$ defined by $x_n=Tx_{n-1}$, for all $n\in\mathbb{N}$. Since, T is α-orbital admissible, we derive

$$\begin{aligned} &\alpha(x_0,x_1,a)=\alpha(x_0,Tx_0,a)\geq 1 \\ \Longrightarrow\ \ &\alpha(Tx_0,T^2x_0,a)=\alpha(x_1,x_2,a)\geq 1. \end{aligned}$$

Recursively, we have

$$\alpha(x_{n-1},x_n,a)\geq 1, \text{ for all } n\in\mathbb{N}_0 \text{ and } a\in\ X \qquad (3.7)$$

Theorem 3.6 Let (X,d_θ) be T-orbitally complete extended b_2-metric space and T be an orbitally continuous self-mapping on X.Postulate that there exist $k\in[0,1)$ and $b\ \geq 1$, such that

$$\begin{aligned} &\alpha(x,y,a)\min\{d_\theta(Tx,Ty,a),d_\theta(x,Tx,a),d_\theta(y,Ty,a)\} \\ &\quad -b\min\{d_\theta(x,Ty,a),d_\theta(Tx,y,a)\} \\ &\quad \leq kd_\theta(x,y,a), \end{aligned} \qquad (3.8)$$

for all $x,y,a\in X$ and $m,n\in\mathbb{N}$. Moreover, we assume that:

i. T is α-orbital admissible ,

ii. there exists $x_0\in X$ such that $\alpha(x_0,Tx_0,a)\geq 1$,

iii. $\lim_{m,n\to\infty}\theta(x_n,x_m,a)\leq\frac{1}{k}$.

Then, for each $x_0\in X$ the sequence $\{T^nx_0\}$ converges to a fixed point of T.

Proof By presumption ii, there exists $x_0\in X$, such that $\alpha(x_0,Tx_0,a)\geq 1$. We build up the sequence $\{x_n\}$ such that, $x_{n+1}=Tx_n$, for all $n\in\mathbb{N}_0$. If $x_{n_0}=x_{n_{0+1}}=Tx_{n_0}$, for some $n_0\in\mathbb{N}_0$, then x_{n_0} is a fixed point of T and hence the proof finishes. Therefore, suppose that $x_n\neq x_{n+1}$, for each $n\in\mathbb{N}_0$. By presumptions i and ii with Lemma 3.5, we have

$$\alpha(x_{n-1},x_n,a)\geq 1,\qquad \text{for all } n\in\mathbb{N}_0.$$

Now, replacing x by x_{n-1} and y by x_n. in In. (3.8), we get

$$\min\begin{Bmatrix} d_\theta(Tx_{n-1},Tx_n,a), \\ d_\theta(x_{n-1},Tx_{n-1},a), \\ d_\theta(x_n,Tx_n,a)\end{Bmatrix}-b\min\begin{Bmatrix} d_\theta(x_{n-1},Tx_n,a) \\ ,d_\theta(Tx_{n-1},x_n,a)\end{Bmatrix}$$

$$\leq\alpha(x_{n-1},x_n,a)\min\begin{Bmatrix} d_\theta(Tx_{n-1},Tx_n,a), \\ d_\theta(x_{n-1},Tx_{n-1},a), \\ d_\theta(x_n,Tx_n,a)\end{Bmatrix}$$

$$-b\min\begin{Bmatrix} d_\theta(Tx_n,x_{n-1},a), \\ d_\theta(Tx_{n-1},x_n,a)\end{Bmatrix}$$

Hence,

$\min\{d_\theta(x_n,x_{n+1},a),d_\theta(x_{n-1},x_n,a)\}\leq kd_\theta(x_{n-1},x_n,a)$. Since $k\ \in\ [0,1)$, the case $d_\theta(x_{n-1},x_n,a)\leq kd_\theta(x_{n-1},x_n,a)$ is impossible. Thus, we infer that

$$d_\theta(x_n,x_{n+1},a)\leq kd_\theta(x_{n-1},x_n,a)$$

By consideration of Lemma 3.4, we get that $\{x_n\}$ is a Cauchy sequence. Owing to the construction $x_n=T^nx_0$ and (X,d_θ) is T-orbitally complete, there is $u\in\ X$, such that $x_n\to u$, as $n\ \to\infty$. By the orbital continuity of T, we deduce that $x_n\to Tu$. Hence, $u=Tu$, which concludes the proof.

