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Totally Volume Integral of Fluxes for Discontinuous Galerkin Method (TVI-DG) II-One Dimensional System of Equations

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ABSTRACT

This is the second paper in a series in which we construct the totally volume integrals of Riemann flux mimicking Godunov flux for discontinuous Galerkin method. In this work the boundaries integrals of the Riemann fluxes are transformed into volume integral. The new family of DG method is accomplished by applying the divergence theorem to the boundaries integrals of the Riemann fluxes. Therefore, the discontinuous Galerkin (DG) method is independent on the boundaries integrals of fluxes at the cell (element) boundaries as in the classical discontinuous Galerkin (DG) methods. The modified streamline upwind Petrov-Galerkin method is used to capture the oscillation of unphysical flow for shocked flow problems. The numerical results of applying totally volume integral discontinuous Galerkin method (TVI-DG) are presented for Euler's equations in one-dimensional cases. The numerical finding of this scheme is very accurate as compared to exact solutions.

Keywords: *Scalar conservation laws; Higher order methods, Discontinuous Galerkin; Divergence theorem.*

1. Introduction

It is well known that nature is governed by many conservation laws. The convection plays an important role in the real word applications as weather forecasting, turbo machinery, gas dynamics and aero acoustic. The devising of robust, accurate and efficient methods is of considerable important. There has been a surge of researches activities in high order methods as spectral volume (SV) method, spectral difference (SD) method, the weighted essentially non-oscillatory (WENO) method, streamline upwind Petrov-Galerkin (SUPG) method, staggered grid (SG) method and the discontinuous Galerkin (DG) method. In this work, we concerned ourselves to the compact and weighted scheme, which is the DG method.

The DG method is introduced by Reed and Hill in 1973 for neutron transport problems and it is developed for fluid dynamics by Cockburn and Shu in series of papers among them [1-3]. The discontinuous Galerkin methods have recently become popular for the solution of systems of conservation laws. The discontinuous Galerkin methods combine two advantageous features commonly associated to finite element and finite volume methods. As in classical finite element methods, accuracy is obtained by means of high-order polynomial approximation within an element rather than by wide stencils as in case of finite volume methods. The fluxes through the element boundaries are then computed using an approximate Riemann solver, mimicking the successful Godunov finite volume method. Due to the use of Riemann fluxes across element boundaries, the DG method is conservative at the element level. Huynh [4] introduced a flux reconstruction (FR) approach, in which the formulation is capable of unifying several popular methods including the discontinuous Galerkin method, staggered-grid method, spectral difference method and spectral volume method into a single formula. The final mathematical form of the discretized governingequation is in the differential form. After that, Wang [5] extended (FR) approach to multidimensional flow and unstructured mesh under the named lifting collocation penalty (LCP) formulation. Therefore, the differences between DG and other methods lies in the definition of degrees of freedom (DOFs) and how the DOFs are updated [5]. It is well known that, the discontinuous Galerkin method is as an efficient and low error magnitude than the other methods. In the DG formulation, the boundary flux is integrated over the boundary of the cell as traditional methods like finite volume (FV) methods. While for its development, the weighted function at the boundary of the cell can be transformed into the correction function $g(\xi)$ for (FR) or lifting coefficients$\alpha_{i,j}$ for LCP formulations. Thus $g(\xi)$and $\alpha_{i,j}$ are dependent on the weighted functions over the boundaries [5] and [6]. Therefore, the weighted functions at the boundary play an important role for boundary flux calculation in the DG method and its development. In general, there are two types of flux integrals, the first one is the volume integral of the physical flux over the entire element domain and the second type of integral is the boundary integral of the Godunov flux over the boundaries of the elements.

This difficulty motivated ourselves to introduce a new family of DG method independent on the weighting functions at the boundaries. Therefore, no boundary integral is needed for this new formulation. The paper is organized as follows. Section 2 introduced the new DG method formulations. The verification of the new formulation is introduced in section 3. Finally, conclusion remarked is introduced into section 4.

