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Asymptotic stability of periodic solutions for a nonlinear neutral first-order differential equation with functional delay

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Highlights

- **Introduce the concept of the fixed-point method and its importance.**
- **We have introduced some important hypotheses, which help us achieve our aim.**
- **Present the main result and prove it.**

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ABSTRACT

We consider the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}\sum_{i=1}^{p} Q_i(t, x(t-g(t))) + \int_{-\infty}^{t}[D(t,s)f(x(s)) + h(s)]\,ds,$$

and use the contraction mapping principle to show the asymptotic stability of the zero solution provided that $Q(t,0) = f(0) = 0$.

1. Introduction

There have been widely varied solutions for stabilization, Lyapunov's direct methodology being the most known. The Lyapunov methodology for the broadly differential equations has been terribly effective in establishing the result for stability see (Burton., 1985; Hatvani., 1997; Seifert., 1973), as well as in establishing the existence of periodic solutions of differential equations with functional delays see (Althubiti *et al.*, 2013). However, there have been certain issues despite the efficacy of Lyapunov's technique if the functions of equations are unbounded with time and the derivative of the delay is not small and the complexity of generating the Lyapunov function, it is a kind of art for finding this function. Researchers have been working on discovering fresh ways of avoiding those problems. Burton *et al.* (2002) noted that some of these issues disappearing when implementing the fixed-point theory. Due to the simplicity of a fixed-point method in comparison with the Lyapunov method, the fixed-point method has become an important instrument to show the existence and uniqueness of solutions and to study the solution's stability in a multitude of mathematical problems.

Makhzoum *et al.* (2018) discussed the existence and uniqueness of periodic solutions of

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}\sum_{i=1}^{p} Q_i\left(t, x(t-g(t))\right) + \int_{-\infty}^{t}[D(t,s)f(x(s)) + h(s)]\,ds, \qquad (1.1)$$

by assuming $a(t)$ is a continuous real-valued function. Taking into consideration $Q: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $D: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, $x: \mathbb{R} \to \mathbb{R}$ *and* $h: \mathbb{R} \to \mathbb{R}$ are continuous functions and to ensure periodicity the following assumption has been made $a(t)$, $g(t)$, $D(t,x)$, $Q(t,x)$ are periodic functions. Also, let C_T stand for the set of all continuous scalar functions $x(t)$ periodic in t of period T.

This paper is mainly concerned with the asymptotic stability of the zero solution on the Eq. (1.1), as follows we have to mutate Eq. (1.1) to an integral mapping equation appropriate for the contraction mapping theorems. This article organized as follows: Section 2 presents the hypotheses to be used in the later sections it also introduces Lemma 2.1 that converts Eq. (1.1) into an essential equation, and Section 3 presents the main results.

2. Preliminaries

In this section, we introduce some significant notations. It is appropriate to assume the following conditions. Let $Q(t,x)$ and $f(x)$ be globally Lipschitz. So for E_1, E_2, E_3, E_4 and K are positive constants such that,

$$\sum_{i=1}^{p} |Q_i(t,x) - Q_i(t,y_i)| \le E_1\|x-y\| \qquad (2.1)$$

$$|f(x) - f(y)| \le E_2\|x-y\| \qquad (2.2)$$

$$\int_{-\infty}^{t} |D(t,s)|\,ds \le E_3 < \infty, h(s) \le E_4 \le KE_4 \qquad (2.3)$$

The following lemma helps transform Eq. (1.1) to an integral corresponding equation,

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Lemma 2.1

Let $Q(t,x)$, $D(t,s)$, $a(t)$, $f(t)$, $x(t)$, $g(t)$ and $h(t)$ are defined as above, then $x(t)$ is a solution of Eq. (1.1) if and only if

$$x(t) = \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right) + \left[x(0) - \sum_{i=1}^{p} Q_i\left(0, x\big(0-g(0)\big)\right)\right] e^{-\int_0^t a(k)dk} + \int_0^t \left[-a(u)\sum_{i=1}^{p} Q_i\left(u, x\big(u-g(u)\big)\right)\right] e^{-\int_u^t a(k)dk} + \int_{-\infty}^{u} \left[D(u,s) f\big(x(s)\big) + h(s)\right] ds\, e^{-\int_u^t a(k)dk} du \quad (2.4)$$

Proof

Let $x(t) \in C_T$ be a solution of Eq. (1.1). Now, by writing Eq. (1.1) as

$$\frac{d}{dt}[x(t) - \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)] = -a(t)x(t) + \int_{-\infty}^{t} \left[D(t,s) f\big(x(s)\big) + h(s)\right] ds.$$

Adding $a(t)\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)$ both sides of the last equation, we obtain

$$\frac{d}{dt}\left[x(t) - \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)\right] = -a(t)\left[x(t) - \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)\right] - a(t)\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right) + \int_{-\infty}^{t} \left[D(t,s) f\big(x(s)\big) + h(s)\right] ds \quad (2.5)$$

