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The existence and uniqueness of periodic solutions for nonlinear neutral first order differential equation with functional delay

Haitham A. Makhzoum[a,*], Rafik. A. Elmansouri[b]

[a] *Department of mathematic, Faculty of Science, University of Benghazi, Benghazi, Libya*

[b] *College of Electrical and Electronic Technology, Benghazi, Libya*

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* Corresponding author:
E-mail address: haali1788@gmail.com
H. A. Makhzoum

ABSTRACT

We employ the fixed point theorem of Krasnoselskii, to show the existence and uniqueness of periodic solutions of the nonlinear neutral differential equation:

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right) + \int_{-\infty}^{t}\big(D(t,s)f(x(s)) + h(s)\big)ds$$

By modifying the given neutral differential equation into an equivalent integral equation using lemma (2.1). This is done by constructing appropriate operators, one is a contraction and the other is compact, which allow us to prove the existence of periodic solutions. In addition, we used the Banach fixed-point theorem to guarantee a unique periodic solution.

1. Introduction

Over the last few decades, the fixed-point theorem was a useful tool to show the existence and uniqueness of solutions in broad range of mathematical problems. One of the captivating results is a fixed point theorem of Krasnoselskii which is established in 1958 (Krasnoselskii M. A., 1958). This theory is characterized by combining both Banach contraction principle, "which named after the polish mathematician Stefan Banach in 1922, known as BCP. BCP is one of the most important results in analysis and considered as the main source of metric fixed point theory" (Banach, 1922) to gether with Schauder's fixed point theorem. "which is produced by the famous scholer J. Schauder in 1930, it has a big effect on the fixed point theory"(Schauder, 1930). Krasnoselskii theorem attracted a lot of scholars and researchers in this area. Solutions of neutral differential equations by using a periodic solution has been studied by a wide range of scholars, see (Krasnosel'skii, 1954; HOA and SCHMITT, 1995; Burton, 1998; Burton and Kirk, 1998; Raffoul, 2003; Maroun and Raffoul, 2005)

The authors (Althubiti, Makhzoum and Raffoul, 2013) studied the existence of periodic solutions of the nonlinear of differential equations:

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}Q\left(t, x\big(t-g(t)\big)\right) + \int_{-\infty}^{t} D(t,s)f\big(x(s)\big)ds,$$

This paper discusses the existence and uniqueness of periodic solutions of the form:

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right) + \int_{-\infty}^{t}\big(D(t,s)f(x(s)) + h(s)\big)ds \qquad (1.1)$$

By assuming $a(t)$ is a continuous real-valued function. Taking into consideration $Q: \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *and* $D: \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *and* $f: \mathbb{R}\to\mathbb{R}$ *and* $h: \mathbb{R}\to\mathbb{R}$ are continuous functions. The neutral term $\frac{d}{dt}\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)$ in (1.1) produces non-linearity in the derivative term, which is more general compared to the neutral term provided in [4]. Also, Eq. (1.1) contains a non-constant function $g(t)$ as the delay term unlike other studies, where they are dealing with constant delay. So we provide new conditions to construct the mappings to employ fixed point theorems.

Krasnoselskii's fixed-point theorem asks for $z = Az + Bz$ yields $z \in M$ where M is a convex set and Az is continuous and compact. Bz is a contraction. The technique used in this paper is mutated (1.1) into an integral equation that help us to create two mappings and it is the requirement of the fixed point theorem of Krasnoselskii and this done in lemma(2.1).Thereafter, as shown in lemma (3.2) and lemma (3.3) we proved that Az is continuous and compact. Bz is a contraction. It enabled us to apply Krasnoselskii's theorem and to grant us to prove the existence of periodic solutions. At the end, we show the uniqueness of the periodic solution by using the contraction mapping principle. This article organized as follows: section 2 presents the assumptions that will be used in the later sections; also, it provides lemma 2 which transforms (1.1) to an integral equation and section 3 the main results have been presented.