2. Totally Volume Integral (TVI) DG Method Formulation

2.1. Space discretization

For convenience of discussion, a review for DG semi-discretization for partial differential equations (PDE)is introduced. This can be done by firstly considering the conservative laws in divergence form:

$$Q_t = \nabla . F(Q) = 0 \quad \text{.................................... (2.1)}$$

The numerical solution of Eqn. (2.1) is sought on the computational domain Ω subject to proper initial and boundary conditions. Where Q is the conservative variable and F is the conservative flux vector. In Eqn. (2.1), Q and F are scalar or column, representing scalar or system of equations. For example, Eqn. (2.1) represents system of equations, which are one-dimensional Euler equations, if:

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}, F(Q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(\rho E + p) \end{bmatrix}, \text{..................... (2.2)}$$

with $p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho u^2\right)$

The weighted residual formulation is obtained by multiplying Eqn. (2.1) by a scalar test function (weighted function) W and integrating by parts over the domain Ω

$$\int_{\Omega} [W Q_t - \nabla W \cdot F(Q)] d\Omega + \int_{\partial \Gamma} W F(Q) \cdot \boldsymbol{n} d\Gamma = 0 \text{.... (2.3)}$$

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A discretization analogue of Eqn. (2.3) over each element can be obtained by subdivided the computational domain Ω into N non-overlapping elements$\Omega = \bigcup_{j=1}^{N} \Omega_h$. By applying Eqn. (2.3) to each elementΩ_h, the semi-discrete analogue of Eqn. (2.3) over the computational grid yields:

$$\int_{\Omega_h} \left[W_h \frac{\partial Q_h}{\partial t} - \nabla W_h . F(Q_h)\right] d\Omega_h + \int_{\partial\Gamma_h} W_h\, F(Q_h).\boldsymbol{n} d\Gamma_h = 0.0 \ldots (2.4)$$

Γ_h denotes the boundary of the element Ω_h and **n** is outward vector normal to the boundary. Let Q_h and W_h represents the finite element approximation to the analytical solution Q and the test function W,respectively. Where Q_h and W_h are approximated by a piecewise polynomial function of degrees k, which is continuous within each element and discontinuous between the elements interfaces

$$Q_h(x,t) = \sum_{j=1}^{j=n} \varphi_j Q_j(t) \text{ and} W_h(x) = \sum_{j=1}^{j=n} \varphi_j W_j \ldots (2.5)$$

where n is the dimension of the polynomial spaceP^K and φ_j is the basis of the polynomial. The expansion coefficients$Q_i(t)$ and W_j denotes the degrees of freedom (DOFs) of the numerical solution and of the test function in element Ω_h, respectively. Thus, the summation in eqn. (2.4) is equivalent to the following system of n equations:

$$\int_{\Omega_h} \varphi_j\, Q_t - \nabla\varphi_j F(Q_h) d\Omega_h + \int_{\Gamma_h} \varphi_j\, F(Q_h).\boldsymbol{n} d\Gamma_h = 0 (2.6)$$

Where, $(1 \leq j \leq n)$

Since in the DG method, the discontinuities are permitted at the interfaces of elements. Because the approximated solution is discontinuous at the element boundaries, the interface flux is not uniquely defined. In this stage, the Riemann fluxes used in the Godunov finite volume method are borrowed.

