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Exact Solution of Time-Fractional Cauchy Reaction-Diffusion Equation Using Jafari Transform

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Abstract:

Fractional calculus techniques are extensively employed in science and engineering including the new Jafari transform iterative method (NJTIM), which has not been studied by researchers using the Caputo fractional derivative. The new methodology demonstrates how two strong methods, the new iterative method and the Jafari transform method, may be combined and applied to provide exact solutions to fractional partial differential equations. Three distinct examples are also provided to demonstrate the accuracy and efficacy of my methodology.

Keywords: Fractional Cauchy reaction-diffusion equation, Caputo fractional operator, Jafari transform, New iterative method.

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الحل الدقيق لمعادلة كوشي ذات الرتبة الكسرية للتفاعل الانتشار باستخدام تحويل جعفري

د. أحمد عبد الله حسين مطول [1].

الملخص:

تُستخدم تقنيات حساب التفاضل والتكامل الكسري على نطاق واسع في العلوم والهندسة، بما في ذلك طريقة تحويل جعفري التكرارية الجديدة (NJTIM)، التي لم يدرسها الباحثون الذين يستخدمون مشتقة كابوتو الكسرية. توضح المنهجية الجديدة كيفية دمج طريقتين قويتين، الطريقة التكرارية الجديدة وطريقة تحويل جعفري، وتطبيقهما لتوفير حلول دقيقة للمعادلات التفاضلية الجزئية ذات الرتبة الكسرية. كما تم تقديم ثلاثة أمثلة مميزة لإثبات دقة وفعالية منهجيتي.

الكلمات المفتاحية: معادلة كوشي الكسرية للتفاعل والانتشار، عامل كابوتو الكسري، تحويل جعفري، الطريقة التكرارية الجديدة.

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1. Introduction:

In the last two decades, physicists, mathematicians, and engineers have focused heavily on fractional differential equations [1-4]. Fractional derivatives may be used to simulate a wide range of trans disciplinary issues. The majority of fractional differential equations, however, are extremely difficult for us to solve exactly, necessitating the use of numerical and approximation techniques. Both linear and nonlinear fractional differential equations have been solved thus far using a variety of techniques. As an illustration, the homotopy perturbation method [5-6], the homotopy analysis method [7], the Adomain decomposition method [8], the variational iteration method [9-10], the variational homotopy perturbation method [11], the homotopy perturbation Sumudu transform method [12], and Sumudu variational iteration method [13]. One of the most significant fractional partial differential equations is the time-fractional Cauchy reaction–diffusion equation [14-16]. Numerous linear and nonlinear systems in physics, biology, and engineering may be described by the time-fractional Cauchy reaction diffusion equations [17]. Using the homotopy perturbation transform method with the aid of the Laplace transform, the approximate solutions of the time fractional Cauchy reaction–diffusion equations have been derived in the reference [18]. Furthermore, using a new iterative method with the aid of the Sumudu transform, the approximate solutions of the time fractional Cauchy reaction–diffusion equations have been derived in the reference [19]. In reference [20], NIM is proposed to solve linear and nonlinear integral and differential equations. NIM is simple to apply and gives superior and more accurate final results than other methods. This paper presents a novel approach to solving time-fractional Cauchy reaction-diffusion equations, which I term the New Jafari Transform Iterative Method (NJTIM). The approach is based on the Jafari transform [21] and can yield analytical and numerical solutions. Thanks to [19-20], my iterative method expands NIM in a novel way. This new approach, which the researcher suggested, has the benefit of making the computation straightforward and extremely accurate in order to estimate the exact result.

2. Fundamentals of fractional calculus

This section provides some fundamental concepts and characteristics of the Jafari transform and fractional calculus that will be used in this paper:

Definition 1. [2] The left sided Riemann-Liouville fractional integral of order $\theta \geq 0$, of a function $\xi \in C_{\sigma}, \sigma \geq -1$ is defined as:

$$J^{\theta}\,\xi(\vartheta,\tau)=\frac{1}{\Gamma(\theta)}\int_{0}^{\tau}(t-s)^{\theta-1}\,\xi(\vartheta,\tau)\,d\,s, \qquad \text{.......(1)}$$

where $\theta>0, \tau>0$ and $\Gamma(\theta)$ is the Gamma function.