2. Preliminaries

This section introduces some significant notations. We start by supposing that for $T > 0$ define C_T be the set of all continuous scalar functions $x\,(t)$, periodic in t of the period. Afterwards $(C_T\,; \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = sup_{t\in[0,T]}|x|$$

It is appropriate to assume the following conditions

$$a(t+T)=a(t),\quad g(t+T)=g(t),\quad D(t,x)=D(t+T,x) \tag{2.1}$$

With $g(t)$ being scalar, continuous, and $g(t)>0$. Also, we assume that

$$\int_0^T a(s)ds>0 \tag{2.2}$$

We also assume that the function $Q(t,x)$ is periodic in t of period T,

$$Q(t,x)=Q(t+T,x) \tag{2.3}$$

As long as we are looking for periodic solutions, it is necessary to assume $Q(t,x)$ and $f(x)$ globally Lipschitz functions. So for E_1 and E_2 are positive constants such that,

$$\sum_{i=1}^{p}|Q_i(t,x)-Q_i(t,y)|\le E_1\|x-y\| \tag{2.4}$$

and,

$$|f(x)-f(y)|\le E_2\|x-y\| \tag{2.5}$$

Also, there is E_3, E_4 such that,

$$\int_{-\infty}^{t}|D(t,s)|\,ds\le E_3<\infty,\qquad h(s)\le E_4 \tag{2.6}$$

Now, the following lemma helps to convert (1.1) to an equivalent integral equation.

Lemma 2.1. If $x(t)\in C_T$ **,** and the conditions (2.1) and (2.3) hold. Then $x(t)$ is a solution of Eq. (1.1) if and only if

$$x(t)=\sum_{i=1}^{p}Q_i(t,x(t-g(t)))+(1-e^{-\int_0^T a(k)dk})^{-1}\Big[\int_{t-T}^{t}-a(u)\sum_{i=1}^{p}Q_i\big(u,x(u-g(u))\big)e^{-\int_u^t a(k)dk}du+\int_{t-T}^{t}\int_{-\infty}^{u}\big(D(u,s)f(x(s))+h(s)\big)ds\,e^{-\int_u^t a(k)dk}du\Big]$$

Proof. Let $x(t)\in C_T$ be a solution of (1.1). By writing (1.1) as

$$\frac{d}{dt}\Big[x(t)-\sum_{i=1}^{p}Q\big(t,x(t-g(t))\big)\Big]=-a(t)x(t)+\int_{-\infty}^{t}[D(t,s)f(x(s))+h(s)]ds$$

Adding $a(t)\sum_{i=1}^{p}Q((t,x(t-g(t)))$ to both sides of the last equation we obtain

$$\frac{d}{dt}\Big[x(t)-\sum_{i=1}^{p}Q\big(t,x(t-g(t))\big)\Big]=-a(t)\Big[x(t)-\sum_{i=1}^{p}Q\big(t,x(t-g(t))\big)\Big]-a(t)\sum_{i=1}^{p}Q\big(t,x(t-g(t))\big)+\int_{-\infty}^{t}[D(t,s)f(x(s))+h(s)]ds \tag{2.7}$$

Multiply both sides of (2.7) by $e^{\int_0^t a(k)dk}$, and then integrate from $t-T$ to t to get

$$\Big[x(t)-\sum_{i=1}^{p}Q_i\big(t,x(t-g(t))\big)\Big]e^{\int_0^t a(k)dk}-\Big[x(t-T)-\sum_{i=1}^{p}Q_i(t-T,x(t-T-g(t-T)))\Big]e^{\int_0^{t-T}a(k)dk}=$$
$$\int_{t-T}^{t}\Big[-a(u)\sum_{i=1}^{p}Q_i\big(u,x(u-g(u))\big)+\int_{-\infty}^{u}(D(u,s)f(x(s))+h(s))ds\Big]e^{\int_0^u a(k)dk}du$$