The normal flux function F(Q_h).n appearing in the last terms of eqn.(2.6) is replaced by a numerical Riemann flux function Fup = F(Q_L, Q_R,n) that depends on Q_L and Q_R which are the approximated solutions of the conservative state variables Q_h at the left and right side of the element boundary, respectively. In order to guarantee consistency and conservation, the Riemann flux must satisfy:

$F(Q_L,Q_R,\mathbf{n}) = F_{up} = F(Q_h).\mathbf{n}$, and $F(Q_L,Q_R,-\mathbf{n}) = -F_{up} = -F(Q_h).\mathbf{n}$. (2.7)

In the present work, the Riemann flux is approximated by using Lax and Friedrich (LF) flux. This scheme is called discontinuous Galerkin method of degree k asgiven in the classical form, or in short notation DG (k) method. The surface and volume integrals in Eqn. (2.6) are calculated in case of DG method by using 2k and 2k+1 order accurate Gauss quadrature formulas, respectively. In order to unify the integrals (surface integral and volume integral), the totally volume integral of the upwind flux scheme for DG method is used for this purpose. This can be done by using the relation between surface and volume integrals for any vector A, which is given by the divergence theorem as:

$$\oiint_s A \cdot d\Gamma = \oiint_s A \cdot n d\Gamma = \iiint_v \nabla \cdot A dV \ldots\ldots\ldots\ldots (2.8)$$

where Γ and V are surface and volume of the problem domain. The totally volume integral DG method is accomplished by applying the divergence theorem to the last term of Eqn. (2.6) and rearrangement to give the following form

$$\int_{\Omega_h} \left[\varphi_j \frac{\partial Q_h}{\partial t} - \frac{\partial \varphi_j}{\partial x} F(Q_h) + \frac{\partial \varphi_j}{\partial x} F_{up} + \varphi_j \frac{\partial F_{up}}{\partial x}\right] d\Omega_h = 0 \ldots (2.9)$$

The Riemann or upwind flux vectors are approximated by polynomial of order k as done for the state variable in Eqn. (2.5).$F(Q_h) = \sum_{i=1}^{i=n} \varphi_i F_i(Q_h)$, $F_{up} = \sum_{i=1}^{i=n} \varphi_i F_{up,i}$the last two terms of Eqns. (2.9) can be companied into one term as follows:

$$\int_{\Omega_h} \left[\varphi_j \frac{\partial Q_h}{\partial t} - \frac{\partial \varphi_j}{\partial x} \varphi_i F_i(Q_h) + \left(\frac{\partial \varphi_j}{\partial x}\varphi_i + \varphi_j \frac{\partial \varphi_i}{\partial x}\right) F_{up,i}\right] d\Omega_h = 0 \ldots (2.10)$$

Equation (2.10) is the DG method in totally volume integral form.

2.2. Coordinate Transformation

In order to achieve an efficient implementation, all the elements are transformed from the computational space (x, y, z) into standard space(ζ, η, ξ). Consequently, all the partial derivatives with respect to the standard space are related to the partial derivative in the computational space as in the finite element methods. For one-dimensional case, the value of the x can be obtained as:

$$x = \sum_{i=1}^{i=n} x_j \varphi_j(\zeta) \ldots\ldots\ldots\ldots (2.11)$$

The derivative of x with respect$\zeta$is obtained as:

$$\frac{\partial x}{\partial \zeta} = x_\zeta = \sum_{i=1}^{i=n} x_j \frac{\partial \varphi_j(\zeta)}{\partial \zeta} \ldots\ldots\ldots (2.12)$$

The derivatives of any function with respect to the standard coordinate can be written as:

$$\frac{\partial(\ )}{\partial \zeta} = \frac{\partial(\ )}{\partial x}\frac{\partial x}{\partial \zeta} = \frac{\partial(\ )}{\partial x} x_\zeta \text{ with} |J| = |x_\zeta| \ldots\ldots (2.13)$$

Where |J| is the determinant of Jacobian matrix. In addition, the derivatives of any function with respect to physical coordinates can be written as:

$$\frac{\partial(\ )}{\partial x} = \frac{\partial(\ )}{\partial \zeta}\frac{\partial \zeta}{\partial x} = \frac{\partial(\ )}{\partial \zeta} \zeta_x, \text{ with } |J^{-1}| = |\zeta_x| \ldots\ldots (2.14)$$

