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Categorical Forms of Green's Function; The One-Dimensional Time Response Function: Forced Harmonic Oscillator

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Highlights

- **Simple analysis of Green's Method to solve an inhomogeneous ordinary differential equation.**
- **The method of variations of parameters can provide specific formats of Green's functions to particular physical problems involving differential equations.**
- **Transformation of 2nd order differential equation into Green's differential equation must involve delta function.**

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ABSTRACT

An introductory technique to solve one-dimensional time-dependent boundary value problems using Green's function, aiming at postgraduate students, and science researchers who have no prior experience with the method. The aim here is to have a rather simple look at Green's functions as a solution to boundary value problems in their explicit functional forms, rather than their explicit expressions, and finally to be acquainted with them. Using the method of variations of parameters, one can provide a satisfactory format of Green's functions, which would make it easier to introduce Green's functions and accept their abstract format in more complex problems including higher dimensions.

1. Introduction

We often come across Sturm-Liouville boundary value problems in many different areas of science, particularly in Physics and Chemistry. In the mathematical sense, it has the form of Lu=f(t), where L is the Sturm-Liouville linear operator, and f(t) is a mathematical function that satisfies the operand. One may approach the solution of this problem by different methods if f(t) is simple enough. On the other hand, in situations where f(t) is an impulse function i.e., acted in a very short time, these methods often fail to provide straightforward solutions. In this paper, we introduce a powerful technique to deal with such problems. By considering the right-hand side of the nonhomogeneous equation, termed as impulse response that acts in infinitesimal time. The equation will be transformed into a new form LG=δ(t-τ), where G is known as Green's function and δ(t-τ) is a distribution function known as Dirac's delta function which has many special properties, whereas its integral over the entire real line is equal to one. The main constructional property, its value blows up at t=τ, and is equal to zero everywhere else. Other important properties are shown in Table 1 cited from Spanier and Oldham (2009).

Once we find these Green's functions then it is possible to find the solutions of Lu=f(x), from u(t)= ∫G(t,τ)f(τ)dτ, since G and u are connected by convolution property; G*u. In this sense, Green's function is the solution to the ordinary differential equation with a non-homogenous Dirac's point source, and the same boundary conditions are applied, as in the original problem. The idea is to find the solutions of the differential equation for a single point source then use them to construct the full solutions of the original differential equation by superposition principle. Although the method is reviewed in many papers in the few past decades (Dennery and Krzywicki, 2009; Witten and McCormick, 1975; Flores-Hidalgo and Barone, 1975), the objective here is to further simplify the approach and have an elicited structural format of Green's functions and understand how they form the backbone of the general solution of the main problem.

Table 1

Some properties of δ-functions

	Dirac's δ-function property
1	$\delta(t-\tau)=0 \qquad t\neq\tau$
2	$\int_0^{+\infty}\delta(t)dt=1$
3	$\delta(\omega t)=\omega^{-1}\delta(x) \qquad \omega\neq 0$
4	$\int_0^{+\infty}\delta(T_1-t)\delta(t-T_2)dx=\delta(T_1-T_2)$
5	$\int_0^{+\infty}f(t)\delta(t-\tau)dx=f(\tau)$

2. Mathematical Approach

It is relatively rare to avoid the use of differential equations to solve physical problems. The development of the mathematical model to solve a particular physical problem often leads to a boundary value problem (Edwards *et al.*, 2014). As an example, the motion of an object of mass, m executing simple harmonic motion, as shown in Fig. 1, leads to the following *homogeneous* ordinary differential equation:

$$Lu(t)=0 \tag{1}$$

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where the Sturm-Liouville operator

$$L \equiv \frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + {\omega_o}^2$$

acted on time-dependent function$u(t)$ which represents the displacement of the particle at any time t; $\omega_o = \frac{k}{m}$ is the natural angular frequency,$k$is the restoring force constant; $\gamma = \frac{b}{2m}$ is part of the resistance force $R = -b\frac{du}{dt}$ and b is a constant that depends on surfaces of contact between the sliding object and ground.