Example 3.7 Let $X=\{1,2,3,4\}$. Define, $\theta: X\times X\times X\longrightarrow[1,\infty)$, by $\theta(x,y,z)=1+x+y+z$, and $d_\theta: X\times X\times X\longrightarrow[0,\infty)$ by

$$d_\theta\,(x,y,z\,)=[\min\{|x-y|,|x-z|,|y-z|\}]^2.$$

Therefore, (X,d_θ) is an extended b_2-metric space. Let, $b\ =\ 8,\ k\ =\frac{1}{8}$ and define $T: X\to X$ by $T1=T3\ =1$ and $T2\ =\ T4\ =\ 3.$
Define *also* $\alpha: X^3\to\mathbb{R}_0^+$, by

$$\alpha\,(x,y,z)\ =\begin{cases}0, & if\ (x,y,z)\in A \\ 1, & \text{otherwise}\end{cases},$$

where, $A=\{(2,3,4),\ (3,2,4),\ (3,4,2),\ (4,3,2)\}$. For each, $x\ \in\ X$, the sequence $\{T^nx\}\to 1$, as $n\ \to\infty$. Hence, the mapping T is orbitally continuous and

$$\lim_{m,n\to\infty}\theta(T^nx\,,T^mx,a\,)<8=\frac{1}{k}\,.$$

Therefore all conditions of Theorem 3.6 are satisfied, thus T has a fixed point $x=1$.

Theorem 3.8 Let (X,d_θ) be T-orbitally complete extended b_2-metric space and T be an orbitally continuous self-mapping on X. Assume that there exists $k\in\ [0,1)$, such that

$$\alpha(x,y,a)\,J(x,y,a)\leq kd_\theta(x,y,a)\,; \qquad (3.9)$$

where,

$$J(x,y,a)=\left[\frac{P(x,y,a)-Q(x,y,a)}{R(x,y,a)}\right],$$

$$P(x,y,a)=\min\begin{Bmatrix} d_\theta(Tx,Ty,a)d_\theta(x,y,a), \\ d_\theta(x,Tx,a)d_\theta(y,Ty,a)\end{Bmatrix},$$

$$Q(x,y,a)=\min\begin{Bmatrix} d_\theta(x,Tx,a)d_\theta(x,Ty,a), \\ d_\theta(y,Ty,a)d_\theta(Tx,y,a)\end{Bmatrix},$$

$R(x,y,a) = \min\{d_\theta(x,Tx,a),\ d_\theta(y,Ty,a)\} \neq 0,$

for all $x,y,a \in X$. Additionally, we suppose that:

i. T is an α-orbital admissible,

ii. there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,a) \geq 1$,

iii. $\lim_{m,n\to\infty} \theta(x_n,x_m,a) \leq \frac{1}{k}$.

Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof As discussed in the proof of Theorem 3.6 by ii, there exists $x_0 \in X$, such that $\alpha(x_0,Tx_0,a) \geq 1$. We construct the sequence $\{x_n\}$ and define, $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. Suppose that, $x_n \neq x_{n-1}$, for each $n \in \mathbb{N}$.

By presumptions i and ii with Lemma 3.4, we have

$$\alpha(x_{n-1},x_n,a) \geq 1, \qquad \text{for all } n \in \mathbb{N}.$$

If we replace x by x_{n-1} and y by x_n in In. (3.9), we get

$$\begin{aligned} J(x_{n-1},x_n,a) &\leq \alpha(x_{n-1},x_n,a)\, J(x_{n-1},x_n,a) \\ &\leq k d_\theta(x_{n-1},x_n,a)\,. \end{aligned} \tag{3.10}$$