Definition 2. Let $\xi \in C_{m}^{n}, n \in \mathrm{N}\cup\{0\}$. The left sided Caputo fractional derivative of ξ in the Caputo sense is defined by [22] as follows:

$$D_{\tau}^{\theta}\xi(\vartheta,\tau)=\begin{cases}\dfrac{1}{\Gamma(n-\theta)}\displaystyle\int_{0}^{\tau}(t-s)^{n-\theta-1}\,\xi^{(n)}(\vartheta,\tau)\;d\,s, & n-1<\theta\leq n,\\ D_{\tau}^{\theta}\,\xi(\vartheta,\tau), & \theta=n,\end{cases} \qquad \text{.......(2)}$$

Also one has the following properties:

$D^{\theta}\,C=0,$ (C is constant),

$$D^{\theta}\tau^{\gamma}=\begin{cases}\dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\theta+1)}\tau^{\gamma-\theta}, & \gamma>\theta-1,\\ 0, & \gamma\leq\theta-1,\end{cases} \qquad \text{.......(3)}$$

$$J^{\theta}D^{\theta}\xi(\vartheta,\tau)=\xi(\vartheta,\tau)-\sum_{k=0}^{m-1}\xi^{(k)}(\vartheta,0)\frac{\tau^{k}}{k\,!},\quad m-1<\theta\leq m, \qquad \text{.......(4)}$$

$$D^{\theta}J^{\theta}\xi(\vartheta,\tau)=\xi(\vartheta,\tau).$$

Definition 3. [24-25] In fractal space, the Mittag-Leffler function is defined as:

$$\mathrm{E}_{\theta}\left(\tau^{\theta}\right)=\sum_{j=0}^{\infty}\frac{\tau^{j\,\theta}}{\Gamma\left(j\,\theta+1\right)},\quad 0<\theta\leq 1, \qquad \text{.......(5)}$$

Definition 4: [21,26] The Jafari transform is defined over a group of functions B.

$$B = \left\{ \xi(\tau) / \exists C, n_1, n_2 > 0, \left|\xi(\tau)\right| < C\ E\left(\frac{|\tau|}{n_i}\right), if\ \ \tau \in (-1)^i \times [0,\infty), i = 1,2 \right\}. \qquad ..(6)$$

By the following integral

$$\Upsilon[\xi(\tau);s] = T(s) = \varpi(s)\int_0^\infty \mathrm{e}^{-\omega(s)\tau}\ \ \xi(\tau)\ d\tau, \ldots\ldots (7)$$

where $\tau \geq 0, \varpi(s) \neq 0,$ and $\omega(s)$ are positive real functions.

Some special properties of the Jafari transform are as follows:

$$\Upsilon[1] = \frac{\varpi(s)}{\omega(s)}, \qquad \ldots\ldots (8$$

$$\Upsilon[\tau] = \frac{\varpi(s)}{\omega^2(s)},$$

$$\Upsilon[\tau^{j\theta}] = \frac{\Gamma(\theta+1)\,\varpi(s)}{\omega^{j\theta+1}(s)}, \theta > 0.$$

Definition 5: [21, 26] The Jafari transform for the Caputo fractional operator for $m-1 < \theta \leq m$ is defined as:

$$\Upsilon\left[{}_0^C D_\tau^\theta\, \xi(\vartheta,\tau)\right] = \omega^\theta(s) E\left[\xi(\vartheta,\tau)\right] - \varpi(s)\sum_{k=0}^{m-1} \omega^{\theta-1-k}(s)\,\xi^{(k)}(\vartheta,0). \ldots\ldots (9)$$

3. Analysis of NJTIM

In Caputo operator sense, consider the following, a general nonlinear fractional partial differential equation with the initial condition:

$${}_0^C D_\tau^\theta\, \xi(\vartheta,\ \tau) + L\,\xi(\vartheta,\ \tau) + N\,\xi(\vartheta,\ \tau) = g(\vartheta,\ \tau), \quad 0 < \theta \leq 1. \ldots\ldots (10)$$

$$\xi(\vartheta,\ 0) = \xi_0(\vartheta), \ldots\ldots (11)$$

where ${}_0^C D_\tau^\theta$ is the θ order fractional Caputo derivative, ξ is the unknown function, L and N are linear and nonlinear operators, and g is the source term.