By dividing both sides of the above equation by $e^{\int_0^t a(k)}$, and due to the fact, that $x(t)$ is a periodic function of period T and using Eqs. (2.1), (2.3) we arrive at

$$\Big[\int_{t-T}^{t}-a(u)\sum_{i=1}^{p}Q_i\big(u,x(u-g(u))\big)e^{-\int_u^t a(k)dk}du+\int_{t-T}^{t}\int_{-\infty}^{u}\big(D(u,s)f(x(s))+h(s)\big)ds\,e^{-\int_u^t a(k)dk}du\Big]$$

3. Existence and uniqueness of periodic solutions

In this section, we will present the state of Krasnoselskii's fixed-point theorem and apply this theorem to prove the existence of a periodic solution.

Theorem 3.1. (Krasnoselskii). Let $\mathcal{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathcal{B},\|\,.\,\|)$. suppose that A and B map $\mathcal{M}$ into $\mathcal{M}$ such that

(i) $x,y\in\mathcal{M}$, implies $Ax+By\in\mathcal{M}$,

(ii) A is continuous and $A\mathcal{M}$ is contained in a compact set subset of $\mathcal{M}$,

(iii) B is a contraction mapping.

Then there exists $z\in\mathcal{M}$ with $z=Az+Bz$.

As theorem 3.1 states there are two mappings, one is a contraction and the other is compact. Therefore, we will define the operator $P: C_T\to C_T$ by

$$(P\varphi)(x)=\sum_{i=1}^{p}Q_i\big(t,\varphi(t-g(t))\big)+\big(1-e^{-\int_0^T a(k)dk}\big)^{-1}\Big[\int_{t-T}^{t}-a(u)\sum_{i=1}^{p}Q_i\big(u,\varphi(u-g(u))\big)e^{-\int_u^t a(k)dk}du+\int_{t-T}^{t}\int_{-\infty}^{u}\big(D(u,s)f(\varphi(s))+h(s)\big)ds\,e^{-\int_u^t a(k)dk}du\Big] \tag{3.1}$$

And by rewriting (3.1) as follows

$$(P\varphi)(t)=(B\varphi)(t)+(A\varphi)(t),$$

Where $A,B: C_T\to C_T$ are given by

$$(B\varphi)(t)=\sum_{i=1}^{p}Q_i(t,\varphi(t-g(t))) \tag{3.2}$$

And,

$$(A\varphi)(t)=\big(1-e^{-\int_0^T a(k)dk}\big)^{-1}\Big[\int_{t-T}^{t}-a(u)\sum_{i=1}^{p}Q_i\big(u,\varphi(u-g(u))\big)e^{-\int_u^t a(k)dk}du+\int_{t-T}^{t}\int_{-\infty}^{u}\big(D(u,s)f(\varphi(s))+h(s)\big)ds\,e^{-\int_u^t a(k)dk}du\Big]$$

The goal here is to show that $(B\varphi)(t)$ is contraction and $(A\varphi)(t)$ is compact. The analysis is introduced in these two lemmas

Lemma 3.2. If B is given by (3.2) with $E_1<1$, and (2.4) hold, then B is a contraction.

Proof. Let B be defined by (3.2). Then for $\varphi,\psi\in C_T$ we have

$$\|(B\varphi)(t)-(B\psi)(t)\|=sup_{t\in[0,T]}|(B\varphi)(t)-(B\psi)(t)|$$

$$= sup_{t\in[0,T]} \sum_{i=1}^{p} |Q_i(t,\varphi(t-g(t))) - Q_i(t,\psi(t-g(t)))|$$

By using (2.4), then

$$\leq E_1 sup_{t\in[0,T]} \|\varphi(t-g(t)) - \psi(t-g(t))\|$$

As $E_1 < 1$. Therefore, B defines a contraction.

