Libyan Journal of Science & Technology 14:2 (2023) 114-126

Faculty of Science - University of Benghazi

Libyan Journal of Science & Technology

journal home page: www.sc.uob.edu.ly/pages/page/77

A Review on the Properties of Jacobi Polynomials.

Haniyah A. Ben Hamdin

Department of Mathematics, Faculty of Science, University of Sirt

E-mail address: h.saed1717@su.edu.ly

Highlights

- **It is desirable to consider the Jacobi polynomials because their properties can be easily reduced to the corresponding properties of some well-known classical polynomials such as Gegenbauer polynomials, Legendre polynomials, first and second kinds Chebyshev polynomials and Zernike polynomials.**
- **It is advantageous to make use of the hypergeometric representations of the Jacobi polynomials to effectively obtain most of their properties.**

ARTICLE INFO

Article history:

Received 29 May 2022
Revised 03 January 2023
Accepted 07 June 2023

Keywords:

Generalized Jacobi Polynomials, Classical Jacobi Polynomials, Gegenbauer Polynomials, Legendre Polynomials, First and second kinds Chebyshev polynomials, Jacobi-Gaussian quadrature, Hypergeometric Function, Orthogonality, zeros.

ABSTRACT

This review is dedicated to Jacobi polynomials which are a generalization of some well-known classical polynomials such as Gegenbauer polynomials, Legendre polynomials, first and second kinds Chebyshev polynomials and Zernike polynomials. To save effort and time, it is advantageous and sufficient to investigate the properties of Jacobi polynomials rather than considering their special cases separately. For demonstration purposes, we show how to reduce most of the obtained properties of Jacobi polynomials to the corresponding properties of Legendre polynomials such as the generating function, Rodrigues formula, special values and the orthogonality property. Most of the properties of Jacobi polynomials are obtained through their hypergeometric representations such as differential recurrence relations, generating functions, some special values (exact and asymptotic) and some integral expansions of positive integrand. The standard orthogonality property of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for the values of the parameters $\alpha, \beta > -1$ is discussed and some applications of such important property are pointed out. The orthogonality property of Jacobi polynomials considerably affects the positions of their zeros. These zeros are essential in any type of numerical quadrature such as Legendre-Gaussian quadrature, first kind Chebyshev-Gaussian quadrature and second kind Chebyshev-Gaussian quadrature. Moreover the recurrence relations of Jacobi polynomials play an important role in computing the zeros of Jacobi polynomials which are needed in the Gaussian-Jacobi quadrature. A narrative review is provided on some attempts were done in extending the orthogonality property of Jacobi polynomials to non-standard values of the indexes $\alpha, \beta \in \mathbb{C}$ by amending some of the orthogonality conditions such as Sobolev and non-hermitian orthogonality. Integral expansions of Jacobi polynomials with a prominent feature (positive integrand) were presented in terms of other Jacobi polynomials. Such integral expansions should allow a variety of applications for Jacobi polynomials.

1. Introduction

The orthogonal polynomials (Szegö, 1975; Sansone, 1991; Gautsch, 2004; Vilmos, 2005; Koornwinder *et al.*, 2010) are set of polynomials that are mutually orthogonal to each other with respect to a measure of weighting function under certain inner product. In fact the orthogonal polynomials are the Eigen-functions of a symmetric second-order differential operator, thus such polynomials are widely used in the theory of moments, continued fractions (Mirevski *et al.*, 2007) and spectral theory (Brychkov, 2008). A substantial historical review on the evolution of the orthogonal polynomials can be found in (Chihara *et al.*, 2001) and references therein. The topic of orthogonal polynomials has become very rich area of research with the emergence of implementing such polynomials in numerical computations as bases functions (Gautschi, 2004) and in numerical quadrature (Abramowitz and Stegun, 1968). Further applications of orthogonal functions establish a connection between the theory of representation of Lie groups and the theory of special functions. Strictly speaking there is a term known as algebraic special functions, a good example of this type is the Hermite functions which form a basis on the Hilbert space of square-integrable functions on the whole space $\mathbb{R}$ (Uspensky, 1927). However, this scheme can be generalized to other orthogonal polynomials such as associated Legendre polynomials, Lagurre polynomials, Jacobi polynomials (JPs, henceforth) and spherical harmonics functions. The classical JPs denoted as $P_n^{(\alpha,\beta)}(x)$ of degree n and order $\alpha, \beta \in \mathbb{C}$ were first introduced by Jacobi in 1859. For specific values of the indexes α and β most of the common classical orthogonal polynomials (Beals and Wong, 2010) are in fact just special cases of the JPs such as the classical Legendre polynomials, Chebyshev polynomials of the first and second types and Gegenbauer (Ultra-spherical) polynomials as shown later through this review. Therefore, it is sufficient to dedicate any effort on studying the properties of the JPs rather than individually considering their special cases. Spectral approximation (Canuto *et al.*, 2006) has been proven very efficient tool in solving higher-order differential equations that are encountered in optics theory, stuctural mechanics, astronomy and geophysics (Guo, Shen and Wang, 2009). The Legendre poynomials which are special class of the JPs have many applications, for instance Janecki (Janecki and Stephen, 2005) used such polynomials for the approximation of cylinderical

 ISSN: Online 2663-1407, Print 2663-1393; National Library of Libya, Legal number: 390/2018

surfaces. Gue (Guo, Shen and Wang, 2009) generated a class of JPs for arbitrary real vaues of the indexes α, β, they implemented such generalized JPs to solve partial differential equations with number of homogenouse boundary conditions that corresponds to the indexes values. They claim that implementing such generalized JPs leads to straightforward, stable and well-conditioned numerical procedures and considerably simplifies the error analysis. Gue (Gue, 2000) generated class of JPs in Hilbert space then showed demonstration for solving some singular differential equations in infinite regions. Gue (Guo, Shen and Wang, 2006) introduced some aplications of the generalized JPs. Gue (Guo, Shen and Wang, 2004) considered Jacobi approximations and Jacobi–Gaussian interpolations in certain spaces. Moreover, the authors (Gue, 2000) and (Chihara *et al.*, 2001) generated a class of JPs by considereing the specific values of the indexes $(\alpha, \beta) = (-1,0), (-1,-1)$ on the whole and half lines respectively. Instead of individually condidering special cases of the pair indexes α, β separtely, it would be more beneficial to generate JPs for general values of the indexes to preserve time and effort. The generalized JPs inhirit some properties from the classical JPs which are essential for spectral approximation. The condition on the indexes values $\alpha, \beta > -1$ is imposed to guarantee that the JPs are mutuay orthogonal on the interval $[-1,1]$, because non-orthogonal bases functions are not suitabe for spectral approximation. Moreover, it is worth to emphasize that the orthogonality property of the JPs considerably affect their zeros distribution.

Essentially the zeros of orthogonal poynomials (Driver and Love, 2001; Driver and Möller, 2001, 2002) are of great importance to proceed any Gaussian quadreture. Moreover, the zeros of orthogonal poynomials has a physical interpretation because they are just the stationary points of the potential. Classically for $\alpha, \beta > -1$ the zeros of JPs are all positioned in the open interval $(-1,1)$. But for general values of the indexes $\alpha, \beta \in \mathbb{C}$, the zeros of JPs are no longer in the same interval, but are distributed into the complex pane in a well- organized matter (Duren and Boggs, 2001).

With the remarkable growth of the interest in implementing the JPs in numerical analysis, there has been a considerable interest in expanding the orthogonality property to include more indexes values as the orthogonality property guarantees simple, stable, accurate error estimation and well-conditioned numerical procedures. Recently there have been considerable concerns in what so-called Sobolev orthogonality (Alvarez, Perez and Pinar, 1998) for the classical polynomials such as the Gegenbauer polynomials and JPs for the non-classical values of the indexes α, β. Finkelshtein (Finkelshtein, 1999) studied the zeros distribution of the JPs of non-hermittian orthogonality which is defined by a compex-valued function on a certain paths in the complex z-plane. Alfaroa (Alfaroa, Alvarez and Rezola, 2002) obtained the orthogonality property of the JPs with negative integer values of the indexes α, β. Kuijlaars (Kuijlaars, Finkelshtein and Orive, 2005) imposed some orthogonality conditions for the JPs on certain paths in the complex z-plane for more values of the indexes $\alpha, \beta \in \mathbb{C}$.

This substantial narrative review is dedicated to shed some light on most important properties of the JPs such as their hypergeometric representations, generating function in terms of the hypergeometric function, Rodrigues formula and other properties. Furthermore, we summarize some significant integral expansions of positive kernels for JPs, and point out some remarkable applications of such polynomials.

This paper is structured as follows: The relevant literature review is presented in section 1. In section 2 we briefly set up some concepts and introduce important theorems that we need to use throughout this review. These concepts consist of a brief introduction of some needed formulae on double series manipulations, Pochhammer symbol, gamma function, beta function and the hypergeometric function. In sections 3 and 4 we will introduce the JPs and their hypergeometric representations. Some differential recurrence relations were derived in section 5. Sections 6 and 7 were dedicated to the generating functions and the Rodrigues formula of JPs. Some special values are obtained in section 8. In section 9 we-show how to obtain the Chebyshev polynomials of both kinds as a special case of JPs. Section 10 is devoted to the most important property of JPs followed by some applications of JPs. In section 11 we summarize some integral expansions of JPs and point out some applications. Finally, a conclusion is drawn in section 12.

2. Preliminaries.

Some necessary concepts that are needed throughout this review are presented here.

2.1. Double Series Manipulations.

Some double-series manipulations are presented here to deal with double series appearing later in this review.

Theorem 1 (Rainville, 1960)**:** For a convergent power series, one has

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\varphi(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\varphi(k,n-k), \tag{1}$$

which can be rewritten in a reverse order as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\varphi(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\varphi(k,n+k), \tag{2}$$

where $\varphi(k,n)$ is the general term of the double series.