From equations (2.12) and (2.14)$x_\zeta = 1/\zeta_x$, with $n_\zeta = \zeta_x/|\zeta_x| = 1$. Thus the Riemann fluxFup = F(Q_L ,Q_R, n) has no negative direction at the boundaries. By substituting into Eqn. (2.9) and rearrangement yields:

$$\int_{\Omega_h} \left[\varphi_j \frac{\partial Q_h}{\partial t} - \frac{\partial \varphi_j}{\partial \zeta}\varphi_i + \left(\frac{\partial \varphi_j}{\partial \zeta}\varphi_i + \varphi_j \frac{\partial \varphi_i}{\partial \zeta}\right)(\zeta_x F_{up,i})\right] d\Omega_h = 0 \ldots\ldots (2.15)$$

Finally, after the spatial discretization is accomplished, equation (2.14) can be written into the following form:

$$M\frac{dQ}{dt} = R(Q), \ldots\ldots\ldots\ldots (2.16)$$

where R(Q) and M are called the residual and consistent mass matrix, respectively.

2.3. Time integral

The semi-discrete equation as eqn. (2.15) can be integrated in time using explicit methods. The explicit three-stage third-order TVD Runge-Kutta scheme RK(3,3) and five-stage forth order RK(5,4) are the widely used methods given in many references among them Ref. [10] and [11] . The RK(3,3) can be expressed in the following form:

$$Q^{(1)} = Q^n + \Delta t M^{-1} R(Q^n) \ldots\ldots\ldots (2.17.a)$$

$$Q^{(2)} = \frac{3}{4}Q^{(n)} + \frac{1}{4}[Q^{(1)} + \Delta t M^{-1} R(Q^{(1)})] \ldots\ldots (2.17.b)$$

$$Q^{n+1} = \frac{1}{3}Q^n + \frac{2}{3}[Q^{(2)} + \Delta t M^{-1} R(Q^{(2)})] \ldots\ldots (2.17.c)$$

This method is linearly stable for a Courant number less than or equal to 1.

3. Numerical Results

As a preliminary test, we apply the totally volume integral discontinuous Galerkin method to Euler equations in one-dimensional cases.

3.1. Numerical tests and comparison.

We apply the TVI-DG method to one-dimensional Euler equations eqn. (2.2) in Cartesian coordinate and in cylindrical coordinate (explosion cylinder).

Example-1

The first example is the Riemann problem of Lax considering in many references among them [9]. Zero gradient boundary conditions are applied at the boundaries. The problem domain x = [-0.5, 0.5] is divided into 100 equally space elements, with the initial conditions given as

$$(\rho, u, P) = \begin{pmatrix} 0.445 & 0.698 & 3.528 & if & x \leqslant 0 \\ 0.5 & 0.0 & 0.571 & if & x > 0 \end{pmatrix}$$

The approximated solutions are constructed from polynomials of orders k from 1 to 4. The lax-Friedrich flux scheme is used to evaluate the boundary fluxes. The RK (3, 3) is used for k = 1 and 2 while RK (5, 4) is used in case of k = 3 and 4, where the RK methods are used for evaluating the time integral part. The numerical results are obtained at time t = 0.13. Figures 1 to 3 display the numerical results of the approximated density, velocity and pressure solutions by using of TVI-DG method with polynomials from 1 to 4. The modified streamline upwind Petrov-Galerkin is used to capture the nonphysical oscillation solutions due to the discontinuity of the flow. Figures 1 to 3 demonstrated that TVI-DG method is very efficient scheme in evaluating discontinuous flow problems. The problem solution consists from left rarefaction, contact and right shock waves. There is no enclosed solution of this problem. Therefore the exact solution can be obtained by using 1500 elements with approximated solution is constructed from polynomial of order k = 2.