Before showing Lemma 3.3. It is appropriate to the following notations:

$$\tau = max_{t\in[0,T]} \left|(1-e^{-\int_0^T a(k)dk})^{-1}\right|, \rho = max_{t\in[0,T]}|a(t)|, \nu = max_{u\in[t-T,t]} e^{-\int_u^t a(k)dk}$$

Lemma 3.3. If A is defined by (3.3), then A is continuous and the image of A is contained in a compact set.

Proof. We will start by proving A is continuous we define A as (3.3). Let $\varphi, \psi \in C_T$,

for a given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{\mathrm{N}}$ with $N = \tau\nu T[\rho E_1 + E_2E_3]$, now for $\|\varphi - \psi\| < \delta$, and by using (2.4) into (3.3) ,we get

$$\|A_\varphi - A_\psi\| \leq \tau\nu T[\rho E_1 + E_2E_3]\|\varphi - \psi\| \leq \mathrm{N}\|\varphi - \psi\| \leq \mathrm{N}\delta \leq \varepsilon.$$

This is show that A is continuous. The second step is showing A is a compact set using Ascoli-Arzela's theorem [5] which states that for $A \subset X$, A is compact if and only if A is bounded, and equicontinuous.

Let $\Omega = \{\varphi \in C_T: \|\varphi\| \leq \Upsilon\}$, where Υ is any fixed positive constant, from (2.4), (2.5) we have,

$$\sum_{i=1}^{p} |Q_i(t,x)| = \sum_{i=1}^{p} |Q_i(t,x) - Q_i(t,0) + Q_i(t,0)|$$

$$\leq \sum_{i=1}^{p} [|Q_i(t,x) - Q_i(t,0)| + |Q_i(t,0)|]$$

$$\leq E_1\|x\| + \alpha$$

where $\alpha = sup_{t\in[0,T]} \sum_{i=1}^{p} |Q_i(t,0)|$.

In the same way,

$$|f(x)| = |f(x) - f(0)|$$
$$\leq |f(x) - f(0)|$$
$$\leq E_2\|x\|.$$

Taking into consideration, $f(0) = 0$. Let $\varphi_n \in \Omega$ where n is a positive integer with

$L = \tau\nu T[\rho E_1(\Upsilon + \alpha) + \Upsilon E_2E_3 + E_4]$ where $L > 0$, Therefore,

$$\|A_{\varphi_n}\| =$$

$$|(1 - e^{-\int_0^T a(k)dk})^{-1}[\int_{t-T}^{t} -a(u) \sum_{i=1}^{p} Q_i\left(u, \varphi_n(u - g(u))\right) e^{-\int_u^t a(k)dk} du + \int_{t-T}^{t}\int_{-\infty}^{u} \left(D(u,s)f(\varphi_n(s)) + h(s)\right) ds\, e^{-\int_u^t a(k)dk} du\,]|$$

$$\leq max_{t\in[0,T]}|(1 - e^{-\int_0^T a(k)dk})^{-1} \int_{t-T}^{t} [-a(u) \sum_{i=1}^{p} Q_i\left(u, \varphi_n(u - g(u))\right) + \int_{-\infty}^{u} (D(u,s)f(\varphi_n(s)) + h(s))ds]e^{-\int_u^t a(k)dk} du|$$

$$\leq \tau\nu \int_{t-T}^{t} [\rho E_1(\|\varphi_n\| + \alpha) + \int_{-\infty}^{u} |D(u,s)f(\varphi_n(s)) + h(s)|ds]\, du$$

$$\leq \tau\nu \int_{t-T}^{t} [\rho E_1(\|\varphi_n\| + \alpha) + E_2E_3\|\varphi_n\| + E_4]\, du$$