By applying the Jafari transform to Eq. (10), the result is obtained

$$\Upsilon\left[{}_{0}^{C}D_{\tau}^{\theta}\,\xi(\vartheta,\,\tau)\right]=\Upsilon\left[g\left(\vartheta,\,\tau\right)-L\,\xi(\vartheta,\,\tau)-N\,\xi(\vartheta,\,\tau)\right], \tag{12}$$

from Definition 5 and Eq. (12)

$$\Upsilon\left[\xi(\vartheta,\,\tau)\right]=\frac{\varpi\left(s\right)}{\omega\left(s\right)}\xi\left(\vartheta,0\right)+\omega^{-\theta}\left(s\right)\Upsilon\left[g\left(\vartheta,\,\tau\right)-L\,\xi(\vartheta,\,\tau)-N\,\xi(\vartheta,\,\tau)\right]. \tag{13}$$

By using the inverse Jafari transform, the following result is obtained

$$\xi(\vartheta,\,\tau)=\Upsilon^{-1}\left[\frac{\varpi\left(s\right)}{\omega\left(s\right)}\xi\left(\vartheta,0\right)+\omega^{-\theta}\left(s\right)\Upsilon\left[g\left(\vartheta,\,\tau\right)-L\,\xi(\vartheta,\,\tau)-N\,\xi(\vartheta,\,\tau)\right]\right]. \tag{14}$$

Then suppose that:

$$\begin{cases} f\left(\vartheta,\,\tau\right)=\Upsilon^{-1}\left[\dfrac{\varpi\left(s\right)}{\omega\left(s\right)}\xi\left(\vartheta,0\right)+\omega^{-\theta}\left(s\right)\,\Upsilon\left[g\left(\vartheta,\,\tau\right)\right]\right], \\ L\,\xi(\vartheta,\,\tau)=-\Upsilon^{-1}\left[\omega^{-\theta}\left(s\right)\,\Upsilon\left[L\,\xi(\vartheta,\,\tau)\right]\right], \\ N\,\xi(\vartheta,\,\tau)=-\Upsilon^{-1}\left[\omega^{-\theta}\left(s\right)\,\Upsilon\left[N\,\xi(\vartheta,\,\tau)\right]\right]. \end{cases} \tag{15}$$

Thus, Eq. (15) may be expressed in the following form:

$$\xi(\vartheta,\,\tau)=f\left(\vartheta,\,\tau\right)+L\,\xi(\vartheta,\,\tau)+N\,\xi(\vartheta,\,\tau). \tag{16}$$

It is possible to express the solution to Eq. (16) in series form.

$$\xi(\vartheta,\,\tau)=\sum_{j=0}^{\infty}\xi_j\left(\vartheta,\,\tau\right). \tag{17}$$

The decomposition of the nonlinear term $N\left[\xi(\vartheta,\,\tau)\right]$ by a new iterative method as follows

$$N\left[\xi(\vartheta,\,\tau)\right]=\sum_{j=0}^{\infty}A_j=N\left[\xi_0(\vartheta,\,\tau)\right]+\sum_{j=1}^{\infty}\left(N\left(\sum_{i=0}^{j}\xi_i\right)-N\left(\sum_{i=0}^{j-1}\xi_i\right)\right). \tag{18}$$

Equations (17), (18) and (15) can be substituted into Eq. (16) to produce

$$\sum_{j=0}^{\infty}\xi_j\left(\vartheta,\,\tau\right)=f\left(\vartheta,\,\tau\right)-\Upsilon^{-1}\left[\omega^{-\theta}\,\Upsilon\left[L\,\sum_{j=0}^{\infty}\xi_j\left(\vartheta,\,\tau\right)+\sum_{j=0}^{\infty}A_j\right]\right]. \tag{19}$$

Based on the comparison in Eq. (19), we can attain the following:

$$\xi_0 = f\left(\vartheta,\tau\right) = \Upsilon^{-1}\left[\frac{\varpi\left(s\right)}{\omega\left(s\right)}\xi\left(\vartheta,0\right) + \omega^{-\theta}\left(s\right)\Upsilon\left[g\left(\vartheta,\tau\right)\right]\right],$$

$$\xi_1 = -\Upsilon^{-1}\left[\omega^{-\theta}\left(s\right)\Upsilon\left[\xi_0 + A_0\right]\right],$$

$$\xi_2 = -\Upsilon^{-1}\left[\omega^{-\theta}\left(s\right)\Upsilon\left[\xi_1 + A_1\right]\right],$$

$$\xi_{j+1} = -\Upsilon^{-1}\left[\omega^{-\theta}\left(s\right)\Upsilon\left[\xi_j + A_j\right]\right], \quad j = 0,1,2,\dots \,. \qquad \text{.................................}(20)$$

The approximate j-term solution for Eq. (20) may be found by

$$\xi(\vartheta,\tau) = \sum_{j=0}^{\infty}\xi_j(\vartheta,\tau). \qquad \text{..}(21)$$

4. Applications

This section evaluates the applicability and accuracy of the fractional new Jafari transform iterative method through three examples.