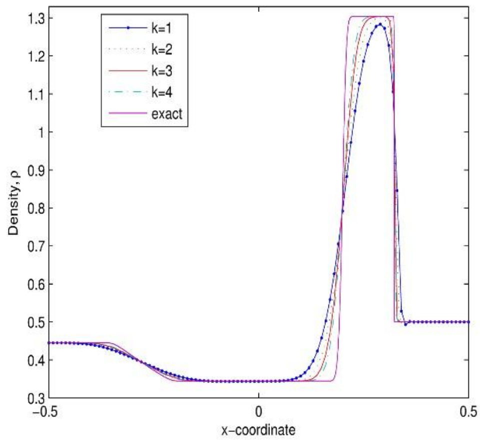

Fig. 1: Density distribution of example 1 by using TVI-DG, at time t=.13 with polynomials of order k=1 to 4.

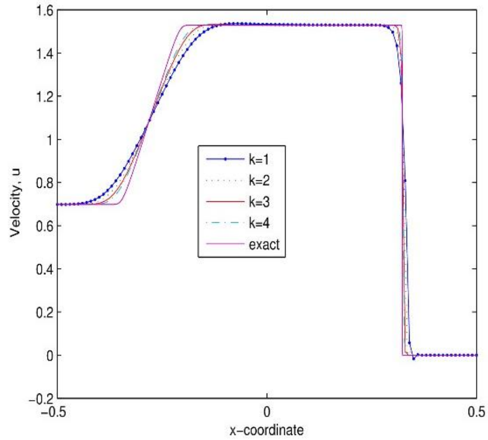

Fig. 2: Velocity distribution of example 1 by using TVI-DG, at time t=.13 with polynomials of order k=1 to 4.

Example-2

The second example is considering in [7], with the initial conditions given as:

$$(\rho, u, P) = \begin{pmatrix} 1.0 & -2.0 & 0.4 & if & 0 \leqslant x \leqslant 0.5 \\ 1.0 & 2.0 & 0.4 & if & 0.5 < x \leqslant 1 \end{pmatrix}$$

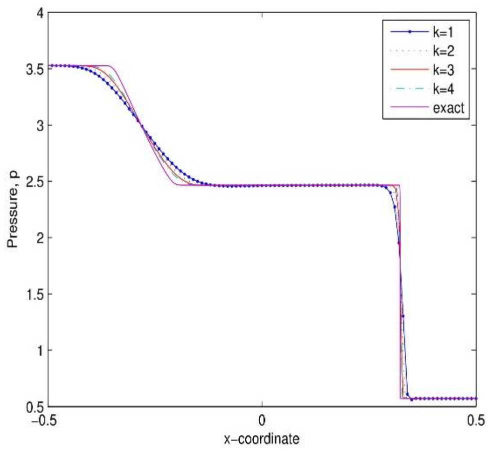

Fig. 3: Pressure distribution of example 1 by using TVI-DG, at time t=.13 with polynomials of order k=1 to 4.

The reflecting boundary conditions are applied at the boundaries. The problem domain x= [0, 1] is divided into 200 equally space elements. The approximated solutions are constructed from polynomials of orders k from 2 to 4. The lax-Friedrich flux scheme is used to evaluate the boundary fluxes. The RK (3, 3) is used for k = 2 while RK (5, 4) is used in case of k = 3 and 4. The numerical results are obtained at time t = 0.15. Figures 4 to 6 display the numerical results of density, velocity and pressure solutions by using TVI-DG method with polynomials from 2 to 4. The modified streamline upwind Petrov-Galerkin is used to capture the nonphysical oscillation solutions due to the discontinuity of the flow. Figures 4 to 6 reveal that TVI-DG method is very efficient scheme in evaluating discontinuous flow problems.

The solution is consisted from left rarefaction, contact and right rarefaction waves. The solutions demonstrated that the pressure is very small closed to vacuum pressure with low-density profile. There is no enclosed solution of this problem. Therefore, the exact solution can be obtained by using 800 element with approximated solution is constructed from polynomial of order k = 5.