Multiply both sides of Eq. (2.5) by $e^{\int_0^t a(k)dk}$, and then integrate from 0 to t to get

$$[x(t) - \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)] e^{\int_0^t a(k)dk} - [x(0) - \sum_{i=1}^{p} Q_i(0, x(0-g(0)))]\, e^{\int_0^0 a(k)dk} = \int_0^t [-a(u)\sum_{i=1}^{p} Q_i\left(u, x\big(u-g(u)\big)\right) + \int_{-\infty}^{u} \left[D(u,s) f\big(x(s)\big) + h(s)\right] ds]\, e^{\int_0^u a(k)dk} du$$

By dividing both sides of the above equation by $e^{\int_0^t a(k)dk}$, we arrive at

$$x(t) = \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right) + [x(0) - \sum_{i=1}^{p} Q_i\left(0, x\big(0-g(0)\big)\right)]\, e^{-\int_0^t a(k)dk} + \int_0^t \left[-a(u)\sum_{i=1}^{p} Q_i\left(u, x\big(u-g(u)\big)\right)\right] e^{-\int_u^t a(k)dk} + \int_{-\infty}^{u} [D(u,s) f\big(x(s)\big) + h(s)]\, ds]\, e^{-\int_u^t a(k)dk} du$$

Thus, we see that $x(t)$ is a solution of Eq. (1.1)

3. Main result

The methods employed in this section are adapted from the paper (Althubiti *et al.*, 2013). We are supposing that $g(t)$ and $f(t)$ are continuous, $g(t) > 0$ and $Q(t,0) = f(0) = 0$. Let, $\psi(t): (-\infty, 0] \to \mathbb{R}$ gives continuous bounded initial function. We say $x(t) = x(t, 0, \psi)$ is a solution of Eq. (1.1), if $x(t) = \psi(t)$ for $t \le 0$ and satisfies Eq. (1.1) for $t \ge 0$. We assume the zero solution of Eq. (1.1) is stable at t_0, if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that $\psi: (-\infty, t_0] \to \mathbb{R}$ with $\|\psi\| < \delta$ on $(-\infty, t_0]$, implies $|x(t, t_0, \psi)| < \epsilon$.

We will state and prove our results without loss of generality by beginning at $t_0 = 0$, Let $C(\mathbb{R})$ be the space of all continuous functions from R → R and define the set U by $U = \{\varphi: \mathbb{R} \to \mathbb{R}: \varphi(t) = \psi(t)$, if $t \le 0, \varphi(t) \to 0$ as $t \to \infty, \varphi \in C(\mathbb{R})$, and φ is bounded$\}$. Then $(U, \|\cdot\|)$ is a complete metric space where $(\|\cdot\|)$ is the supremum norm. We impose the following requirements on the next theorem

$$\int_0^t a(s)ds > 0 \ \ and\ e^{-\int_0^t a(s)ds} \to 0, \ \ as\ \ t \to \infty \quad (3.1)$$

There is an $\alpha > 0$ such that

$$E_1 + \int_0^t [|a(u)|\, E_1 + E_2E_3 + E_4]\ e^{-\int_u^t a(k)dk}\ du \le \alpha < 1, \quad (3.2)$$

$$t - g(t) \to \infty, \text{as } t \to \infty, \quad (3.3)$$

$$Q(t,0) \to 0, \text{as } t \to \infty, \quad (3.4)$$

Where, E_1, E_2, E_3 and E_4 are defined in inequalities (2.1)–(2.3).

Theorem 3.1

If the inequalities (2.1)–(2.3) and the conditions (3.1)–(3.4) hold, then every solution $x(t, 0, \psi)$ of Eq. (1.1) with small continuous initial function $\psi(t)$ is bounded and approaches zero as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.

Proof

Define the mapping $P: U \to U$ by

$$(P\varphi)(t) = \begin{cases} \psi(t) & if\ t \le 0, \\ \sum_{i=1}^{p} Q_i\left(t, \varphi\big(t-g(t)\big)\right) + [\psi(0) + \sum_{i=1}^{p} Q_i\left(0, \psi\big(-g(0)\big)\right)]\, e^{\int_0^t a(k)dk} + \int_0^t -a(u)\sum_{i=1}^{p} Q_i\left(u, \varphi\big(u-g(u)\big)\right) e^{-\int_u^t a(k)dk} + \int_0^t \int_{-\infty}^{u} \left[D(u,s) f\big(\varphi(s)\big) + h(s)\right] ds\, e^{-\int_u^t a(k)dk} du & if\ t > 0 \end{cases}$$