We will consider here the analysis of the damping case where$\omega_o > \gamma$. The solution of this equation which can be easily obtained by a variety of techniques is given by (Bosa, 2006; Arfken *et al.*, 2012):

$$u(t) = exp(-\gamma t)[A\cos\omega t + B\sin\omega t] \tag{2}$$

where A and B are two arbitrary constants reflecting the fact that we have two initial conditions (i.e., position; $u(0)$ and velocity; $u'(0)$); $\omega^2 = \gamma^2 - \omega_o^2$. However, when this system is subjected to an abrupt forcing term $f(t)$ (force per unit mass), we obtain an *inhomogeneous* differential equation of the form:

$$Lu(t) = -f(t) \tag{3}$$

Here the solution of this differential equation composes of two parts, a complementary solution; $u_c(t)$ obtained earlier by Eq. (2) and a particular solution; $u_p(t)$ satisfying Eq (3). In this case, the general solution is given by:

$$u(t) = u_c(t) + u_p(t) \tag{4}$$

To find this particular solution for this abstract form of forcing term$f(t)$ (often referred to as source), we introduce an equivalent equation with unit impulse represented by delta function $\delta(t - t_o)$ which acts as a distribution function (per unit time) and has the property (Oliver *et al.*, 2010):

$$\int_{-\infty}^{\infty} \delta(t - t_o)dt = 1 \tag{5}$$

The corresponding equation for which a unit impulse represented by delta function, $\delta(t - t_o)$ is applied at a reference timet_o,

$$LG(t, t_o) = -\delta(t - t_o) \tag{6}$$

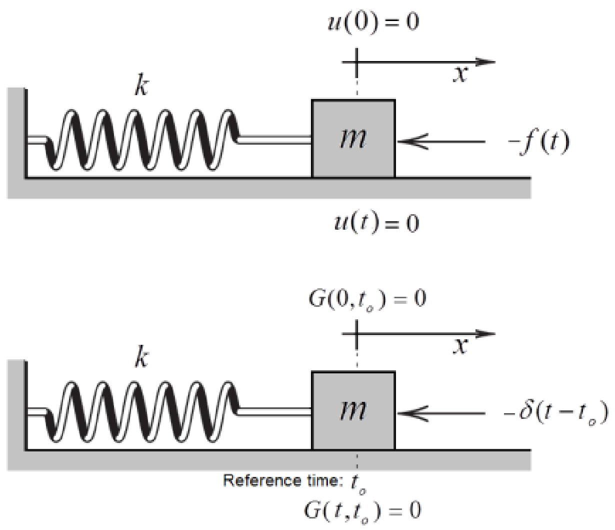

Fig. 1. A damped oscillating system with a forcing term $f(t)$ shown above, and an equivalent system with a source term given by Dirac delta function acted at a reference timet_o.

Here$G(t, t_o)$ is Green's function, which corresponds to the displacement $u(t)$ in the original differential Eq. (3).

Our aim here is to develop a connection between $u(t)$ and $G(t, t_o)$. By multiplying Eq. (3) by $G(t, t_o)$ and Eq. (6) by $u(t)$ and subtract;

$$G(t,t_o)\frac{d^2u}{dt^2} - u(t)\frac{d^2G(t,t_o)}{dt^2} + 2\gamma\left[G(t,t_o)\frac{du}{dt} - u(t)\frac{dG(t,t_o)}{dt}\right] = -G(t,t_o)f(t) + u(t)\delta(t,t_o)$$

Then integrate over the time of action of force $f(t)$ from the start of the motion at $t = 0$ to time $t = \tau$ imply that;

$$\left[G(t,t_o)\frac{du}{dt} - u\frac{dG(t,t_o)}{dt}\right]_0^\tau + 2\gamma\int_0^\tau \left[G(t,t_o)\frac{du}{dt} - u\frac{dG(t,t_o)}{dt}\right]dt = u(t_o) - \int_0^\tau f(t)G(t,t_o)dt$$

As $G(t, t_o)$ and $u(t)$ vanish at $t = 0$ and $t = \tau$, the two terms on the left-hand of the equation must have vanished, thus we obtain a solution represented by an integral equation given by:

$$u(t) = \int_0^\tau f(t')G(t', t)dt' \tag{7}$$

which should replace the earlier differential Eq. (3). This abstract form of solution does not provide any additional insight into our problem. In order to understand this form of integral, one should find an explicit form for$G(t, t_o)$.