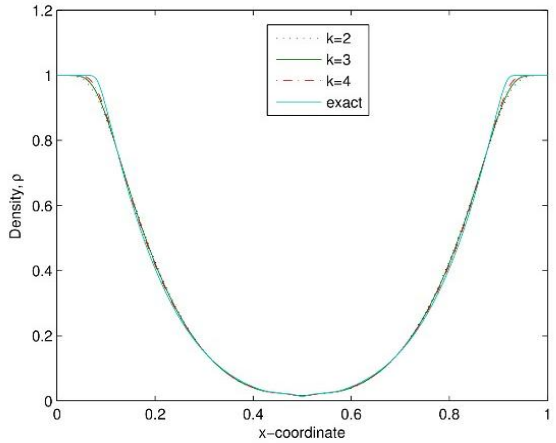

Fig. 4: Density distribution of example 2 by using TVI-DG, at time t=.15 with polynomials of order k=2 to 4.

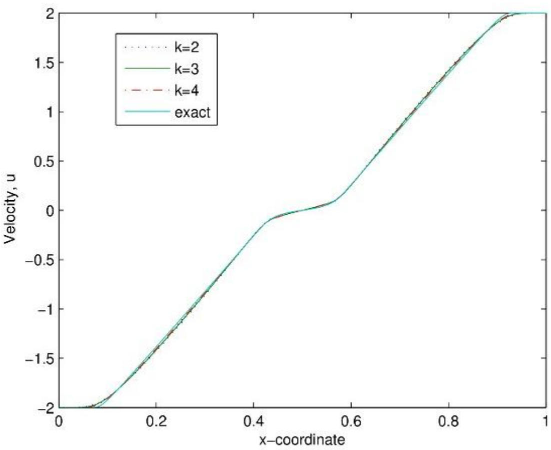

Fig. 5: Velocity distribution of example 2 by using TVI-DG, at time t=.15 with polynomials of order k=2 to 4.

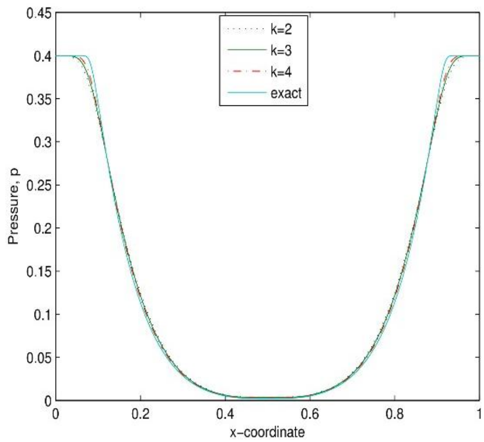

Fig. 6: Pressure distribution of example 2 by using TVI-DG, at time t=.15 with polynomials of order k=2 to 4

Example-3

The third example is a shock sod tube in cylindrical coordinate given in [8]. The two dimensional cylindrical Euler equations can be reduced into one-dimensional Euler equations with source term as follows:

$$\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial r} = S(Q),$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}, F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{bmatrix} \text{ and } S(Q) = \frac{-\alpha}{r} \begin{bmatrix} \rho u \\ \rho u^2 \\ u(\rho E + p) \end{bmatrix}$$

with r is the radial distance from origin and α = 1 for cylindrical symmetric flows, with the initial conditions are given as:

$$(\rho, u, P) = \begin{pmatrix} 1.0 & 0.0 & 1.0 & if & 0 \leqslant x \leqslant 0.4 \\ 0.125 & 0.0 & 0.1 & if & 0.4 < x \leqslant 1 \end{pmatrix}$$