It is noticeable that $\varphi \in U$, $(P\varphi)(t)$ is continuous. Let $\varphi \in U$ with $\|\varphi\| \leq K$ for some positive constant K. Let, $\psi(t)$ be small given continuous initial function with $\delta > 0$, $\|\psi\| < \delta$, then using (3.2) in the definition of $(P\varphi)(t)$ we have

$$\begin{aligned}\|(P\varphi)(t)\| &\leq E_1K + (1+E_1)\delta \\ &\quad + \int_0^t [|a(u)|E_1\,K + E_2E_3K + KE_4] \\ &\leq (1+E_1)\delta \\ &\quad + K\Bigg[E_1 \\ &\quad + \int_0^t [|a(u)|E_1 + E_2E_3 + E_4]\; e^{-\int_u^t a(k)dk}\; du\Bigg], \\ &\leq (1+E_1)\delta + K\alpha,\end{aligned}$$

which implies $\|(P\varphi)(t)\| \leq K$ for the right δ. Thus, $(P\varphi)(t)$ is bounded. Next, we show that $(P\varphi)(t) \to 0 \; as \; t \to \infty$. The second term on the right side of $P(\varphi)(t)$ tends to zero, by condition (3.1) also the first term on the right side tends to zero, because of (3.3), (3.4), and the fact that $\varphi \in U$. Left to show that the integral term goes to zero as $t \to \infty$.

Let, $\epsilon > 0$ be given and $\varphi \in U$ with $\|\varphi\| \leq K$, $K > 0$, then there exist, $t_1 > 0$, so that for $t > t_1$, $|\varphi(t - g(t))| < \epsilon$. Due to condition (3.1) there exists $t_2 > t_1$, such that for $t > t_2$ implies that $e^{-\int_{t_1}^t a(k)\,dk} < \frac{\epsilon}{\alpha K}$, thus for $t > t_2$ we have

$$\begin{aligned}&\Bigg|\int_0^t [-a(u)\sum_{i=1}^p Q_i\big(u, \varphi(u-g(u))\big) \\ &\quad + \int_{-\infty}^u \big[D(u,s)f(\varphi(s)) \\ &\quad + h(s)\big]\,ds]\, e^{-\int_u^t a(k)dk} du\Bigg| \\ &\leq \int_0^{t_1} [|a(u)|E_1\,K + E_2E_3K \\ &\quad + E_4K]\, e^{-\int_u^t a(k)dk}\; du \\ &\quad + \int_{t_1}^t [|a(u)|E_1\epsilon + E_2E_3\epsilon + E_4\epsilon]\, e^{-\int_u^t a(k)dk}\; du \\ &\leq K\int_0^{t_1} [|a(u)|E_1 + E_2E_3 + E_4]\; e^{-\int_u^t a(k)dk}\; du \\ &\quad + \epsilon\int_{t_1}^t [|a(u)|E_1 + E_2E_3 \\ &\quad + E_4]\; e^{-\int_u^t a(k)dk}\; du, \\ &\leq Ke^{-\int_{t_1}^t a(k)dk} \int_0^{t_1} [|a(u)|E_1 + E_2E_3 + E_4]\; e^{-\int_u^{t_1} a(k)dk}\; du + \alpha\epsilon, \\ &\leq \alpha K e^{-\int_{t_1}^t a(k)dk} + \alpha\epsilon, \\ &\leq \epsilon + \alpha\epsilon.\end{aligned}$$

Hence, $(P\varphi)(t) \to 0$, as $t \to \infty$.

It remains to show that $(P\varphi)(t)$ is a contraction under the supremum norm.

Theorem 3.2

Let J be a positive constant satisfying the inequality

$$E_1 + \int_0^t [|a(u)|\,E_1 + E_2E_3]\; e^{-\int_u^t a(k)dk}\; du \leq J \leq 1 \qquad (3.5)$$

Then, $(P\varphi)(t)$ is a contraction under the supremum norm.

Proof

Let, $\mu, \tau \in U$. Then

$$\begin{aligned}|(P\mu)(t) - (P\tau)(t)| &\leq E_1 \\ &\quad + \int_0^t [|a(u)|E_1 + E_2E_3]\; e^{-\int_u^t a(k)dk}\; du\|\mu - \tau\| \\ &\leq J\|\mu - \tau\|.\end{aligned}$$

Therefore, according to the principle of contraction mapping, $(P\varphi)(t)$ is bound and tends to be zero since t is infinite, moreover, $(P\varphi)(t)$ has a unique fixed point in U that resolves Eq. (1.1). The stability of the zero solution at $t_0 = 0$ resulted from simply replacing K with ϵ. That ends the proof.

4. Conclusion

In this paper, the nonlinear neutral first-order differential equation with functional delay Eq. (1.1) has been transformed into an integral equation by using Lemma 2.1 The integral equation allows us to create a map that enables us to apply the concept of the contraction-map, which ensures us the stability of periodic solutions for a nonlinear neutral first-order differential equation with functional delay.

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