2.2. Pochhammer Symbol, Gamma and Beta Functions

It is necessary here to introduce the Pochhammer symbol $(\alpha)_n$ as,

$$(\alpha)_n = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{k=1}^{n}(\alpha + k - 1), & \text{if } n = 1,2,3,\dots \end{cases} \tag{3}$$

Definition 1 (Arfken, 1985)**:** The gamma function $\Gamma(\alpha)$ is given by the following Euler integral,

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt\,, \qquad \alpha > 0. \tag{4}$$

Theorem 2 (Abramowitz and Stegun, 1968): The gamma function is related to the Pochhammer symbol by the this identity,

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \alpha \neq 0, \pm 1, \pm 2, \dots, n = 0,1,2,\dots \tag{5}$$

Theorem 3 (Arfken, 1985): For positive and large values of , we have the following approximation value of gamma function as,

$$\Gamma(n+1) \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n, \qquad n \gg 1, \tag{6}$$

which is known as Stirling's formula for gamma function.

Definition 2 (Abramowitz and Stegun, 1968): For a non-negative real numbers α, β the beta function $B(\alpha, \beta)$ is defined by the following Euler integral of first kind as,

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \quad \alpha,\beta > 0. \tag{7}$$

Theorem 4 (Abramowitz and Stegun, 1968): For a non-negative real numbers α, β the beta function $B(\alpha, \beta)$ is related to the gamma function by the following relation,

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{8}$$

Next, we introduce some convenient identities that we will be used throughout this review.

Theorem 5 (Rainville, 1960): If n is a non-negative integer and a is any real number with $n \leq a$, then,

$$\binom{a}{n} = \frac{(-1)^n(-a)_n}{n!}, a \in \mathbb{R}, \qquad n \leq a, n = 0,1,2 \tag{9}$$

Remark: The relation (11) can be rewritten in the following form,

$$(-a)_n = \frac{(-1)^n a!}{(a-n)!}, a \in \mathbb{R}, \qquad n \le a, n = 0,1,2 \tag{10}$$

Lemma 1: Let n be a positive integer and a any real number,

$$(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n \tag{11}$$

Proof: since,

$$(a)_{2n} = a(a+1)(a+2)(a+3)\dots(a+2n-1),$$
$$= 2^{2n}\left[\frac{a}{2}\left(\frac{a}{2}+\frac{1}{2}\right)\left(\frac{a}{2}+1\right)\left(\frac{a}{2}+\frac{3}{2}\right)\dots\left(\frac{a}{2}+n-\frac{1}{2}\right)\left(\frac{a}{2}+n-1\right)\right],$$
$$= 2^{2n}\frac{a}{2}\left(\frac{a}{2}+1\right)\dots\left(\frac{a}{2}+n-1\right)\left(\frac{a+1}{2}\right)\left(\frac{a+3}{2}\right)\dots\left(\frac{a}{2}+n-\frac{1}{2}\right).$$

Thus,

$$(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n$$

In a similar fashion to lemma 1, we can prove the following beneficial formulae,

$$(a)_{kn} = k^{nk}\left(\frac{a}{k}\right)_n\left(\frac{a+1}{k}\right)_n\dots\left(\frac{a+k-1}{k}\right)_n, \quad k = 1,2,3,\dots; n = 0,1,2,\dots \tag{12}$$

$$\frac{(-n)_k}{n!} = \frac{(-1)^k}{(n-k)!}, \qquad \forall\, k,n \in \mathbb{Z}, 0 \le k \le n. \tag{13}$$

$$(a)_{n+m} = (a)_m(a+m)_n, \quad \forall\, m,n \in \mathbb{N}. \tag{14}$$

$$(a)_{n-k} = \frac{(-1)^k(a)_n}{(1-a-n)_k}, 0 \le k \le n. \tag{15}$$

$$(a)_n = (-1)^n(1-a-n)_n\,. \tag{16}$$

$$2^{2n}\left(\frac{1}{2}\right)_n = \frac{(2n)!}{n!}. \tag{17}$$

$$\left(\frac{3}{2}\right)_n = \frac{(2n+1)!}{2^n n!}. \tag{18}$$

It is necessary to introduce a very important function that we need to derive most of the properties of the Jacobi polynomials as shown in the next section.

4. The Hypergeometric Function

Consider the series

$$1 + \sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)\ \ n!}z^n \tag{19}$$

where z is a complex variable, α or β and γ are parameters, which can take arbitrary real or complex values provided that $\gamma \ne 0,-1,-2,$. If we let $\alpha = 1$ and $\beta = \gamma$, then the series (19) is reduced to the elementary geometric series $\sum_{n=0}^{\infty} z^n$.

In terms of the Pochhammer symbol (3) we can simplify the hypergeometric series (19) in the following form,

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n\,.$$

We shall denote the convergent hypergeometric series (19) by the notation $F(\alpha,\beta;\gamma;z)$ that is,

$$F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n, \quad |z| < 1, \gamma \ne 0,-1,-2, \tag{20}$$

Lemma 4: The derivative of the hypergeometric series (12) is defined as,

$$\frac{d}{dz}F(\alpha,\beta;\gamma;z) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1;\gamma+1;z). \tag{21}$$

Proof: Since,

$$\frac{d}{dz}F(\alpha,\beta;\gamma;z) = \frac{d}{dz}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n,$$
$$= \sum_{n=1}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n(n-1)!}z^{n-1},$$
$$= \frac{\alpha\beta}{\gamma}\sum_{n=1}^{\infty}\frac{(\alpha+1)_{n-1}(\beta+1)_{n-1}}{(\gamma+1)_{n-1}(n-1)!}z^{n-1},$$

Finally replace n by $n+1$ to end the proof.

Next, we show how to represent a function in terms of the hypergeometric function

Example 2: Rewrite the function $f(z) = (1-z)^a$ in terms of the hypergeometric function.

We can rewrite the function $f(z)$ in terms of the hypergeometric function as follows,

$$(1-z)^{-a} = 1 + az + \dots + a(a+1)(a+n-1)\frac{z^n}{n!} + \dots$$

Thus,

$$(1-z)^{-a} = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{n!} = \mathrm{F}(a,b;b;z), \quad |x| < 1. \tag{22}$$

Next, we introduce very important integral representations of the hypergeometric function which are essential for deriving some important integral formulae of the JPs.

5. Integral Representations of the Hypergeometric Function.

The Euler integral representation of the hypergeometric function is defined as shown in the following theorem.

Theorem 6 (Rainville, 1960)**:** If $|x| < 1$, and $Re(\gamma) > Re(\beta) > 0$ then,

$$F(\alpha,\beta;\gamma;x) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}dt \tag{23}$$

which is known as the Euler integral representation of the hypergeometric function

Proof: Since $|tx| < 1$, thus according to the relation (22) we have,

$$(1-tx)^{-a} = \sum_{n=0}^{\infty}\frac{(a)_n(tx)^n}{n!}, \qquad |tx| < 1.$$

Substitute this uniformly convergent series in the right hand side of equation (23) allows us to interchange the order of summation and integration, thus

$$\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(a)_n x^n}{n!}\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt\,.$$

Now recall the definition of the beta function (7) with the assumption that $Re(\gamma) > Re(\beta) > 0$ to guarantee the Euler integral is convergent at the end-points. So, one has

$$\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(a)_n x^n}{n!}B(\beta+n,\gamma-\beta).$$

Here recall the relation between the gamma and beta function (8) to obtain,

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(a)_n x^n}{n!}\frac{\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\gamma+n)}.$$

Finally, we need to recall the relation between gamma function and Pochhammer symbol (5) to obtain the function $F(\alpha,\beta;\gamma;x)$ as required.

Furthermore, we shall introduce the following theorems that we need later in this article in the derivations of the integral formulae of the JPs.

From the main integral formula (23) we can derive various linear transformations relating the hypergeometric functions as shown in the theorems below.

Theorem 7 (Abramowitz and Stegun, 1968)**:** For $\left|\frac{x}{x-1}\right| < 1$, we have

$$F(\alpha,\beta;\gamma;x) = (1-x)^{-\alpha}F\left(\alpha,\gamma-\beta;\gamma;\frac{x}{x-1}\right), \left|\frac{x}{x-1}\right| < 1 \tag{24}$$

Proof: Making the variable change $t = 1-w$ in the formula (23), then do some calculations to obtain the desired transformation.

Theorem 8 (Abramowitz and Stegun, 1968): For $|x| < 1$, we have

$$F(\alpha,\beta;\gamma;x) = (1-x)^{\gamma-\alpha-\beta} \times F(\gamma-\alpha,\gamma-\beta;\gamma;x). \quad (25)$$

Proof: Interchanging α and β in the formula (24) and use the symmetry property of the hypergeometric function, we obtain the following transformation,

$$F(\alpha,\beta;\gamma;x) = (1-x)^{-\beta} F\left(\gamma-\alpha,\beta;\gamma;\frac{x}{x-1}\right), \left|\frac{x}{x-1}\right| < 1 \quad (26)$$

Now applying the transformation (24) for the right hand side of equation (26) to obtain this important transformation,

$$F(\alpha,\beta;\gamma;x) = (1-x)^{\gamma-\alpha-\beta} \times F(\gamma-\alpha,\gamma-\beta;\gamma;x) \quad (27)$$

Theorem 9 (Askey and Fitch, 1969): If $|x| < 1$, and $\rho > 0, \gamma > 0$ then,

$$F(\alpha,\beta;\gamma+\rho;x) = \frac{1}{B(\gamma,\rho)} \int_0^1 t^{\gamma-1} (1-t)^{\rho-1} F(\alpha,\beta;\gamma;xt)dt \quad (28)$$

Proof: The proof is similar to the proof of theorem 6.

Theorem 10 (Askey and Fitch, 1969): For $|x| < 1$, and $\beta > \rho > 0$ then,

$$F(\alpha,\beta-\rho;\gamma;x) = \frac{1}{B(\rho,\beta-\rho)} \int_0^1 t^{\beta-\rho-1} (1-t)^{\rho-1} F(\alpha,\beta;\gamma;xt)dt \quad (29)$$

Proof: The proof is similar to the proof of theorem 6.