Example 1: Consider the following time-fractional Cauchy reaction–diffusion equation in Caputo operator sense.

$${}_0^C D_\tau^\theta\, \xi(\vartheta,\tau) = \xi_{\vartheta\vartheta}(\vartheta,\tau) - \xi(\vartheta,\tau), \quad 0 < \theta \le 1, \qquad \text{..}(22)$$

Subject to the initial condition

$$\xi(\vartheta,0) = e^{-\vartheta} + \vartheta. \qquad \text{..}(23)$$

The Jafari transform is applied to Eq. (22), and the initial condition (23) is utilized to obtain

$$\Upsilon\left[\xi(\vartheta,\tau)\right] = \frac{\varpi\left(s\right)}{\omega\left(s\right)}\left(e^{-\vartheta} + \vartheta\right) + \omega^{-\theta}\left(s\right)\Upsilon\left[\xi_{\vartheta\vartheta}(\vartheta,\tau) - \xi(\vartheta,\tau)\right]. \qquad \text{........................}(24)$$

The inverse Jafari transform is applied to Eq. (24), and the result is obtained

$$\xi(\vartheta,\tau) = \Upsilon^{-1}\left[\frac{\varpi\left(s\right)}{\omega\left(s\right)}\left(e^{-\vartheta} + \vartheta\right) + \omega^{-\theta}\left(s\right)\Upsilon\left[\xi_{\vartheta\vartheta}(\vartheta,\tau) - \xi(\vartheta,\tau)\right]\right]. \qquad \text{...................}(25)$$

Based on the NJTIM, the result is obtained

$$\begin{cases} f(\vartheta,\tau)=\xi(\vartheta,0)=\Upsilon^{-1}\left[\dfrac{\varpi(s)}{\omega(s)}\left(e^{-\vartheta}+\vartheta\right)\right], \\ L\left[\xi(\vartheta,\tau)\right]=\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{\vartheta\vartheta}(\vartheta,\tau)-\xi(\vartheta,\tau)\right]\right]. \end{cases} \qquad \text{...(26)}$$

Equations (17) and (26) can be substituted into Eq. (16) to produce

$$\sum_{j=0}^{\infty}\xi_j(\vartheta,\tau)=\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega(s)}\left(e^{-\vartheta}+\vartheta\right)\right]+\Upsilon^{-1}\left[\omega^{-\theta}\,\Upsilon\left[\sum_{j=0}^{\infty}\left[\xi_{j\,\vartheta\vartheta}(\vartheta,\tau)-\xi_j(\vartheta,\tau)\right]\right]\right].$$

....(27)

The following findings are acquired by iteration:

$$\xi_0=e^{-\vartheta}+\vartheta,$$

$$\begin{aligned}\xi_1&=\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{0\,\vartheta\vartheta}-\xi_0\right]\right]\\ &=\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[e^{-\vartheta}-e^{-\vartheta}-\vartheta\right]\right]\\ &=-\vartheta\,\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega^{\theta+1}(s)}\right]\\ &=-\frac{\vartheta\,\tau^{\theta}}{\Gamma(\theta+1)},\end{aligned}$$

$$\begin{aligned}\xi_2&=\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{1\,\vartheta\vartheta}-\xi_1\right]\right]\\ &=\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\frac{\vartheta\,\tau^{\theta}}{\Gamma(\theta+1)}\right]\right]\\ &=\vartheta\,\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega^{2\theta+1}(s)}\right]\\ &=\frac{\vartheta\,\tau^{2\theta}}{\Gamma(2\theta+1)},\end{aligned}$$

$$\begin{aligned}\xi_3&=\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{2\,\vartheta\vartheta}-\xi_2\right]\right]\\ &=-\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\frac{\vartheta\,\tau^{2\theta}}{\Gamma(2\theta+1)}\right]\right]\\ &=-\vartheta\,\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega^{3\theta+1}(s)}\right]\\ &=-\frac{\vartheta\,\tau^{3\theta}}{\Gamma(3\theta+1)},\end{aligned}$$

$$\vdots$$

$$\begin{aligned}\xi_{j+1}&=\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{j\,\vartheta\vartheta}-\xi_j\right]\right]\\ &=\frac{(-1)^j\,\vartheta\,\tau^{j\theta}}{\Gamma(j\,\theta+1)}.\end{aligned}$$