Fortunately, the method of variation of parameters provides solutions to Eq. (6), which would indeed give us an indication of how these functions look. The proposed solution of this equation can be written in terms of the solution of the homogeneous Eq. (2) as:

$$G(t,t_o) = \alpha(t,t_o)\,exp(-\gamma t)\cos(\omega t) + \beta(t,t_o)\,exp(-\gamma t)\sin(\omega t) \tag{8}$$

where $\alpha(t, t_o)$ and $\beta(t, t_o)$ are variable coefficients to be determined by the method. Uposubstitution of Eq. (8) and its 1st and 2nd derivatives into our differential Eq. (6) we get the following equations (here for simplicity, we have dropped arguments on both α and β):

$$\frac{d\alpha}{dt}exp(-\gamma t)\cos(\omega t) + \frac{d\beta}{dt}exp(-\gamma t)\sin(\omega t) = 0 \tag{9}$$

$$exp(-\gamma t)\left[\frac{d\alpha}{dt}\{-\gamma\cos(\omega t) - \omega\sin(\omega t)\} + \frac{d\beta}{dt}\{-\gamma\sin(\omega t) + \omega\cos(\omega t)\}\right] = -\delta(t - t_o) \tag{10}$$

Using Cramer's rule (Lay *et al.*, 2015), one can solve this system which is composed of two linear equations, by simply rewriting Eq's. (9) and (10) in matrix form:

$$exp(-\gamma t)\begin{bmatrix} -\gamma\cos(\omega t) - \omega\sin(\omega t) & -\gamma\sin(\omega t) + \omega\cos(\omega t) \\ \cos(\omega t) & \sin(\omega t) \end{bmatrix}\begin{bmatrix} \frac{d\alpha}{dt} \\ \frac{d\beta}{dt} \end{bmatrix} = \begin{bmatrix} -\delta(t - t_o) \\ 0 \end{bmatrix} \tag{11}$$

We can easily solve this equation to find $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$. First, we estimate the non-zero determinant of the 2x2 matrix on the left-hand side, which is also known as the Wronskian, Was follow:

$$W = exp(-2\gamma t)\begin{vmatrix} -\gamma\cos(\omega t) - \omega\sin(\omega t) & -\gamma\sin(\omega t) + \omega\cos(\omega t) \\ \cos(\omega t) & \sin(\omega t) \end{vmatrix} = -\omega\,exp(-2\gamma t) \tag{12}$$

and hence the solutions of Eq. (11):

$$\frac{d\alpha}{dt} = \frac{\begin{vmatrix} -\delta(t - t_o) & exp(-\gamma t)\{-\gamma\sin(\omega t) + \omega\cos(\omega t)\} \\ 0 & exp(-\gamma t)\sin(\omega t) \end{vmatrix}}{W} = \frac{1}{\omega}exp(\gamma t)\sin(\omega t)\delta(t - t_o) \tag{13}$$

$$\frac{d\beta}{dt} = \frac{\begin{vmatrix} exp(-\gamma t)\{-\gamma\cos(\omega t) - \omega\sin(\omega t)\} & -\delta(t - t_o) \\ exp(-\gamma t)\cos(\omega t) & 0 \end{vmatrix}}{W} = -\frac{1}{\omega}exp(\gamma t)\cos(\omega t)\delta(t - t_o) \tag{14}$$

Using the "sifting property" of integration involving delta function, (Flores-Hidalgo and Barone, 2011):

$$F(t) = \int_{-\infty}^{\infty} F(t')\delta(t - t')\,dt' \tag{15}$$

allow us to find the integrands of Eqs (13) and (14):

$$\alpha(t,t_o) = A + \begin{Bmatrix} \frac{1}{\omega} exp(\gamma t_o)\, sin(\omega t_o) & t < t_o \\ 0 & t > t_o \end{Bmatrix} \tag{16}$$

$$\beta(t,t_o) = B - \begin{Bmatrix} \frac{1}{\omega} exp(\gamma t_o)\, cos(\omega t_o) & t < t_o \\ 0 & t > t_o \end{Bmatrix} \tag{17}$$