Theorem 11 (Askey and Fitch, 1969): For $\rho > 0, \gamma > 0$, and $0 < x < 1$ then,

$$x^{\gamma+\rho-1}(1-x)^{\alpha-\gamma} F(\alpha,\beta+\rho;\gamma+\rho;x) = \frac{1}{B(\gamma,\rho)} \int_0^x t^{\gamma-1} (1-t)^{\alpha-\gamma-\rho} (x-t)^{\rho-1} F(\alpha,\beta;\gamma;t)dt \quad (30)$$

Proof: The relation can be proved easily by plugging equation (24) into equation (28), so,

$$(1-x)^{-\alpha} F\left(\alpha,\gamma+\rho-\beta;\gamma+\rho;\frac{x}{x-1}\right) = \frac{1}{B(\gamma,\rho)} \int_0^1 t^{\gamma-1} (1-t)^{\rho-1} (1-xt)^{-\alpha} \times F\left(\alpha,\gamma-\beta;\gamma;\frac{xt}{xt-1}\right) dt \quad (31)$$

Now do the following variable transformations,

$$w = \frac{x}{x-1}, \qquad u = \frac{xt}{xt-1},$$

Finally, replace $\gamma-\beta$ by β to arrive at the desired formula (31). After introducing all the needed concepts, now we shall introduce the Jacobi polynomials.

5. The Jacobi differential equation

The Jacobi polynomials are the eigen-functions of a singular Sturm-liouville operator given as,

$$L_n^{\alpha,\beta} y(x) = -(1-x)^{-\alpha}(1+x)^{-\beta} \times \frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1} \frac{d}{dx} y(x)\right], \quad (32)$$
$$= (x^2-1)y'' + [\alpha-\beta+(\alpha+\beta+2)x]y'$$

which correspond to the eigenvalues,

$$\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1).$$

That is, we have the following eigen-value problem,

$$L_n^{\alpha,\beta} y(x) = \lambda_n^{\alpha,\beta} y(x).$$

Thus the Jacobi differential equation takes the form,

$$(1-x^2)y'' + [\beta-\alpha-(\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0.$$

This equation can be reduced to the Legendre differential equation f by setting the indexes values α, β in equation (32) to zeros to obtain

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

Solving the second-order, linear and homogenous Jacobi differential equation by the Frobenius method (Arfken, 1985), one obtains the Jacobi polynomials denoted as $P_n^{(\alpha,\beta)}(x)$ of degree n and order (α,β) as,

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+k+1)(-1)^k}{(n-k)!\,k!\,\Gamma(n+\alpha+\beta+1)\Gamma(k+\alpha+1)} \left(\frac{1-x}{2}\right)^k \quad (33)$$

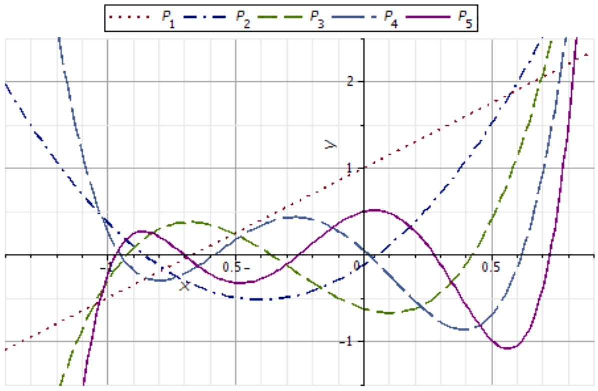

Fig. 1. First few Jacobi polynomials for $\alpha = 1.5, \beta = -0.5, n = 1..5$

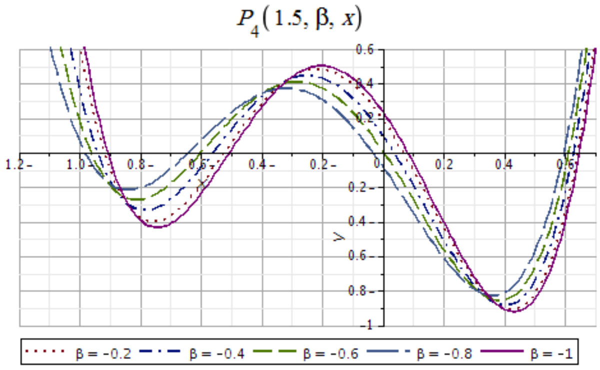

Fig. 2. Jacobi polynomials for different values of β.

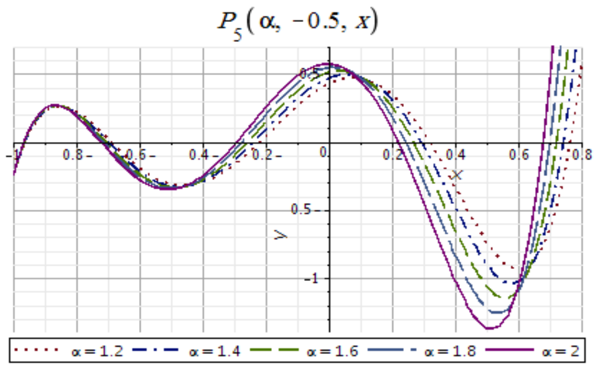

Fig. 3. Jacobi polynomials for different values of α.

Fig. 1 shows the first five JPs for fixed values of the indexes α and β. Fig. 2 show $P_4(1.5,\beta,x)$ for fixed value of the parameter α and changing value of the other parameter β, whereas Fig. 3 shows $P_4(\alpha,-0.5,x)$ for fixed value of the parameter β and changing value of the other parameter α.

The Jacobi polynomial series (33) can be rewritten in more elegant and professional forms as it will be shown in the next section

4. Hypergeometric Representations of the Jacobi Polynomials.

Here we shall introduce some hypergemetric representations of Jacobi polynomials (Rainville, 1960). To achieve that we use the relation (5) for the terms containing gamma function and the identity (13) in equation (33) to obtain,

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\sum_{k=0}^{n}\frac{(n+\alpha+\beta+1)_k(-n)_k}{k!\,(1+\alpha)_k}\left(\frac{1-x}{2}\right)^k \tag{34}$$

Rewriting this equation using the notation of the hypergeometric function (20), yields

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\times F\left(-n, n+\alpha+\beta+1; 1+\alpha; \frac{1-x}{2}\right). \tag{35}$$

This is one of the versatile hypergemetric representations of JPs which we will be of multiple use later. For example but not limited to, this hypergeometric representation of the Jacobi polynomials will be beneficial in obtaining some integral expansions of Jacobi polynomials as it will be shown in section 11. Furthermore, if we recall the identity (14) for the term $(n+\alpha+\beta+1)_k$ in equation (34), then we have

$$(n+\alpha+\beta+1)_k = \frac{(\alpha+\beta+1)_{n+k}}{(\alpha+\beta+1)_n}.$$

Plugging this identity in equation (34) leads to the following formula of JPs

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\frac{(1+\alpha)_n(1+\alpha+\beta)_{n+k}}{k!\,(n-k)!\,(1+\alpha)_k(1+\alpha+\beta)_n}\left(\frac{x-1}{2}\right)^k \tag{36}$$

Now use the relation between the Pochhammer symbol (3) and gamma function (5) in the last equation to obtain,

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)}\times \sum_{k=0}^{n}\frac{\Gamma(n+\alpha+\beta+k+1)(-1)^k}{(n-k)!\,k!\,\Gamma(k+\alpha+1)}\left(\frac{1-x}{2}\right)^k \tag{37}$$

which can be arranged by using the identity (9)as,

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(n+\alpha+\beta+1)} \times\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(k+\alpha+1)}\left(\frac{x-1}{2}\right)^k \tag{38}$$

Also equation (37) can be rearranged by using the relations (5) and (10) to obtain,

$$\frac{(-1)^k}{(n-k)!} = \frac{(-n)_k}{n!}.$$

Thus, one has

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(1+\alpha)_k}\left(\frac{1-x}{2}\right)^k \tag{39}$$

Now using the notation of the hypergeometric function (20), yields

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}F\left(-n, n+\alpha+\beta+1; 1+\alpha; \frac{1-x}{2}\right).$$

By exploiting the integral linear transformation (24) for the hypergeometric function in equation (39) we obtain,

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\left(\frac{1+x}{2}\right)^n \times F\left(-n, -n-\beta; 1+\alpha; \frac{x-1}{x+1}\right) \tag{40}$$

Now replace x by $-x$ and interchange α and β in the formula for the Jacobi polynomials (39), then employ the symmetry property of JPs (58) to obtain,

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(1+\beta)_n}{n!} \times F\left(-n, n+\alpha+\beta+1; 1+\beta; \frac{1+x}{2}\right) \tag{41}$$

Using the relations (13) and (14) in the formula for the Jacobi polynomials (41) leads to,

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\frac{(-1)^{n-k}(1+\beta)_n(\alpha+\beta+1)_{n+k}}{k!\,(n-k)!\,(1+\beta)_k(\alpha+\beta+1)_n}\left(\frac{1+x}{2}\right)^k \tag{42}$$

From the previous formula for the Jacobi Polynomials (40), one can obtain another representation of the JPs as

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\left(\frac{1+x}{2}\right)^n\sum_{k=0}^{n}\frac{(-n)_k(-\beta-n)_k}{k!\,(1+\alpha)_k}\left(\frac{x-1}{x+1}\right)^k \tag{43}$$

Now using the relation (11) for the factors $(-n)_k$ and $(-\beta-n)_k$ to obtain,

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\frac{(1+\alpha)_n(1+\beta)_n}{k!\,(n-k)!\,(1+\alpha)_k(1+\beta)_{n-k}}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k} \tag{44}$$

Remark: It should be noted that all the hypergeometric relations of JPs can be easily reduced to the hypergeometric relations of Legendre and Chebyshev polynomials. To prevent repetition and due to page limitation we do not show the straightforward reduction process here.

5. Differential Recurrence Relations for the Jacobi Polynomials.

The recurrence relations for the Jacobi polynomials have great importance in any numerical computations involving JPs, because they are used to compute the zeros of JPs and the associated weights used in the numerical quadrature.