Since,

$$J(x_{n-1},x_n,a) = \left[\frac{P(x_{n-1},x_n,a) - Q(x_{n-1},x_n,a)}{R(x_{n-1},x_n,a)}\right],$$

$$P(x_{n-1},x_n,a) = \min\begin{Bmatrix} d_\theta(Tx_{n-1},Tx_n,a)d_\theta(x_{n-1},x_n,a), \\ d_\theta(x_{n-1},Tx_{n-1},a)d_\theta(x_n,Tx_n,a) \end{Bmatrix},$$

$$= d_\theta(x_n,x_{n+1},a)d_\theta(x_{n-1},x_n,a),$$

$$Q(x_{n-1},x_n,a) = \min\begin{Bmatrix} d_\theta(x_{n-1},Tx_{n-1},a)d_\theta(x_{n-1},Tx_n,a), \\ d_\theta(x_n,Tx_n,a)d_\theta(Tx_{n-1},x_n,a) \end{Bmatrix},$$

$$= 0,$$

$$R(x_{n-1},x_n,a) = \min\{d_\theta(x_{n-1},Tx_{n-1},a), d_\theta(x_n,Tx_n,a)\},$$

$$= \min\{d_\theta(x_{n-1},x_n,a), d_\theta(x_n,x_{n+1},a)\}.$$

The In (3.10) becomes

$$\left[\frac{d_\theta(x_n,x_{n+1},a)d_\theta(x_{n-1},x_n,a)}{\min\{d_\theta(x_{n-1},x_n,a), d_\theta(x_n,x_{n+1},a)\}}\right] \leq k d_\theta(x_{n-1},x_n,a)\,.$$

If $\min\{d_\theta(x_{n-1},x_n,a), d_\theta(x_n,x_{n+1},a)\} = d_\theta(x_n,x_{n+1},a)$,

then we get

$$d_\theta(x_{n-1},x_n,a) \leq k d_\theta(x_{n-1},x_n,a) < d_\theta(x_{n-1},x_n,a)\,,$$

which is a contradiction, since $k \in [0,1)$. Thus, we conclude that

$$d_\theta(x_n,x_{n+1},a) \leq k\, d_\theta(x_{n-1},x_n,a). \tag{3.11}$$

Applying In. (3.11) recurrently, we find that

$$\begin{aligned} d_\theta(x_n,x_{n+1},a) &\leq k\, d_\theta(x_{n-1},x_n,a), \\ &\leq k^2\, d_\theta(x_{n-2},x_{n-1},a), \\ &\ \ \vdots \\ d_\theta(x_n,x_{n+1},a) &\leq k^n\, d_\theta(x_0,x_1,a). \end{aligned}$$

In order to complete this proof, a similar restatement will be required from the related lines in proving Theorem 3.6.

Theorem 3.9 Let (X,d_θ) be T-orbitally complete extended b_2-metric space and T be an orbitally continuous self-mapping on X. Assume that there exist, $k \in [0,1)$ and $b \geq 1$, such that

$$\alpha(x,y,a)\, P(x,y,a) - b\, Q(x,y,a) \leq k\, R(x,y,a)\,, \tag{3.12}$$

where,

$$P(x,y,a) = \min\begin{Bmatrix} d_\theta(Tx,Ty,a),\ d_\theta(x,Tx,a), \\ d_\theta(y,Ty,a),\ d_\theta(x,y,a) \end{Bmatrix},$$

$$Q(x,y,a) = \min\{d_\theta(x,Ty,a),\ d_\theta(Tx,y,a)\},$$

$$R(x,y,a) = \max\{d_\theta(x,y,a), d_\theta(x,Tx,a)\} \neq 0,$$

for all $x,y,a \in X$. Additionally, we postulate that:

i. T is α-orbital admissible ,

ii. there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,a) \geq 1$,

iii. $\lim_{m,n\to\infty} \theta(x_n,x_m,a) \leq \frac{1}{k}$.

Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof We use the same construction as in Theorem 3.6 by assumption ii, there exists, $x_0 \in X$, such that $\alpha(x_0,Tx_0,a) \geq 1$. We set up the Sequence $\{x_n\}$ such that, $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. Suppose that, $x_n \neq x_{n-1}$, for each $n \in \mathbb{N}$. By hypothesis i and ii with Lemma 3.4, we have

$$\alpha(x_{n-1},x_n,a) \geq 1, \qquad \text{for all } n \in \mathbb{N}_0.$$

Replacing x by x_{n-1} and y by x_n. in In (3.12), we get

$$\begin{aligned} P(x_{n-1},x_n,a) - b\, Q(x_{n-1},x_n,a) &\leq \alpha(x_{n-1},x_n,a)\, P(x_{n-1},x_n,a) \\ &\quad - b\, Q(x_{n-1},x_n,a) \\ &\leq k\, R(x_{n-1},x_n,a). \end{aligned} \tag{3.13}$$

Since,

$$P(x_{n-1},x_n,a) = \min\begin{Bmatrix} d_\theta(Tx_{n-1},Tx_n,a)d_\theta(x_{n-1},x_n,a), \\ d_\theta(x_{n-1},Tx_{n-1},a)d_\theta(x_n,Tx_n,a) \end{Bmatrix},$$

$$= d_\theta(x_n,x_{n+1},a)d_\theta(x_{n-1},x_n,a),$$

$$Q(x_{n-1},x_n,a) = \min\{d_\theta(x_{n-1},Tx_n,a), d_\theta(Tx_{n-1},x_n,a)\},$$

$$= 0,$$

$$R(x_{n-1},x_n,a) = \max\{d_\theta(x_{n-1},x_n,a)\,, d_\theta(x_{n-1},Tx_{n-1},a)\},$$

$$= d_\theta(x_{n-1},x_n,a).$$

Which yields that, the In. (3.13) becomes,

$$\min\{d_\theta(x_{n-1},x_n,a), d_\theta(x_n,x_{n+1},a)\} \leq k\, d_\theta(x_{n-1},x_n,a)\,.$$

If, $\min\{d_\theta(x_{n-1},x_n,a), d_\theta(x_n,x_{n+1},a)\} = d_\theta(x_n,x_{n+1},a)$, then we get

$$d_\theta(x_{n-1},x_n,a) \leq k d_\theta(x_{n-1},x_n,a) < d_\theta(x_{n-1},x_n,a)\,,$$

which is a contradiction, since $k \in [0,1)$. Thus, we infer that

$$d_\theta(x_n,x_{n+1},a) \leq k d_\theta(x_{n-1},x_n,a). \tag{3.14}$$

Applying In (3.14) recurrently, we find that

$$\begin{aligned} d_\theta(x_n,x_{n+1},a) &\leq k d_\theta(x_{n-1},x_n,a) \\ &\leq k^2 d_\theta(x_{n-2},x_{n-1},a), \\ d_\theta(x_n,x_{n+1},a) &\leq k^n d_\theta(x_0,x_1,a). \end{aligned}$$

By completing the derivation, all steps were validated as originally presented in the proof of Theorem 3.6.

Theorem 3.10 Let (X,d_θ) be T-orbitally complete extended b_2-metric space and T be an orbitally continuous self-mapping on X. Assume that there exist $k \in [0,1)$ and $b \geq 1$, such that

$$\alpha(x,y,a)m(x,y,a) - bn(x,y,a) \leq k d_\theta(x,Tx,a)d_\theta(y,Ty,a) \tag{3.15}$$

where,

$$m(x,y,a) = \min\begin{Bmatrix} [d_\theta(Tx,Ty,a)]^2, \\ d_\theta(x,y,a)\, d_\theta(Tx,Ty,a), \\ [d_\theta(y,Ty,a)]^2 \end{Bmatrix},$$

$$n(x,y,a) = \min\begin{Bmatrix} d_\theta(x,Tx,a)d_\theta(y,Ty,a)\,, \\ d_\theta(x,Ty,a)d_\theta(Tx,y,a) \end{Bmatrix},$$

for all $x,y,a \in X$. Furthermore, we assume that:

i. T is α-orbital admissible ,

ii. there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,a) \geq 1$,

iii. $\lim_{m,n\to\infty} \theta(x_n,x_m,a) \leq \frac{1}{k}$.

Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof As in the proof of Theorem 3.6, we starting from an arbitrary, $x_0 \in X$ and construct the sequence $\{x_n\}$ such that, $x_n =$

Tx_{n-1}, for all $n \in \mathbb{N}$. Suppose that, $x_n \neq x_{n-1}$, for each $n \in \mathbb{N}$. By presumptions i and ii with Lemma 3.4, we have

$$\alpha(x_{n-1}, x_n, a) \geq 1, \qquad \text{for all } n \in \mathbb{N}.$$

Now for, $x = x_{n-1}$ and $y = x_n$. in In. (3.15), implies that

$$\begin{aligned} m(x_{n-1}, x_n, a) - b\, n(x_{n-1}, x_n, a) &\leq \alpha(x_{n-1}, x_n, a)\, m(x_{n-1}, x_n, a) \\ &\quad - b\, n(x_{n-1}, x_n, a), \\ &\leq k[d_\theta(x_{n-1}, Tx_{n-1}, a) d_\theta(x_n, Tx_n, a)] \end{aligned} \tag{3.16}$$

Since,

$$m(x_{n-1}, x_n, a) = \min\left\{\begin{matrix} [d_\theta(Tx_{n-1}, Tx_n, a)]^2, \\ d_\theta(x_{n-1}, x_n, a)\, d_\theta(Tx_{n-1}, Tx_n, a), \\ [d_\theta(x_n, Tx_n, a)]^2 \end{matrix}\right\},$$

$$= \min\left\{\begin{matrix} [d_\theta(x_n, x_{n+1}, a)]^2, \\ d_\theta(x_{n-1}, x_n, a)\, d_\theta(x_n, x_{n+1}, a), \\ [d_\theta(x_n, x_{n+1}, a)]^2 \end{matrix}\right\},$$

$$= \min\left\{\begin{matrix} [d_\theta(x_n, x_{n+1}, a)]^2, \\ d_\theta(x_{n-1}, x_n, a)\, d_\theta(x_n, x_{n+1}, a) \end{matrix}\right\},$$

$$n(x_{n-1}, x_n, a) = \min\left\{\begin{matrix} d_\theta(x_{n-1}, Tx_{n-1}, a) d_\theta(x_n, Tx_n, a), \\ d_\theta(x_{n-1}, Tx_n, a) d_\theta(Tx_{n-1}, x_n, a) \end{matrix}\right\},$$

$$= \min\left\{\begin{matrix} d_\theta(x_{n-1}, x_n, a) d_\theta(x_n, x_{n+1}, a), \\ d_\theta(x_{n-1}, x_{n+1}, a) d_\theta(x_n, x_n, a) \end{matrix}\right\},$$

$$= 0.$$

But,

$$\min\left\{\begin{matrix} [d_\theta(x_n, x_{n+1}, a)]^2, \\ d_\theta(x_{n-1}, x_n, a) d_\theta(x_n, x_{n+1}, a) \end{matrix}\right\} = d_\theta(x_{n-1}, x_n, a) d_\theta(x_n, x_{n+1}, a),$$

is impossible because in this case In. (3.16) becomes

$$\begin{aligned} d_\theta(x_{n-1}, x_n, a)\, d_\theta(x_n, x_{n+1}, a) &\leq k d_\theta(x_{n-1}, x_n, a)\, d_\theta(x_n, x_{n+1}, a) \\ &< d_\theta(x_{n-1}, x_n, a)\, d_\theta(x_n, x_{n+1}, a), \end{aligned}$$

which is a contradiction. Consequently, we deduce that

$$[d_\theta(x_n, x_{n+1}, a)]^2 \leq k d_\theta(x_{n-1}, x_n, a)\, d_\theta(x_n, x_{n+1}, a), \tag{3.17}$$

since, $x_n \neq x_{n+1}$, then we get

$$d_\theta(x_n, x_{n+1}, a) \leq k d_\theta(x_{n-1}, x_n, a)$$

Iteratively, we get that

$$\begin{aligned} d_\theta(x_n, x_{n+1}, a) &\leq k d_\theta(x_{n-1}, x_n, a) \\ &\leq k^2 d_\theta(x_{n-2}, x_{n-1}, a), \\ d_\theta(x_n, x_{n+1}, a) &\leq k^n d_\theta(x_0, x_1, a). \end{aligned}$$

Whereas from the corresponding lines of the Theorem 3.6 proof remaining proof can be obtained.