....(28)

Thus, the solution of Eq. (28) is given as

$$\begin{aligned}\xi(\vartheta,\tau) &= \xi_0(\vartheta,\tau)+\xi_1(\vartheta,\tau)+\xi_2(\vartheta,\tau)+\cdots\\ &= e^{-\vartheta}+\vartheta\left(1-\frac{\tau^{\theta}}{\Gamma(\theta+1)}+\frac{\tau^{2\theta}}{\Gamma(2\theta+1)}-\frac{\tau^{3\theta}}{\Gamma(3\theta+1)}+\cdots+\frac{\tau^{j\theta}}{\Gamma(j\theta+1)}+\cdots\right)\\ &= e^{-\vartheta}+\vartheta\sum_{j=0}^{\infty}\frac{(-1)^{j}\,\tau^{j\theta}}{\Gamma(j\theta+1)}\\ &= e^{-\vartheta}+\vartheta\,\mathrm{E}_{\theta}\left(-\tau^{\theta}\right). \qquad \text{......(29)}\end{aligned}$$

If $\theta=1$ is substituted into Eq. (29), the exact solution is obtained

$$\xi(\vartheta,\tau)=e^{-\vartheta}+\vartheta\,\mathrm{E}(-\tau)=e^{-\vartheta}+\vartheta\,e^{-\tau}.$$

This is entirely consistent with the findings in Reference [18-19].

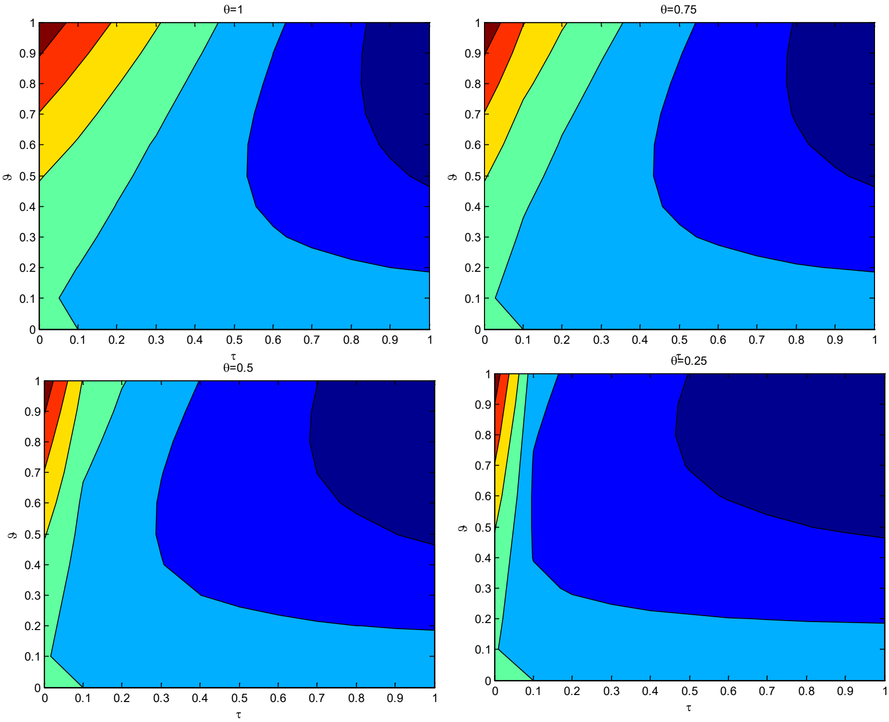

Fig 1. The three-dimensional contours solution of $\xi(\vartheta,\tau)$.