Taking the derivative of the JPs in its hypergeometric form given by equation (39) with respect to the variable x. Thus with the aid of the identity (21) to do the derivative of the hypergeometric function we obtain,

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{n(1+\alpha+\beta+n)(1+\alpha)_n}{2\,n!\,(1+\alpha)}\times F\left(1-n, n+\alpha+\beta+2; 2+\alpha; \frac{1-x}{2}\right).$$

This equation can rearranged as,

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta+n)(2+\alpha)_{n-1}}{2(n-1)!}\times F\left(-(n-1), (n-1)+(\alpha+1)+(\beta+1)+1; 1+(\alpha+1); \frac{1-x}{2}\right).$$

That is,

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(1+\alpha+\beta+n)P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{45}$$

This relation can be generalized for $0 < k \le n$ as,

$$\left(\frac{d}{dx}\right)^k P_n^{(\alpha,\beta)}(x) = 2^{-k}(1+\alpha+\beta+n)_k P_{n-k}^{(\alpha+k,\beta+k)}(x) \tag{46}$$

By following similar manner we can derive some more recurrence relations for the JPs (Rainville, 1960). For instance, Taking the derivative of the JPs in its hypergeometric form given by equation (40) with respect to the variable x . Thus with the aid of the identity (21) to do the derivative of the hypergeometric function we obtain,

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \left(\frac{n}{x+1}\right)P_n^{(\alpha,\beta)}(x) + \frac{(n+\beta)(2+\alpha)_{n-1}}{(n-1)!\,(x+1)}\left(\frac{1+x}{2}\right)^{n-1}\times F\left(1-n, 1-n-\beta; 1+(\alpha+1); \frac{x-1}{x+1}\right),$$

which can be written as the following,

$$(x+1)\left[P_n^{(\alpha,\beta)}(x)\right]' = nP_n^{(\alpha,\beta)}(x) + (n+\beta)P_{n-1}^{(\alpha+1,\beta)}(x). \tag{47}$$

Further to these recurrence differential relations, there are other relations that can be derived in a similar fashion to the derivations above (Rainville, 1960).

6. Generating Function for the Jacobi Polynomials.

The generating function for the Jacobi polynomials can be derived from the formula (36) of the JPs (Rainville, 1960). We start by considering the following series,

$$\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n}h^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(1+\alpha+\beta)_{n+k}}{k!\,(n-k)!\,(1+\alpha)_k}\left(\frac{x-1}{2}\right)^k h^n, |h| < 1.$$

The double series on the right hand side can be arranged using the identity (2) to obtain,

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1+\alpha+\beta)_{n+2k}}{k!\,n!\,(1+\alpha)_k 2^k}(x-1)^k h^{n+k}.$$

Now use the identity (14) for the factor $(1+\alpha+\beta)_{n+2k}$ to obtain

$$(1+\alpha+\beta)_{n+2k}=(1+\alpha+\beta+2k)_n(1+\alpha+\beta)_{2k}.$$

Hence,

$$\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(x)h^n}{(1+\alpha)_n} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta+2k)_n h^n}{n!}\frac{(1+\alpha+\beta)_{2k}(x-1)^k h^k}{k!\,(1+\alpha)_k 2^k}.$$

Employing the identity (22) in the last equation leads to,

$$\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n}h^n = \sum_{k=0}^{\infty}(1-h)^{-(1+\alpha+\beta+2k)}\frac{(1+\alpha+\beta)_{2k}(x-1)^k h^k}{k!\,(1+\alpha)_k 2^k}.$$

Finally, recall the identity (11) to obtain the generating function for the Jacobi polynomials as,

$$\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n}h^n = (1-h)^{-(1+\alpha+\beta)}\times F\left(\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2};1+\alpha;\frac{2h(x-1)}{(1-h)^2}\right) \quad (48)$$

Moreover, there is the Bateman's generating function of the Jacobi polynomials (Bateman, 1950) which can be obtained directly from the formula (44) of Jacobi polynomials as we show here (Rainville, 1960).

By using the formula (44) for the Jacobi polynomials we consider the following series as,

$$\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x)h^n}{(1+\alpha)_n\,(1+\beta)_n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}h^n}{k!\,(n-k)!\,(1+\alpha)_k\,(1+\beta)_{n-k}}.$$

The double series on the right hand side can be arranged using the identity (2) to obtain,

$$\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x)h^n}{(1+\alpha)_n\,(1+\beta)_n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n}h^{n+k}}{k!\,n!\,(1+\alpha)_k\,(1+\beta)_n}.$$

Again the double series on the right hand side can be rearranged as,

$$\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x)h^n}{(1+\alpha)_n\,(1+\beta)_n} = \left(\sum_{k=0}^{\infty}\frac{\left(\frac{x-1}{2}\right)^k h^k}{k!\,(1+\alpha)_k}\right)\left(\sum_{n=0}^{\infty}\frac{\left(\frac{x+1}{2}\right)^n h^n}{n!\,\,(1+\beta)_n}\right).$$

Now exploit the notation of the hypergeometric function (20) to obtain the Bateman's generating function of the Jacobi polynomials as,

$$\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x)h^n}{(1+\alpha)_n\,(1+\beta)_n} = F\left(-,-;1+\alpha;\frac{h(x-1)}{2}\right) + F\left(-,-;1+\beta;\frac{h(x+1)}{2}\right), \qquad |h|<1 \quad (49)$$

Remarks: 1- It should be noted that by replacing x by $-x$ and t by $-t$ into the generating function (49) for the Jacobi polynomials we obtain the symmetry property of JPs (58).
2- The generating function for the JPs (49) can be reduced to the generating function for the Legendre polynomials $P_n(x)$ by setting the indexes values α,β in equation (49) to zeros, thus

$$\sum_{n=0}^{\infty}\frac{P_n(x)h^n}{(n!)^2} = F\left(-,-;1;\frac{h(x-1)}{2}\right) + F\left(-,-;1;\frac{h(x+1)}{2}\right) \quad (50)$$

7. Rodrigues formula for the Jacobi Polynomials.

Theorem 12 (Bell, 1968): The Jacobi polynomials of degree n and order α,β for $x\in(-1,1)$ are defined by the following formula of Rodrigues type as,

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\times\left(\frac{d}{dx}\right)^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \quad (51)$$

Proof: This formula can be derived easily with the aid of the Leibniz's theorem for the n^{th} derivative of the product of two functions that appear in the right hand side of the equation (51), so

$$\left(\frac{d}{dx}\right)^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{d}{dx}\right)^k(1+x)^{n+\beta}\left(\frac{d}{dx}\right)^{n-k}(1-x)^{n+\alpha}.$$

Now by taking the actual derivatives in the last expression, one has

$$\left(\frac{d}{dx}\right)^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] = \sum_{k=0}^{n}\binom{n}{k}(n+\beta)(n+\beta-1)\dots(n+\beta-k+1)(1+x)^{n+\beta-k}\times (-1)^{n-k}(n+\alpha)(n+\alpha-1)\dots(n+\alpha-n+k+1)(1-x)^{n+\alpha-n+k}.$$

To simplify the coefficients in this expansion we use the relation (5) to obtain,

$$(n+\alpha)(n+\alpha-1)\dots(\alpha+k+1)=\frac{\Gamma(\alpha+\mathrm{n}+1)/\Gamma(\alpha+1)}{\Gamma(\alpha+\mathrm{k}+1)/\Gamma(\alpha+1)} = \frac{(\alpha+1)_n}{(\alpha+1)_k},\quad \alpha>-1, k\le n. \quad (52)$$

And similarly, we have,

$$(n+\beta)(n+\beta-1)\dots(n+\beta-k+1) = \frac{\Gamma(\beta+\mathrm{n}+1)/\Gamma(\beta+1)}{\Gamma(\beta+n-\mathrm{k}+1)/\Gamma(\beta+1)} = \frac{(\beta+1)_n}{(\beta+1)_{n-k}},\quad \beta>-1, k\le n. \quad (53)$$

Now substituting the relations (52) and (53) into equation (51) to obtain

$$\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\times\left(\frac{d}{dx}\right)^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] \quad (54)$$

But the right hand side of equation (54) is just JPs $P_n^{(\alpha,\beta)}(x)$ according to equation (44). Thus, we end the proof of the Rodrigues formula of JPs (51).

Remarks:

1- The Rodrigues formula of JPs (51) is very beneficial because it shows that the JPs $P_n^{(\alpha,\beta)}(x)$ are analytic functions of their parameters $\alpha, \beta \in \mathbb{C}$.

2- The Rodrigues formula of JPs (51) can be reduced to the Rodrigues formula of Legendre polynomials by setting the indexes values α, β in equation (51) to zeros, thus

$$P_n(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n [(1-x)^n(1+x)^n] \quad (55)$$

3- The Rodrigues formula of JPs (51) can be reduced to the Rodrigues formula of the first and second kind Chebyshev polynomials by setting the indexes values respectively to $\alpha = \beta = -\frac{1}{2}$, and $\alpha = \beta = \frac{1}{2}$ in equation (51), thus

$$T_n(x) = \frac{(-1)^n}{2^n n!}\sqrt{1-x^2}\left(\frac{d}{dx}\right)^n (1-x^2)^{n-\frac{1}{2}} \quad (56)$$

and,

$$U_n(x) = \frac{(-1)^n(n+1)\sqrt{\pi}}{2^{n+1}(n+\frac{1}{2})!}(1-x^2)^{-\frac{1}{2}} \times \left(\frac{d}{dx}\right)^n (1-x^2)^{n+1/2} \quad (57)$$

Theorem 13 (Bell, 1968) The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ obey the following symmetry relation,

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x) \quad (58)$$

Proof:- Replace x by $-x$ in the Rodrigues formula for the Jacobi polynomials (51) to obtain,

$$P_n^{(\alpha,\beta)}(-x) = \frac{(-1)^n}{2^n n!}(1+x)^{-\alpha}(1-x)^{-\beta} \times \left[\frac{d}{d(-x)}\right]^n [(1+x)^{n+\alpha}(1-x)^{n+\beta}].$$

$$P_n^{(\alpha,\beta)}(-x) = (-1)^{2n}\frac{(-1)^n}{2^n n!}(1-x)^{-\beta}(1+x)^{-\alpha} \times \left(\frac{d}{dx}\right)^n [(1-x)^{n+\beta}(1+x)^{n+\alpha}].$$

Again calling the Rodrigues formula of JPs (51) leads to,

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x).$$

This symmetry property of the Jacobi polynomials will be beneficial in obtaining some integral representations of Jacobi polynomials as it will be shown in section 11.