By substituting $\alpha(t,t_o)$ and $\beta(t,t_o)$ in Eq. (8) we get:

$$G(t,t_o) = [A\,cos(\omega t) + B\,sin(\omega t)]\,exp(-\gamma t) - \begin{Bmatrix} \frac{exp[-\gamma(t-t_o)]}{\omega}(sin[\omega(t-t_o)]) & t > t_o \\ 0 & t < t_o \end{Bmatrix} \tag{18}$$

Applying the boundary conditions $G(0,t_o) = 0$and $G(\tau,t_o) = 0$ which signals the initial and final positions while the applied force still on the action in the interval $(0,\tau)$. These conditions imply that $A = 0$ and $B = exp[\gamma t_o)]\, sin[\omega(\tau - t_o)]\,/\omega\, sin(\omega\tau)$.

Rewriting Eq. (18):

$$G(t,t_o) = \begin{Bmatrix} \frac{exp[-\gamma(t-t_o)]\, sin[\omega(\tau-t)]\, sin(\omega t_o)}{\omega\, sin(\omega\tau)} & t > t_o \\ \frac{exp[-\gamma(t-t_o)]\, sin[\omega(\tau-t_o)]\, sin(\omega t)}{\omega\, sin(\omega\tau)} & t < t_o \end{Bmatrix} \tag{19}$$

Here we solve Eq. (3) once more by the method of variation of parameters, to make a connection with the solution of Green's differential Eq. (6), to make a connection between $u(t)$ and $G(t,t_o)$.

Following same previous steps, we get the parameters $\alpha(t)$ and $\beta(t)$ which are in fact similar to $\alpha(t,t_o)$ and $\beta(t,t_o)$, obtained earlier by Eqs. (13) and (14), which are given by:

$$\alpha(t) = A + \frac{1}{\omega}\int_0^t f(t')\, exp(-\gamma t')\, sin(\omega t')dt' \tag{20}$$

$$\beta(t) = B - \frac{1}{\omega}\int_0^t f(t')\, exp(-\gamma t')\, cos(\omega t')dt' \tag{21}$$

The final solution will have again a similar form of Eq. (8):

$$u(t) = exp(-\gamma t)\{\alpha(t)\, cos(\omega t) + \beta(t)\, sin(\omega t)\} \tag{22}$$

Applying the boundary conditions $u(0) = 0$and $u(\tau) = 0$, we get:

$$u(t) = \frac{1}{\omega\, sin(\omega\tau)} \begin{Bmatrix} \int_t^\tau exp(-\gamma(t-t')\, sin(\omega t)\, sin[\omega(\tau - t')]\, f(t')dt' \\ -\int_0^t exp(-\gamma(t-t')\, sin(\omega t')\, sin[\omega(\tau - t)]\, f(t')dt' \end{Bmatrix} \tag{23}$$

Eq. (23) is nothing but the form:

$$u(t) = \int_0^\tau f(t')G(t',t)dt' \tag{24}$$

which has introduced earlier by Eq. (7). Eq. (24) redefined the use of Green's functions in the abstract form. Thus Eq. (19) definitely defines Green's function$G(t,t_o)$, for the forced damped oscillating system, expressed in its explicit form.

We have generated here a clear form $G(t,t_o)$ for which can help to understand the structural format of Green's function for this specific case. Other explicit forms for Green's function for similar differential equations with boundary conditions; $G(0,t_o) = 0$ and $G(\tau,t_o) = 0$ are given in Table 1.

Table 1

Green's function for a variety of one dimensional Green's differential equations.