Theorem 3.11 Let (X, d_θ) be T-orbitally complete extended b_2-metric space and T be an orbitally continuous self-mapping on X. Assume that there exist $k \in [0,1)$ and $b \geq 1$, such that

$$\alpha(x, y, a)\, K(x, y, a) - b\, Q(x, y, a) \leq k\, S(x, y, a) \tag{3.18}$$

where,

$K(x, y, a) = \min\{\, d_\theta(Tx, Ty, a),\ d_\theta(y, Ty, a)\}$,

$Q(x, y, a) = \min\{d_\theta(x, Ty, a),\ d_\theta(Tx, y, a)\}$,

$S(x, y, a) = \max\{d_\theta(x, y, a),\ d_\theta(x, Tx, a),\ d_\theta(y, Ty, a)\}$, for all $x, y, a \in X$. Moreover, we suppose that:

i. T is α-orbital admissible ,
ii. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, a) \geq 1$,
iii. $\lim_{m,n\to\infty} \theta(x_n, x_m, a) \leq \frac{1}{k}$.

Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof Let, $x_0 \in X$, we build the sequence $\{x_n\}$ such that, $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. As discussed in the proof of Theorem 3.6, we suppose that, $x_n \neq x_{n-1}$, for each $n \in \mathbb{N}$. By hypothesis (i) and (ii) with Lemma 3.4, we have

$$\alpha(x_{n-1}, x_n, a) \geq 1, \qquad \text{for all } n \in \mathbb{N}.$$

Now replacing x by x by x_{n-1} and y by x_n. in In. (3.18), we get

$$\begin{aligned} K(x_{n-1}, x_n, a) - b\, Q(x_{n-1}, x_n, a) &\leq \alpha(x_{n-1}, x_n, a) K(x_{n-1}, x_n, a) \\ &\quad - b\, Q(x_{n-1}, x_n, a) \\ &\leq k\, S(x_{n-1}, x_n, a) \end{aligned} \tag{3.19}$$

Since,

$$\begin{aligned} K(x_{n-1}, x_n, a) &= \min\{d_\theta(Tx_{n-1}, Tx_n, a),\ d_\theta(x_n, Tx_n, a)\}, \\ &= \min\{d_\theta(x_n, x_{n+1}, a),\ d_\theta(x_n, x_{n+1}, a)\}, \\ &= d_\theta(x_n, x_{n+1}, a), \\ Q(x_{n-1}, x_n, a) &= \min\{d_\theta(x_{n-1}, Tx_n, a), d_\theta(x_n, Tx_{n-1}, a)\}, \\ &= 0, \end{aligned}$$

$$S(x_{n-1}, x_n, a) = \max\left\{\begin{matrix} d_\theta(x_{n-1}, x_n, a), d_\theta(x_{n-1}, Tx_{n-1}, a), \\ d_\theta(x_n, Tx_n, a) \end{matrix}\right\},$$

$$= \max\{d_\theta(x_{n-1}, x_n, a)\ , d_\theta(x_n, x_{n+1}, a)\}.$$

Thus, In. (3.19) becomes,

$$d_\theta(x_n, x_{n+1}, a) \leq k \max\{d_\theta(x_{n-1}, x_n, a)\ , d_\theta(x_n, x_{n+1}, a)\}.$$

Because, $k \in [0,1)$ the case

$$\max\{d_\theta(x_{n-1}, x_n, a), d_\theta(x_n, x_{n+1}, a)\} = d_\theta(x_n, x_{n+1}, a)$$

is impossible, then In. (3.19) derive that

$$d_\theta(x_n, x_{n+1}, a) \leq k\, d_\theta(x_{n-1}, x_n, a) < d_\theta(x_n, x_{n+1}, a),$$

which is a contradiction. Thus, we infer that

$$d_\theta(x_n, x_{n+1}, a) \leq k d_\theta(x_{n-1}, x_n, a) \tag{3.20}$$

Applying In. (3.20) recurrently, we find that

$$\begin{aligned} d_\theta(x_n, x_{n+1}, a) &\leq k d_\theta(x_{n-1}, x_n, a) \\ &\leq k^2 d_\theta(x_{n-2}, x_{n-1}, a), \\ d_\theta(x_n, x_{n+1}, a) &\leq k^n d_\theta(x_0, x_1, a), \end{aligned}$$

Also, a similar repetition will be required from the related steps in proving Theorem 3.6 in order to complete this proof.