8. Exact and Asymptotic Special Values of Jacobi Polynomials.

For specific values of the indexes α and β most of the common classical orthogonal polynomials are in fact just special cases of the JPs. For instance, the classical Legendre polynomials correspond to the special case $\alpha = \beta = 0$. Furthermore, the Chebyshev polynomials of the first and second types correspond respectively to the special cases $\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$ and Gegenbauer (ultraspherical) polynomials correspond to the special case $\alpha = \beta$. Here we show how to evaluate the JPs at the two end points $x = \pm 1$.

$$P_0^{(\alpha,\beta)}(x) = 1,$$

$$P_1^{(\alpha,\beta)}(x) = (1+\alpha) + (\alpha+\beta+2)\left(\frac{x-1}{2}\right),$$

By substituting $x = 1$ in the formula of Jacobi polynomials given by equation (39), one has

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!} = \binom{\alpha+n}{n} = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)}. \quad (59)$$

Now the indexes values $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$ in the formula (59) for JPs to obtain,

$$P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1) = \frac{\left(\frac{1}{2}\right)_n}{n!}. \quad (60)$$

Similarly, we have

$$P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(1) = \frac{(2n+1)\left(\frac{1}{2}\right)_n}{n!} \quad (61)$$

From the symmetry property of JPs (58) we can obtain some useful special values of the JPs. If we substitute $x = 1$ in the relation (58), then

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n P_n^{(\beta,\alpha)}$$

Now substituting from equation (59) into this equation to obtain

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n\frac{(1+\beta)_n}{n!} = (-1)^n\binom{\beta+n}{n} = \frac{\Gamma(\beta+n+1)}{n!\,\Gamma(\beta+1)} \quad (62)$$

Theorem 14 (Abramowitz and Stegun, 1968): The Gegenbauer polynomials $C_n^{\alpha}(x) = P_n^{(\alpha,\alpha)}(x)$ are even or odd functions respectively for the even or odd values of the index n.

Proof:- Substitute $\alpha = \beta$ in the symmetry property of JPs (58) to obtain,

$$P_n^{(\alpha,\alpha)}(-x) = (-1)^n P_n^{(\alpha,\alpha)}(x)$$

Using the notation of the Gegenbauer polynomials, one has,

$$C_n^{\alpha}(-x) = (-1)^n C_n^{\alpha}(x). \quad (63)$$

Moreover, Substitute $\alpha = \beta = 0$ in the relation (63) to obtain the following well-known property of Legendre polynomials $P_n(x)$ as,

$$P_n(-x) = (-1)^n P_n(x) \quad (64)$$

For large values of the degree n, we can obtain asymptotic values of some special values of the JPs $P_n^{(\alpha,\beta)}(x)$. Thus for large values $n \gg 1$ we can implement the Stirling's approximation of gamma function (17) in equation (59) to obtain,

$$P_n^{(\alpha,\beta)}(1) \sim \frac{\sqrt{2\pi(n+\alpha)}\left(\frac{n+\alpha}{e}\right)^{n+\alpha}}{\alpha!\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}, n \gg 1.$$

Then do some simple computations to obtain the following asymptotic special value of JPs as,

$$P_n^{(\alpha,\beta)}(1) \sim n^{\alpha}, \qquad n \gg 1. \quad (65)$$

By following a similar fashion, we obtain the other asymptotic special value of JPs as,

$$P_n^{(\alpha,\beta)}(-1) \sim n^{\beta}, \qquad n \gg 1. \quad (66)$$

8.1 Some Special Values of the Jacobi Polynomials $P_n^{(\alpha,\beta)}(\cos\theta)$ (Beals and Wong, 2010)

To derive some special values of the Jacobi polynomials of the variable $x = \cos\theta$, we resort to the following useful relations,

$$\cos(n\theta) = \cos(\theta)\, F\left(\frac{1-n}{2}, \frac{1+n}{2}; \frac{1}{2}; \sin^2(\theta)\right) \quad (67)$$

$$\cos(n\theta) = F\left(\frac{n}{2}, \frac{-n}{2}; \frac{1}{2}; \sin^2(\theta)\right) \quad (68)$$

$$\sin(n\theta) = n\,\sin(\theta)\, F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; \sin^2(\theta)\right) \quad (69)$$

$$\sin(n\theta) = n\,\sin(\theta)\cos(\theta) \times F\left(1-\frac{n}{2}, 1+\frac{n}{2}; \frac{3}{2}; \sin^2(\theta)\right) \quad (70)$$

If we replace n by $2(n+1)$ and θ by $\frac{\theta}{2}$ in the relation (70), we obtain

$$\frac{\sin[(n+1)\theta]}{(n+1)\sin(\theta)} = F\left(-n, n+2; \frac{3}{2}; \sin^2\left(\frac{\theta}{2}\right)\right) \quad (71)$$

In addition, if we replace n by $2n$ and θ by $\frac{\theta}{2}$ in the relation (68), we obtain

$$\cos(n\theta) = F\left(n, -n; \frac{1}{2}; \sin^2\left(\frac{\theta}{2}\right)\right) \quad (72)$$

Substitute $x = \cos\theta$ and the indexes values $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$ in the formula for JPs (39) to obtain,

$$P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(\cos\theta) = \frac{\left(\frac{1}{2}\right)_n}{n!} F\left(-n, n+1; \frac{1}{2}; \frac{1-\cos(\theta)}{2}\right).$$

By calling a simple trigonometric relation we have,

$$P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(\cos\theta) = \frac{\left(\frac{1}{2}\right)_n}{n!} F\left(-n, n+1; \frac{1}{2}; \sin^2\left(\frac{\theta}{2}\right)\right) \qquad (73)$$

On the other hand if we replace n by $2n+1$ and θ by $\frac{\theta}{2}$ in the relation (67), we obtain

$$\cos\left[\left(\frac{2n+1}{2}\right)\theta\right] = \cos\left(\frac{\theta}{2}\right) F\left(-n, 1+n; \frac{1}{2}; \sin^2\left(\frac{\theta}{2}\right)\right) \qquad (74)$$

Finally substitute the relation (74) into the equation (73) to obtain

$$P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(\cos\theta) = \frac{\left(\frac{1}{2}\right)_n}{n!} \frac{\cos\left[\left(\frac{2n+1}{2}\right)\theta\right]}{\cos\left(\frac{\theta}{2}\right)} \qquad (75)$$

With the aid of relation (60), this identity can be rewritten as,

$$P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(\cos\theta) \Big/ P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1) = \frac{\cos\left[\left(\frac{2n+1}{2}\right)\theta\right]}{\cos\left(\frac{\theta}{2}\right)} \qquad (76)$$

Moreover, Substitute $x = \cos\theta$ and the indexes values $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$ in formula for JPs (39) to obtain,

$$P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\cos\theta) = \frac{\left(\frac{3}{2}\right)_n}{n!} F\left(-n, n+1; \frac{3}{2}; \frac{1-\cos(\theta)}{2}\right)$$

By calling a simple trigonometric relation we have,

$$P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\cos\theta) = \frac{\left(\frac{3}{2}\right)_n}{n!} F\left(-n, n+1; \frac{3}{2}; \sin^2\left(\frac{\theta}{2}\right)\right) \qquad (77)$$

On the other hand if we replace n by $2n+1$ and θ by $\frac{\theta}{2}$ in the relation (69), we obtain

$$\sin\left[\left(n+\frac{1}{2}\right)\theta\right] = (2n+1)\sin\left(\frac{\theta}{2}\right) F\left(-n, 1+n; \frac{3}{2}; \sin^2(\theta)\right) \qquad (78)$$

Now substitute the relation (78) into the equation (77) to obtain

$$P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\cos\theta) = \frac{\left(\frac{3}{2}\right)_n}{n!} \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{(2n+1)\sin\left(\frac{\theta}{2}\right)} \qquad (79)$$

Since,

$$\left(\frac{3}{2}\right)_n = (2n+1)\left(\frac{1}{2}\right)_n \qquad (80)$$

Finally substitute the relation (80) into the equation (79) to obtain

$$P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\cos\theta) = \frac{\left(\frac{1}{2}\right)_n}{n!} \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{\sin\left(\frac{\theta}{2}\right)} \qquad (81)$$

In a similar manner to the derivation above we can obtain,

$$P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(\cos\theta) = \frac{\left(\frac{3}{2}\right)_n}{n!} \frac{\sin[(n+1)\theta]}{(n+1)\sin(\theta)} \qquad (82)$$

With the aid of relation (61), the identity (81) can be rewritten as,

$$P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\cos\theta) \Big/ P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(1) = \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{(2n+1)\sin\left(\frac{\theta}{2}\right)} \qquad (83)$$

Also, we have the following relations,

$$P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(\cos\theta) \Big/ P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(1) = \frac{\sin[(n+1)\theta]}{(n+1)\sin(\theta)} \qquad (84)$$

And

$$P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(\cos\theta) \Big/ P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1) = \cos(n\theta) \qquad (85)$$

These representations of the special cases of the Jacobi polynomials will be beneficial in obtaining the integral representations of Jacobi polynomials as it will be shown in section 11.

9. Chebyshev Polynomials as Special Cases of Jacobi Polynomials

Exploiting the relations derived in the previous section, we can define the first and second kind Chebyshev polynomials respectively $T_n(x)$ and $U_n(x)$ in terms of the JPs $P_n^{(\alpha,\beta)}(x)$ for special values of the indexes α and β (Beals and Wong, 2010). We know that, the first kind Chebyshev polynomials $T_n(x)$ is defined as,

$$T_n(\cos\theta) = \cos(n\theta)$$

Substituting relation (85) into this equation leads to,

$$T_n(\cos\theta) = P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(\cos\theta) \Big/ P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1) \qquad (86)$$

where the coefficient $P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1)$ is defined as,

$$P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1) = \frac{\left(\frac{1}{2}\right)_n}{n!}.$$

Moreover, we know that, the second kind Chebyshev polynomials $U_n(x)$ is defined as,

$$U_n(\cos\theta) = \frac{\sin[(n+1)\theta]}{\sin(\theta)}.$$

Substituting relation (84) into this equation leads to,

$$U_n(\cos\theta) = \frac{(n+1)}{P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(1)} P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(\cos\theta) \qquad (87)$$

where the coefficient $P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(1)$ is defined by

$$P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(1) = \frac{\left(\frac{3}{2}\right)_n}{n!} \qquad (88)$$

Thus using the relations (6) and (7) with (88) for the coefficient in equation (87) leads to,

$$\frac{(n+1)}{P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(1)} = \frac{(n+1)n!}{\left(\frac{3}{2}\right)_n} = \frac{(n+1)!\,2^n n!}{(2n+1)!} = \frac{(n+1)!\,2^{2n} n!\sqrt{\pi}}{n!\,2^{2n+1}\Gamma\left(n+\frac{3}{2}\right)}.$$

Hence,

$$U_n(x) = \frac{(n+1)!\sqrt{\pi}}{2\Gamma\left(n+\frac{3}{2}\right)} P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x) \qquad (89)$$

10. The Orthogonality and the orthonormality Properties of the Jacobi Polynomials.

In this section we shall prove one of the most important properties of the Jacobi polynomials which are the orthogonlaity and the orthonormality properties.