$$\leq \tau\nu T[\rho E_1(\|\varphi_n\| + \alpha) + E_2E_3\|\varphi_n\| + E_4]$$

$$\leq \tau\nu T[\rho E_1(\Upsilon + \alpha) + \Upsilon E_2E_3 + E_4] \leq L.$$

This is showing that A is bounded. To prove A is equicontinuous we need to find $(A\varphi_n)'(t)$ and prove that it is uniformly bounded. Therefore, after derivative (3.3) with using (2.2), (2.3) we get,

$$(A\varphi_n)'(t) = -a(t)A(\varphi_n(t)) + a(t)\sum_{i=1}^{p} Q_i\left(u, \varphi_n(u - g(u))\right) + \int_{-\infty}^{t} (D(t,s)f(\varphi_n(s)) + h(s))ds$$

The above expression yields $\|(A\varphi_n)'\| \leq Z$ where Z is some positive constant. Hence, by Ascoli-Arzela's theorem $A\varphi$ is compact.

Theorem 3.4. Suppose the hypothesis of Lemma 2.4. Let $\alpha = sup_{t\,[[0,T]} \sum_{i=1}^{p} |Q_i(t,0)|$ and, suppose (2.1)-(2.6) hold. Let J be a positive constant satisfying the inequality

$$\tau\nu T[\rho E_1(J + \alpha) + E_2E_3J + E_4] + E_1J + \alpha \leq J$$

Let $\mathcal{M} = \{\varphi \in C_T: \|\varphi\| \leq J\}$. Then Eq. (1.1) has a solution in $\mathcal{M}$.

Proof: First of all, we will define $\mathcal{M} = \{\varphi \in C_T: \|\varphi\| \leq J\}$, and by knowing that A is continuous and AM is contained in a compact set. Also, the mapping B is a contraction from lemma (3.2), (3.3) and it is clear that $A, B: C_T \to C_T$. The aim is showing that $\|A_\varphi + B_\psi\| \leq J$. Let $\varphi, \psi \in \mathcal{M}$, with $\|\varphi\|, \|\psi\| \leq J$. Then,

$$\|A_\varphi + B_\psi\| \leq \|A_\varphi\| + \|B_\psi\|$$

Lemma 3.3 says that,

$$\|A_{\varphi_n}\| \leq \tau\nu T[\rho E_1(\|\varphi_n\| + \alpha) + E_2E_3\|\varphi_n\| + E_4]$$

Therefore,

$$\|A\| + \|B\| \leq \tau\nu T[\rho E_1(\|\varphi_n\| + \alpha) + E_2E_3\|\varphi_n\| + E_4] + E_1\|\psi\| + \alpha$$
$$\leq \tau\nu T[\rho E_1(J + \alpha) + E_2E_3J + E_4] + E_1J + \alpha \leq J$$

This is proving all conditions of Theorem 3.1. Thus, there exists a fixed-point z in $\mathcal{M}$. By Lemma 2.1, this fixed point is a solution of (1.1). Hence, (1.1) has a T-periodic solution.

Theorem 3.5. Let (2.1)-(2.6) hold if

$$E_1 + \tau\nu T(\rho E_1) + TE_3E_2 < 1$$

Then Eq. (1.1) has a unique T-periodic solution..

Proof. We define P as (3.1). Let $\varphi, \psi \epsilon C_T$, in view of (3.1) we have,

$$\|P_\varphi - P_\psi\| < [E_1 + \tau\nu T(\rho E_1) + TE_3E_2]\|\varphi - \psi\|.$$

This completes the proof of Theorem 3.5.

4. Conclusion

In this paper, we convert the nonlinear neutral Eq. (1.1) into an integral equation, and then we apply the Krasnoselskii's fixed-point theorem, which guarantee the existence of periodic solutions of the resulting equation. Obtaining the integral equation enables us to create two mappings, one is a contraction and the other is completely continuous. This allows us to benefit from the contraction mapping principle to prove the uniqueness of periodic solutions of the nonlinear neutral Eq. (1.1) according to the Krasnoselskii's fixed point.

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