4. Conclusion

We note that several consequences can be observed from the main results in distinct aspects. For example, taking $\theta(x, y, z) = s$, implies corresponding fixed point results in the context of b_2-metric space. In addition, standard version at the given results follow when we take, $\theta(x, y, z) = 1$. Note, also that our results generalize and extend some classical results of non-unique fixed point theorems in the literature, in particular, Theorems 3.6, 3.8, 3.9 and 3.10 are generalization and extension of the corresponding results of Ćirić (1974), Achari (1976), Pachpatte (1979), Ćirić *et al.* (1998), respectively. Indeed, in this present paper we proved some non-unique fixed point theorems in extended b_2-metric space. So far, the results obtained in this article are more general than other preview known results on extended b-metric space Alqahtani *et al.* (2018).

References

Achari, J. (1976) 'On Ćirić's non-unique fixed points, Mat', *Vesnik*, 13, pp. 255-257.

Alqahtani, B., Fulga, A, Karapinar, E. (2018a), 'Non-unique fixed point results in extended b-metric space', *Mathematics*, 6(5) p. 68.

Alqahtani, B., Fulga, A., Karapınar, E., (2018b) 'Common fixed point results in an extended *b*-metric space', *Journal of Inequalities and Applications*, 158, pp. 1-15.

Ćirić, B.L., (1974a) 'A generalization of Banach contraction principle', *Proc. Am. Math. Soc.*, 45, pp. 267-273.

Ćirić, B.L., (1974b) 'On some maps with a non-unique fixed point', *Publ. Inst. Math.*,17, 52-58.

Ćirić. B. L., Jotic. N. (1998) 'A further extension of maps with non-unique fixed points', *Mat. Vesnik*, 50, pp. 1-4.

Elmabrok, S.A., Alkaleeli, R.S., (2018) 'A Generalization of b2-metric spaces and some fixed point Theorems', *Libyan Journal of Science and Technology*, 8(1), pp. 51-55.

Iseki, K. (1975) 'A property of orbitally continuous mapping on 2-metric spaces', *Math. Seminar Notes*, Kobe Univ., 3, pp. 131-132.

Liu, Z., Guo', Z., Kang, S.M., Lee, S.K. (2006) 'On Ćirić type mappings with non-unique fixed and periodic points', *Int. J. Pure Appl. Math*., 26 (3), pp, 399-408.

Pachpatte, B.G. (1979) 'On Ćirić's type maps with a non-unique fixed point', *Indian J. Pure Appl. Math*, 10(8), pp. 1039-1043.

Paliwal, C.Y. (1987) 'Some non-unique fixed point theorems in 2-metric spaces', *Math. Education*, 21, pp. 27-30.

Popescu, O. (2014) 'Some new fixed point theorems for α-Geraghty-contraction type maps in metric spaces', *Fixed Point Theory Appl*, 190, pp. 1-12.

Samet, B., Vetro, C, Vetro, P. (2012) 'Fixed point theorem for $\alpha - \psi$-contractive type mappings', *Nonlinear Anal*, 75, pp. 2154-2165.

WAN Mei-ling, Shu-yi, Xiao-di. ZHENG (2017) 'Non-unique fixed point theorems in 2-metric spaces', *Journal of Zhengzhou University of Light Industry (Natural Science Edition)*, (4), pp. 105-108.

Yuan Lin, Shuyi Zhang (2017) 'Non-unique Fixed Point Theorems with Symmetric Function in 2-Metric Spaces', *Journal of Beihua University (Natural Science)*', (5), pp. 572-575