Theorem 15: The Jacobi polynomials (eigenfunctions of the second-order differential operator (32)) are orthogonal with respect to the weight function (beta density) $w^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$ on the interval $\in (-1,1)$, that is

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)}\delta_{m,n}, \quad Re(\alpha), Re(\beta) > -1, \qquad (90)$$

where $\delta_{m,n}$ is the Kroncker delta symbol defined as,

$$\delta_{m,n} = \begin{cases} 0, & \text{for } m \neq n, \\ 1, & \text{for } m = n. \end{cases}$$

Proof:- We shall begin the proof by multiplying both sides of the Rodrigues formula for the Jacobi polynomials (51) by $P_m^{(\alpha,\beta)}(x)$ and then integrate over the interval $[-1,1]$ to obtain,

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}\left(\frac{d}{dx}\right)^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]P_m^{(\alpha,\beta)}(x)dx,$$

Now, doing the integral on the right hand side by parts leads to,

$$u(x)=P_m^{(\alpha,\beta)},\, dV=\left(\frac{d}{dx}\right)^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]dx,$$

$$du=\frac{dP_m^{(\alpha,\beta)}}{dx},\, V(x)=\left(\frac{d}{dx}\right)^{n-1}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right].$$

Thus, one has

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx = \frac{(-1)^{n+1}}{2^n n!}\int_{-1}^{1}\left(\frac{d}{dx}\right)^{n-1}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]\frac{d}{dx}P_m^{(\alpha,\beta)}dx,$$

Where the first term resulted from integration vanishes at the end points $x=\pm 1$. Repeating the integration by parts n-times, one has

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx = \frac{(-1)^{2n}}{2^n n!}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}\left(\frac{d}{dx}\right)^n P_m^{(\alpha,\beta)}dx,$$

Now we need to classify two cases, the first case we assume that $\neq n$. Since m and n are arbitrary integer numbers, we choose $n>m$ (if $m>n$ then interchange m and n), hence we have,

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx = 0,\, m\neq n, \tag{91}$$

because

$$\left(\frac{d}{dx}\right)^n P_m^{(\alpha,\beta)} = 0,\, m<n.$$

Secondly, we suppose that $m=n$, so

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left[P_n^{(\alpha,\beta)}(x)\right]^2 dx = \frac{(-1)^{2n}}{2^n n!}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}\left(\frac{d}{dx}\right)^n P_n^{(\alpha,\beta)}dx, \tag{92}$$

Since the coefficient $a_n^{(\alpha,\beta)}$ of x^n in $P_n^{(\alpha,\beta)}$ in equation (38) is

$$a_n^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(n+\alpha+\beta+1)}\binom{n}{n}\frac{\Gamma(2n+\alpha+\beta+1)}{2^n\Gamma(n+\alpha+1)},$$

So,

$$a_n^{(\alpha,\beta)} = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n n!\,\Gamma(\alpha+\beta+n+1)} \tag{93}$$

Hence, substitute from equation (93) into equation (92) to obtain,

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left[P_n^{(\alpha,\beta)}(x)\right]^2 dx = \frac{a_n^{(\alpha,\beta)}}{2^n}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx, \tag{94}$$

To evaluate the integral on the right hand side of this equation, we make the variable change $w=1-x$, so

$$\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx = 2^{2n+\alpha+\beta+1}\int_{0}^{1}w^{n+\alpha}(1-w)^{n+\beta}dw, = 2^{2n+\alpha+\beta+1}B(n+\alpha+1,n+\beta+1)$$

Using the relation (8) leads to

$$\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx = \frac{2^{2n+\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2)}. \tag{95}$$

Now substitute from equation (95) into equation (94) to obtain

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left[P_n^{(\alpha,\beta)}(x)\right]^2 dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)}. \tag{96}$$

Thus,

$$\left\|P_n^{(\alpha,\beta)}(x)\right\|^2 = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)}, Re(\alpha), Re(\beta) > -1.$$

Gathering equations (96) and (91) leads to the property (90). Thus the normalized and mutually orthogonal set $\left\{\frac{P_n^{(\alpha,\beta)}(x)}{\left\|P_n^{(\alpha,\beta)}(x)\right\|}\right\}_{n=1}^{\infty}$ with respect to the measure weighting function $w^{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$ over the interval $x\in[-1,1]$ for the indexes $\alpha,\beta>-1$ form a complete set in the Hilbert space $L^2(-1,1)$.

The orthogonality property of JPs (90) can be reduced to the orthogonality property of Legendre polynomials by setting the indexes values α,β in equation (90) to zeros, thus

$$\int_{-1}^{1}P_n(x)P_m(x)dx = \frac{2}{(2n+1)}\delta_{m,n} \tag{97}$$

10.1. Applications of the Orthogonality Property of Jacobi Polynomials

Theorem 16 (Rainville, 1960)**:** If the set of real polynomials $\{f_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to a certain positive norm function $w(x)$ on the interval $[a,b]$, then the zeros of such set are distinct and all are positioned in the interval $[a,b]$. Moreover, each polynomial $f_n(x)$ has n simple zeros in the closed interval $[a,b]$ as shown in Fig. 4 where $P_4^{(\alpha,\beta)}(x)$ has four zeros and $P_5^{(\alpha,\beta)}(x)$ has five zeros. In addition, Fig. 4 shows that all the zeros of JPs lie in the open interval $(-1,1)$

Theorem 3 (Rainville, 1960)**:** If the set of real polynomials $\{f_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to a certain positive norm function $w(x)$ on the interval $[a,b]$, and $m<n\in\mathbb{N}$, then between any two zeros of the polynomial $f_m(x)$ there is a zero of the polynomial $f_n(x)$. That is, the zeros of the polynomials $f_m(x)$ and $f_n(x)$ separate each other.

Fig. 4 shows that between any two zeros of the polynomial $P_5^{(\alpha,\beta)}(x)$ there is a zero of the polynomial $P_4^{(\alpha,\beta)}(x)$. Fig. 5 shows that for the indexes values $\alpha=1.2$ and $\beta=-3.5<-1$, the corresponding JPs $P_4^{(1.2,-0.5)}, P_5^{(1.2,-0.5)}$ do not fulfill theorems 1 and 2. By violating the classical condition for the orthogonality property (90), one can clearly observe that the number of zeros of JPs $P_4^{(1.2,-0.5)}, P_5^{(1.2,-0.5)}$ is not equal to the polynomial degree, and the zeros of $P_5^{(1.2,-0.5)}$ do not separate the ones for $P_4^{(1.2,-0.5)}$.

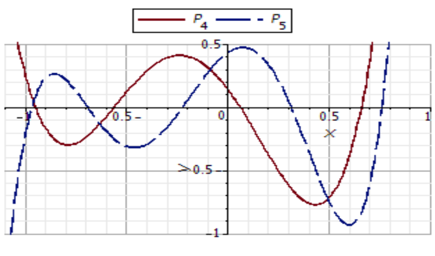

Fig. 4. Jacobi polynomials $P_4^{(1.2,-0.5)}, P_5^{(1.2,-0.5)}$.

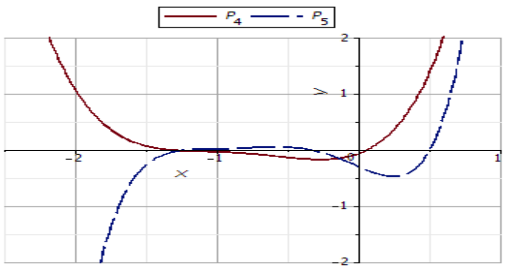

Fig. 5. Jacobi polynomials $P_4^{(1.2,-3.5)}, P_5^{(1.2,-3.5)}$.

Definition 3 (Abramowitz and Stegun, 1968)**:** The Gaussian quadrature to estimate the definite integral of a function $f(x)$ defined as,

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + R_n,$$

where R_n is the approximation error. This quadrature hold exactly for any polynomial of degree less than $2n-1$.
It should be noted that the nodes x_i are the zeros of the orthogonal polynomial of degree n associated with the weight function and the interval $[a,b]$. The quadrature weights w_i are chosen in such a way to enforce the quadrature rule for a polynomial of degree $2n-1$ to be exact $R_n = 0$.

Definition 4 (Abramowitz and Stegun, 1968)**:** The Gaussian-Jacobi quadrature to estimate the definite integral of a function $f(x)$ defined as,

$$\int_{-1}^{1} f(x)\,(1-x)^{\alpha}(1+x)^{\beta}dx \cong \sum_{i=1}^{n} w_i f(x_i) \tag{98}$$

where the nodes x_i are the zeros of the Jacobi polynomial of degree n associated with the following weights,

$$w_i = -\frac{(2n+\alpha+\beta+2)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(n+1)!\,(n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)} \times \frac{2^{\alpha+\beta}}{\left[P_n^{(\alpha,\beta)}(x_i)\right]' P_{n+1}^{(\alpha,\beta)}(x_i)}.$$

The derivative of the JPs in w_i can be computed using the identity (45).

Definition 5 (Abramowitz and Stegun, 1968)**:** The Gaussian-Legendre quadrature to estimate the definite integral of a function $f(x)$ defined as,

$$\int_a^b f(x)dx \cong \sum_{i=1}^{n} w_i f(x_i) \tag{99}$$

where the nodes x_i are the zeros of the Legendre polynomial $P_n(x)$ associated with the weights,

$$w_i = \frac{2[P_n'(x_i)]^{-2}}{\left(1-x_i^2\right)^2}.$$

Definition 6 (Abramowitz and Stegun, 1968)**:** The Gaussian- 1^{st} kind Chebyshev quadrature to estimate the definite integral of a function $f(x)$ defined as,

$$\int_a^b \frac{f(x)}{\sqrt{1-x^2}}dx \cong \sum_{i=1}^{n} w_i f(x_i) \tag{100}$$

where the nodes x_i are given as,

$$x_i = \frac{\cos(2i-1)\pi}{2n},$$

and the associated weights are,

$$w_i = \frac{\pi}{n}.$$

Definition 7 (Abramowitz and Stegun, 1968)**:** The Gaussian- 2^{nd} kind Chebyshev quadrature to estimate the definite integral of a function $f(x)$ defined as,

$$\int_{-1}^{1} \sqrt{1-x^2} f(x)dx \cong \sum_{i=1}^{n} w_i f(x_i) \tag{101}$$

where the nodes x_i are given as,

$$x_i = \cos\left(\frac{i}{n+1}\right)\pi,$$

and the associated weights are,

$$w_i = \frac{\pi}{n+1}\sin^2\left(\frac{i\pi}{n+1}\right).$$

The nodes x_i and associated weights w_i for all types of the Gaussian quadrature listed above are tabulated in (Abramowitz and Stegun, 1968) for different values of n. It should be noted that, the integrals above can be estimated at any other interval $[a,b]$ by a suitable linear transformation.