Point Source ODE's	Wronskian	Solutions; Green's function
$\frac{d^2G}{dt^2} + 2\gamma\frac{dG}{dt} + \omega_o^2 G = -\delta(t-t_o)$ forced under-damped harmonic oscillator equation $\omega_o^2 > \gamma^2$	$-\omega\, exp(-2\gamma t)$	$\frac{exp[-\gamma(t-t_o)]\, sin(\omega t)\, sin(\omega(\tau-t_o))}{\omega\, sin(\omega\tau)}$ $t < t_o$ $\frac{exp[-\gamma(t-t_o)]\, sin(\omega t_o)\, sin(\omega(\tau-t))}{\omega\, sin(\omega\tau)}$ $t > t_o$
$\frac{d^2G}{dt^2} + 2\gamma\frac{dG}{dt} + \omega_o^2 G = -\delta(t-t_o)$ forced over-damped harmonic oscillator equation $\omega_o^2 < \gamma^2$	$-\omega\, exp(-2\gamma t)$	$\frac{exp[-\gamma(t-t_o)]\, sinh(\omega t)\, sinh(\omega(\tau-t_o))}{\omega\, sinh(\omega\tau)}$ $t < t_o$ $\frac{exp[-\gamma(t-t_o)]\, sinh(\omega t_o)\, sinh(\omega(\tau-t))}{\omega\, sinh(\omega\tau)}$ $t > t_o$
$\frac{d^2G}{dt^2} + 2\gamma\frac{dG}{dt} + \omega_o^2 G = -\delta(t-t_o)$ forced critically-damped harmonic oscillator equation $\omega_o^2 = \gamma^2$	$-exp(-2\gamma t)$	$\frac{t(\tau-t_o)\, exp[-\gamma(t-t_o)]}{\tau}$ $t < t_o$ $\frac{t_o(\tau-t)\, exp[-\gamma(t-t_o)]}{\tau}$ $t > t_o$
$\frac{d^2G}{dt^2} - \omega_o^2 G = -\delta(t-t_o)$ grow and decay equation $\omega_o^2 > 0\,,\ \gamma = 0$	$-\omega$	$\frac{sinh(\omega t)\, sinh(\omega(\tau-t_o))}{\omega\, sinh(\omega\tau)}$ $t < t_o$ $\frac{sinh(\omega t_o)\, sinh(\omega(\tau-t))}{\omega\, sinh(\omega\tau)}$ $t > t_o$
$\frac{d^2G}{dt^2} + \omega_o^2 G = -\delta(t-t_o)$ harmonic oscillator equation, $\omega_o^2 > 0,\ \gamma = 0$	$-\omega$	$\frac{sin(\omega t)\, sin(\omega(\tau-t_o))}{\omega\, sin(\omega\tau)}$ $t < t_o$ $\frac{sin(\omega t_o)\, sin(\omega(\tau-t))}{\omega\, sin(\omega\tau)}$ $t > t_o$

References

Arfken George B., Weber, Hans J., and Harris, Frank E. (2012) Mathematical Methods for Physicists: A Comprehensive Guide, 7th edition, Academic Press, ISBN 978-0-12-384654-9.

Boas, M. L. (2006) Mathematical methods in Physical Sciences, 3rd edition, Chapter 8. John Wiley & Sons.

Edwards C. Henry, Penney David E., and Calvis D. (2014) Differential Equations and Boundary Value Problems: Computing and Modeling, Publisher: Pearson; 5th edition.

Flores-Hidalgo, G. and Barone, F. A. (2011) 'The one-dimensional damped forced harmonic oscillator revisited', *Eur. J. Phys*., 32, pp. 377–379

Lay, David C., Lay, Steven R., McDonald, Judi J. (2015) Linear Algebra and Its Applications (5th Edition) Pearson; 5 edition. ISBN-13: 978-0321982384.

Oliver, Frank W. J., Lozier, Daniel W., Boisvert, Ronald F. and Clark, Charles W. (2010) NIST Handbook of Mathematical Functions by (Editors) page 37-38. Cambridge University Press; ISBN-13: 978-0521140638

Philippe Dennery and Andre Krzywicki (2012) Mathematics for Physicists, Ch. 7, pp. 277-291, Dover Books (2012).

Spanier, J. and Oldham, K. B. (2009) "The Dirac Delta Function", Ch. 10 in *An Atlas of Functions.* Washington, DC: Hemisphere.

Witten, R.C. and P.T. McCormick (1975) 'Elementary introduction to the Green's function', *Am J Phys*, 43, p.