The reflecting boundary conditions are applied at the boundaries. The problem domain x= [0, 1] is divided into 200 equally space elements. The approximated solutions are constructed from polynomials of orders k from 1 to 4. The lax-Friedrich flux scheme is used to evaluate the boundary fluxes. The RK (3, 3) is used for k = 1 and 2 while RK (5, 4) is used in case of k = 3 and 4. The numerical results are obtained at time t = 0.2. Figures 7 to 9 display the numerical results of the approximated density, velocity and pressure solutions by using TVI-DG method with polynomials of orders from 1 to 4. The modified streamline upwind Petrov-Galerkin is used to capture the nonphysical oscillation solutions due to the discontinuity of the flow. Figures 7 to 9 demonstrated that TVI-DG method is very efficient scheme in evaluating discontinuous flow problems. The solution consists of a circular shock wave propagating from the origin, followed by a circular contact discontinuity traveling in the same direction, and a circular expansion wave moving towards the origin. The solutions are used as validating for the full model of two-dimensional problem.

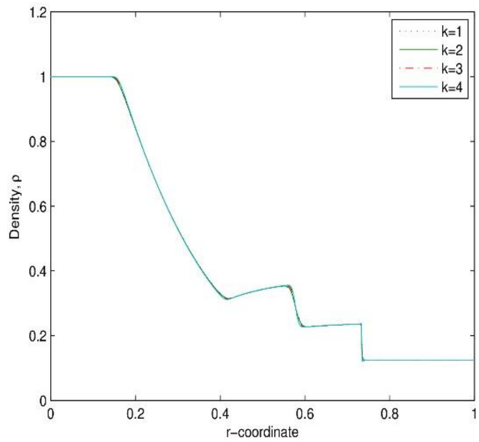

Fig.7: Density distribution of example 3 by using TVI-DG, at time t= 0.2 with polynomials of order k=1 to 4.

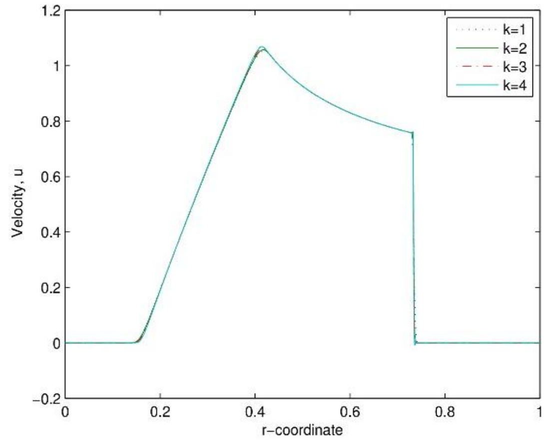

Fig. 8: Velocity distribution of example 3 by using TVI-DG, at time t= 0.2 with polynomials of order k=1 to 4

4. Discussion and conclusions

The transformation of the boundaries integrals into the volume integrals is introduced in this work under the named TVI-DG method. Thus, there is no integration of the test function at the boundaries as in the classical DG method. The totally volume integral discontinuous Galerkin method is used to solve system of equations (Euler equations). The numerical finding presented that the TVI-DG scheme is very efficient for solving problems with shock waves.

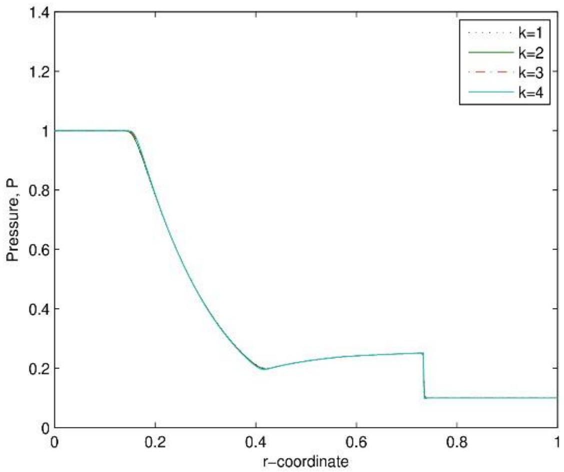

Fig. 9: Pressure distribution of example 3 by using TVI-DG, at time t= 0.2 with polynomials of order k=1 to 4.

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