There are various numerical techniques such as Newton-Raphson method to compute the zeros of orthogonal polynomials. Though now there are various built-in tools in the well-known mathematical packages to easily compute such zeros in an accurate manner. The interest in computing the zeros of JPs comes as a consequence of the increase interest in spectral approximation. The orthogonality property of the JPs considerably affect their zeros distribution. The interest in extending the orthogonality property of JPs come a result of the growth interest in the problem of finding the zeros of JPs which has extreme importance in spectral approximation as shown in the next section.

10.2. Some Orthogonality Conditions of the Jacobi Polynomials

The standard orthogonality property of the Jacobi polynomials is hold for the classical values of the indexes $\alpha, \beta > -1$ over the interval $x \in [-1,1]$. Hence the zeros of JPs are simple and all lie in the open interval $(-1,1)$. But for non-classical values of the indexes $\alpha, \beta \in \mathbb{C}$, the zeros of JPs are complex and may be multiple and lie in a region in the complex z-plane in a well-organized manner. Kuijlaars (Kuijlaars, Finkelshtein and Orive, 2005) established some orthogonality conditions of the Jacobi polynomials on the Riemann surface for more general values of the indexes $\alpha, \beta \in \mathbb{C}$ on some paths in the complex z-plane as stated in the following theorem.

Theorem 18 (Kuijlaars, Finkelshtein and Orive, 2005): For more general values of the indexes $\alpha, \beta \in \mathbb{C}$ and the weight function $w^{\alpha,\beta}(z)$ defined as,

$$w^{\alpha,\beta}(z) = (1-z)^{\alpha}(1+z)^{\beta} = e^{[\alpha\ln(1-z)+\beta\ln(1+z)]}.$$

Then for $s \in \mathbb{Z}$ and $0 \le s \le n$ one has,

$$\int_C z^s\, w^{\alpha,\beta}(z) P_n^{(\alpha,\beta)}(z)dz = \frac{-\pi^2 2^{n+\alpha+\beta+3} e^{i\pi(\alpha+\beta)}\delta_{s,n}}{\Gamma(2n+\alpha+\beta+2)\Gamma(-\alpha-n)\Gamma(-\beta-n)} \tag{102}$$

where C is a contour on Riemann surface for the function $w^{\alpha,\beta}(z)$. It should be noted that, the weight function $w^{\alpha,\beta}(z)$ is a multivalued function with branch points at $z = \pm 1, \infty$, but by extending $w^{\alpha,\beta}(z)$ on the contour C it is enforced to be a single-valued function. For the proof of this theorem, the reader is referred to the main source (Kuijlaars, Finkelshtein and Orive, 2005).

11. Integral Representations of the Jacobi polynomials.

In this section we shall summarize some of remarkable integral expansions of JPs (Askey and Fitch, 1969). These expansions should grant us diversity of applications for JPs. We start with the following trivial identity,

$$\frac{\sin(n\theta)}{n} = \int_0^{\theta} \cos(n\theta)\,d\theta,$$

Employing the relations (84) and (86) into this identity leads to,

$$\frac{P_{n-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(\cos\psi)}{P_{n-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(1)}\sin\psi=\int_0^{\psi}\frac{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(\cos\theta)}{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1)}d\theta$$

Now make the substitution $x=\cos\psi$ to obtain,

$$\frac{P_{n-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(x)}{P_{n-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(1)}(1-x^2)^{\frac{1}{2}}=\int_x^1\frac{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(w)}{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1)}(1-w^2)^{-\frac{1}{2}}dw \tag{103}$$

In fact the integral formula (103) is a special case of the following formula,

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^k(n-k)!}{2^k n!}(1-x)^{-\alpha}(1+x)^{-\beta}\times\left(\frac{d}{dx}\right)^k\left[(1-x)^{k+\alpha}(1+x)^{k+\beta}P_{n-k}^{(\alpha+k,\beta+k)}(x)\right],n\geq k. \tag{104}$$

This Rodrigues-Like formula should be beneficial in any derivations involving the Rodrigues formula. If we substitute by the hypergeometric functions in the integral transformation (28) by its equivalence from equation (39), we arrive at the following integral representation of JPs,

$$(1-x)^{\alpha+\rho}\ P_n^{(\alpha+\rho,\beta-\rho)}(x)=\frac{P_n^{(\alpha+\rho,\beta-\rho)}(1)}{B(\alpha+1,\rho)}\int_x^1\frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(1)}(1-t)^{\alpha}(t-x)^{\rho-1}dt\,. \tag{105}$$

Remark: If we substitute by the indexes values $\rho=1$, $\alpha=-\frac{1}{2},\beta=\frac{1}{2}$ in equation (105), one has

$$\sqrt{1-x}\,P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(x)=\frac{P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(1)}{2P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1)}\int_x^1\frac{P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t)}{\sqrt{1-t}}dt \tag{106}$$

Proof: To derive this special case, we start from the trivial fact,

$$\frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{\left(n+\frac{1}{2}\right)}=\int_0^{\theta}\cos\left[\left(n+\frac{1}{2}\right)\varphi\right]d\varphi. \tag{107}$$

Plugging equations (79) and (83) into equation (107) leads to,

$$\frac{2\,P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\cos\theta)}{P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(1)}\sin\left(\frac{\theta}{2}\right)=\frac{1}{2}\int_0^{\theta}\frac{P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(\cos\varphi)}{P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1)}\cos\left(\frac{\varphi}{2}\right)d\varphi \tag{108}$$

Now, make the variable change $x=\cos\theta$ and $t=\cos\varphi$ respectively in the left and right hand side of the last equation to arrive at the required equation (106).

The hypergeometric representation of the JPs allow us to make use of the Euler's integral representation of the hypergeometric function to obtain some useful integral formulae for the JPs as we show next.

Theorem 19 (Askey and Fitch, 1969): For $x>-1,\ t<1$ and $\rho>0$, we have,

$$(1-x)^{-(n+\beta+1)}\,(1+x)^{\beta+\rho}\,P_n^{(\alpha,\beta+\rho)}(x)=\frac{2^{\rho}P_n^{(\alpha,\beta+\rho)}(-1)}{B(\beta+1,\rho)}\times\int_{-1}^{x}\frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(-1)}(1+t)^{\beta}(1-t)^{-(n+\beta+\rho+1)}(x-t)^{\rho-1}dt.$$

Proof: If we substitute by the hypergeometric functions in equation (28) by its equivalence from equation (39), we arrive at the following integral representation of JPs,

$$(1-x)^{\alpha+\rho}\ P_n^{(\alpha+\rho,\beta-\rho)}(x)=\frac{P_n^{(\alpha+\rho,\beta-\rho)}(1)}{B(\alpha+1,\rho)}\int_x^1\frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(1)}(1-t)^{\alpha}(t-x)^{\rho-1}dt\,,\rho>0,$$

where $x>-1,t<1$. Employing the symmetry property of JPs (58) in the last equation leads to,

$$(1+x)^{\beta+\rho}\ P_n^{(\alpha-\rho,\beta+\rho)}(x)=\frac{P_n^{(\alpha-\rho,\beta+\rho)}(-1)}{B(\beta+1,\rho)}\int_{-1}^{x}\frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(-1)}(1+t)^{\beta}(x-t)^{\rho-1}dt\,.$$

Now employing equation (29) in the last equation leads to,

$$(1+x)^{n+\alpha+\beta}\ P_n^{(\alpha-\rho,\beta)}(x)=\frac{1}{B(n+\alpha+\beta-\rho+1,\rho)}\int_{-1}^{x}P_n^{(\alpha,\beta)}(t)(1+t)^{n+\alpha+\beta-\rho}(x-t)^{\rho-1}dt\,.$$

Again implementing the symmetry property of JPs (58) in the last equation leads to,

$$(1-x)^{n+\alpha+\beta}\ P_n^{(\alpha,\beta-\rho)}(x)=\frac{1}{B(n+\alpha+\beta-\rho+1,\rho)}\int_x^1 P_n^{(\alpha,\beta)}(t)(1-t)^{n+\alpha+\beta-\rho}(t-x)^{\rho-1}dt$$

Using the transformation (30) leads to,

$$(1+x)^{-(n+\alpha+1)}\,(1-x)^{\alpha+\rho}\,P_n^{(\alpha+\rho,\beta)}(x)=\frac{P_n^{(\alpha+\rho,\beta)}(1)}{B(\alpha+1,\rho)}\int_x^1\frac{P_n^{(\alpha,\beta)}(t)(1-t)^{\alpha}}{(1+t)^{n+\alpha+\rho+1}}(t-x)^{\rho-1}dt\,,$$

Recalling the symmetry property of JPs (58) in the last equation leads to,

$$(1-x)^{-(n+\beta+1)}\,(1+x)^{\beta+\rho}\,P_n^{(\alpha,\beta+\rho)}(x)=\frac{2^{\rho}P_n^{(\alpha,\beta+\rho)}(-1)}{B(\beta+1,\rho)}\times\int_{-1}^{x}\frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(-1)}(1+t)^{\beta}(1-t)^{-(n+\beta+\rho+1)}(x-t)^{\rho-1}dt \tag{109}$$

Thus we obtain an integral formula for JPs $\left(\frac{P_n^{(\alpha+\rho,\beta)}(x)}{P_n^{(\alpha+\rho,\beta)}(1)}\right)$ in terms of other JPs $\left(\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}\right)$. Actually, it is advantageous to have an integral expansion of a positive kernel for JPs. These integral expansions are of great importance in obtaining some useful inequalities of positive coefficients sum for JPs such as the following inequality.

Theorem 19 (Feldheim, 1963)**:** For $|x|\leq 1$, and n is a non-negative integer number,

$$\sum_{k=0}^{n}\frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\alpha)}(1)}\geq 0 \tag{110}$$

The following example shows that why it is advantageous to have a series or integral expansion of a positive kernel. This example is designed for Legendre polynomials which are defined by the following series expansion of alternative-sign kernel,

$$P_n(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k\left(\frac{1}{2}\right)_{n-k}(2x)^{n-2k}}{k!\,(n-2k)!} \tag{111}$$

Also, the Legendre polynomials defined by the following series expansion of positive kernel as,

$$P_n(\cos\theta)=\sum_{k=0}^{n}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_{n-k}\cos\,(n-2k)\theta \tag{112}$$

Taking the absolute value of both sides of equation (112) to obtain,

$$|P_n(\cos\theta)|=\left|\sum_{k=0}^{n}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_{n-k}\cos\,(n-2k)\theta\right|.$$

$$|P_n(\cos\theta)|\leq\sum_{k=0}^{n}\left|\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_{n-k}\right||\cos\,(n-2k)\theta|,$$

$$|P_n(\cos\theta)|\leq P_n(1).$$

Thus, one obtains the following important inequality of Legendre polynomials as,

$$|P_n(\cos\theta)|\leq 1 \tag{113}$$

Therefore we obtain this property in a straightforward manner, whereas this cannot be that obvious from the representation (111). Hence, it is very beneficial to have a series expansion of positive kernel rather than a mixed- sign kernel.

12. Discussion and Conclusion

To conclude this review was devoted on very important class of orthogonal polynomials known as Jacobi polynomials. Their importance lies in the generalization feature that such polynomials possess because they implicitly include many common orthogonal polynomials. This narrative review began with deriving some significant hypergeometric representations of JPs as shown in section 4. Such hypergeometric representations were exploited to deduce most of the properties of JPs. Actually the hypergeometric approach have been adopted to gain some differential recurrence relations, generating function in terms of the hypergeometric function and obtain some special values of JPs as shown in section 8. The differential recurrence relations (45), (46) and (47) are useful in computing the nodes and the corresponding weights in any Gaussian quadrature. We show how to reduce the Rodrigues formula of JPs (51) to the corresponding formula for the Legendre and Chebyshev polynomials of the first and second kinds (55), (56) and (57) respectively. The Rodrigues formula of JPs (51) was exploited to derive the symmetry property of JPs (58) which is of great use in deriving some integral expansions of JPs as shown in section 11. Some special values either exact or asymptotic were obtained through some hypergeometric formulae of JPs as shown in section 8. This is followed by showing how to obtain the Chebyshev polynomials of the first and second kinds with the variable $x = \cos(\theta)$.

Section 10 was dedicated to a prominent property of any orthogonal polynomials which is the orthogonality property. In fact the standard orthogonality property of the Jacobi polynomials is hold for the classical values of the indexes $\alpha, \beta > -1$ over the interval $x \in [-1,1]$. Some previous works were summarized on extending the orthogonality property of the Jacobi polynomials to include more values of the indexes $\alpha, \beta \in \mathbb{R}$ or even more values $, \beta \in \mathbb{C}$. Then we show how to take advantage of Jacobi polynomials in approximating a definite integral of certain function using a variety of Gaussian quadrature types such as the Gaussian-Jacobi quadrature (98), Gaussian-Legendre quadrature (99) and Gaussian- 1^{st} and 2^{nd} kinds Chebyshev quadrature respectively (100) and (101). The orthogonality property is an essential feature for any spectral approximation, moreover such property extremely affect the locations of the zeros of any orthogonal polynomials which are of great significance in any Gaussian quadrature. We conclude this review by summarizing some integral expansions of JPs (103)-(109) with a remarkable feature of possessing positive kernel. Such feature should grant us a variety of applications of JPs such as obtaining some useful inequalities of positive coefficients (110) and (113). A future work should be devoted on reducing the JPs to the less common Zernike polynomials, because there is no much research on such polynomials.

References

Arfken, G. (1985) *Mathematical Methods for Physicists*. 3rd edn. Academic Press, Inc.

Andrew, GE., Askey, R. and Roy R. (1999) *Special Functions*. Cambridge University Press.

Bell, W. (1968) *Special Functions for Scientists and Engineers*. New York.

Brychkov, Yu. (2008) *Handbook of Special Functions: Derivatives, Integrals, Series, and other formulas*. CRC Press, Boca Raton, FL.

Chihara, T.S. and Seio, T. (1978) *An Introduction to Orthogonal Polynomials*. Gordon and Breach, New York.

Abramowitz M and Stegun A. (1968) *Handbook of Mathematical Functions*, 9th edn, New York: Dover Publications Inc..

Chihara, T.S. and Seio T. (2001) '45 years of orthogonal polynomials: a view from the wings', *Proceedings of the Fifth International Symposium on Orthogonal Polynomials*.

Chihara, S. and Seio, T. (1999) 'Special Functions and their Applications (Patras,)', *Journal of Computational and Applied Mathematics*, 133 (1), pp. 13–21.

Gautsch, W. (2004) Orthogonal Polynomials: Copmputation and Approximation. Oxford University Press, New York.

Koornwinder, H., Wong, S. C., Koekoek, R. and Swarttouw, F. (2010) 'Orthogonal Polynomials', in Olver, W.J., Lozier, M., Boisvert, F. and Clark, W. (eds.) 'NIST Handbook of Mathematical Functions'. Cambridge University Press.

Beals, R. and Wong, R. (2010) *Special Function: A graduate text*. Cambridgre studies in advanced mathematics 126: Cambridge University Press.

Rainville, E. (1960) *Special Functions*. The Macmillan Co., New York.

Sneddon, I.N. (1980) *Special Functions of Mathematical Physics and Chemistry*. 3rd edn. New York: Longman.

Sansone, G. (1991) *Orthogonal Functions*. Dover, New York.

Szegö, G. (1975) *Orthogonal Polynomials*. Americam Mathematical Society, Providence, RI.

Uspensky, J. V. (1972) 'On the Development of Arbitrary Functions in Series Of Hermite's and Laguerre's Polynomials', *Annals of Math*., 28 (2), pp. 593-619.

Vilmos, T. (2005) *Orthogonal Polynomials*. Surveys in Approximation Theory.

Guo, B. (2000) 'Jacobi spectral approximation and its applications to differential equations on the half line', *J. Comput. Math*. ,18,pp. 95–112.

Guo, B. (2000) 'Jacobi Approximations in Certain Hilbert Spaces and their Applications to Singular Differential Equations, J. Math. Anal. Appl., 243, pp. 373–408.

Guo, B., Shen, J. and Wang, L. (2006) 'Optimal Spectral-Galerkin Methods Using Generalized Jacobi Polynomials', *J. Sci. Comp*., 27, pp. 305–322.

Guo, B. and Wang, L. (2004) 'Jacobi Approximations and Jacobi–Gauss-type Interpolations in Nonuniformly Weighted Sobolev Spaces', *J. Approx. Theory*, 128, pp. 1–41.

Kuijlaars, A.B.J., Mart´Inez-Finkelshtein, A. and Orive. R. (2005) 'Orthogonality of Jacobi Polynomials with General Parameters', *Electronic Transactions on Numerical Analysis*. 19, pp. 1-17.

Alfaroa, M., Alvarez de Morales, M. and Rezola, M. L. (2002) 'Orthogonality of the Jacobi polynomials with negative integer parameter', *Journal of Computational and Applied Mathematics*, 145, pp. 379–386.

Alvarez de Morales, M., Perez, T. E. and Pinar, M.A. (1998) 'Sobolev Orthogonality property of the Gegenbauer polynomials $\left\{C_n^{(-N+1/2)}\right\}_{n\geq 0}$', *Journal of Computational and Applied Mathematics*, 100, pp. 111–120.

Janecki, D. and Stephen, K. (2005) 'Legendre Polynomials Used for The Approximation Of Cyliderical Surfaces', *Common Sci, Lett. Univ. Zilina,* 4, pp. 59-61.

Mirevski S.P., Boyadjiev, L. and Scherer, R. (2007) 'On the Riemann-lioville Fractional Calculus: g-Jacobi Functions and F-Gauss Functions', *Appl. Math. Comput*, 187(1), pp. 315-325.

Guo, B., Shen, J. and Wang, L. (2009) 'Generalized Jacobi polynomials/functions and their applications', *Appied Numerical Mathematics*, 59, pp. 1011-1028.

Gautschi, W. (2004) *Orthogonal polynomials: computation and approximation*. Oxford University Press, New York.

Driver, K. and Love, A. (2001) 'Zeros of ${}_3F_2$ hypergeometric polynomials', *J. Comput. Appl. Math.*, 131, pp. 243-251.

Driver, K. and Möller, M. (2001) ' Zeros of hypergeometric polynomials $F(-n, b; -2n; z)$ ', *J. Approx. Theory*, 110, pp. 74-87.

Driver, K., Möller, M. (2002) 'Quadratic and cubic transformations and zeros of hypergeometric polynomials', *J. Comput. Appl. Math*., 142, pp. 411-417.

Canuto, C., Hussaini, M.Y., Quarteroni, A. and Zang T.A.(2006) *Spectral Methods: Fundamentals in Single Domains*. Springer-Verlag,:Heidelberg.

Duren, P. and Boggs, K. (2001) ' Zeros of hypergeometric functions', *Comput. Methods Func. Theory*, 1, pp. 275-287.

Finkelshtein, M.A., Gonzalez, P.M. and Orive. R. (1999) 'Orthogonality of Jacobi Polynomials with General Parameters', *Proceeding of the international workshop on special functions,* HonKong, 19, pp. 98-113.

Bateman, H. (1950) ' A generalization of the Legendre polynomials', *Proc. London Math. Soc.*, 3(2), pp. 111-123.

Askey, R., and Fitch, J. (1969) 'Integral Representation for Jacobi Polynomials and Some Applications', *Journal of Mathematical Analysis and Applications,* 26, pp. 411-437.

Feldheim, F. (1963) 'On the positivity of certain sums of ultraspherical polynomials', *J. Anal. Math*. 11, pp. 275-284.