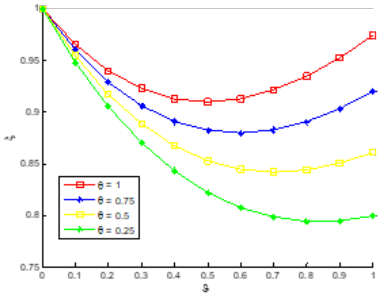

Fig 2. The two-dimensional graphical visual of $\xi(\vartheta,\ \tau)$ with $\tau = 0.5$ at multiple values of θ.

Example 2: Consider the following time-fractional Cauchy reaction–diffusion equation in Caputo operator sense.

$$ {}_{0}^{C}D_{\tau}^{\theta}\,\xi(\vartheta,\ \tau)=\xi_{\vartheta\vartheta}(\vartheta,\ \tau)-\left(1-4\vartheta^{2}\right)\xi(\vartheta,\ \tau),\quad 0<\theta\le 1, \qquad \text{..........(30)} $$

Subject to the initial condition

$$ \xi(\vartheta,\ 0)=e^{\vartheta^{2}}. \qquad \text{..........(31)} $$

The Jafari transform is applied to Eq. (30), and the initial condition (31) is utilized to obtain

$$ \Upsilon\left[\xi(\vartheta,\ \tau)\right]=\frac{\varpi(s)}{\omega(s)}\left(e^{\vartheta^{2}}\right)+\omega^{-\theta}(s)\,\Upsilon\left[\xi_{\vartheta\vartheta}(\vartheta,\ \tau)-\left(1-4\vartheta^{2}\right)\xi(\vartheta,\ \tau)\right]. \qquad \text{..........(32)} $$

The inverse Jafari transform is applied to Eq. (32), and the result is obtained

$$ \xi(\vartheta,\ \tau)=\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega(s)}\left(e^{\vartheta^{2}}\right)+\omega^{-\theta}(s)\,\Upsilon\left[\xi_{\vartheta\vartheta}(\vartheta,\ \tau)-\left(1-4\vartheta^{2}\right)\xi(\vartheta,\ \tau)\right]\right]. \qquad \text{......(33)} $$

Based on the NJTIM, the result is obtained

$$ \begin{cases} f(\vartheta,\ \tau)=\xi(\vartheta,\ 0)=\Upsilon^{-1}\left[\dfrac{\varpi(s)}{\omega(s)}\left(e^{\vartheta^{2}}\right)\right], \\ L\left[\xi(\vartheta,\ \tau)\right]=\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{\vartheta\vartheta}(\vartheta,\ \tau)-\left(1-4\vartheta^{2}\right)\xi(\vartheta,\ \tau)\right]\right]. \end{cases} \qquad \text{..........(34)} $$

Equations (17) and (34) can be substituted into Eq. (16) to produce

$$ \sum_{j=0}^{\infty}\xi_{j}(\vartheta,\ \tau)=\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega(s)}\left(e^{\vartheta^{2}}\right)\right]+\Upsilon^{-1}\left[\omega^{-\theta}\,\Upsilon\left[\sum_{j=0}^{\infty}\left[\xi_{j\,\vartheta\vartheta}(\vartheta,\ \tau)-\left(1-4\vartheta^{2}\right)\xi_{j}(\vartheta,\ \tau)\right]\right]\right]. \qquad \text{.......(35)} $$

The following findings are acquired by iteration:

$$\xi_0 = e^{\vartheta^2},$$

$$\begin{aligned}
\xi_1 &= \Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{0\,\vartheta\vartheta}-\left(1-4\vartheta^2\right)\xi_0\right]\right]\\
&= \Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[4\vartheta^2 e^{\vartheta^2}+2e^{\vartheta^2}-\left(1-4\vartheta^2\right)e^{\vartheta^2}\right]\right]\\
&= e^{\vartheta^2}\,\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega^{\theta+1}(s)}\right]\\
&= \frac{e^{\vartheta^2}\tau^{\theta}}{\Gamma(\theta+1)},
\end{aligned}$$

$$\begin{aligned}
\xi_2 &= \Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{1\,\vartheta\vartheta}-\left(1-4\vartheta^2\right)\xi_1\right]\right]\\
&= \Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\frac{e^{\vartheta^2}\tau^{\theta}}{\Gamma(\theta+1)}\right]\right]\\
&= e^{\vartheta^2}\,\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega^{2\theta+1}(s)}\right]\\
&= \frac{e^{\vartheta^2}\tau^{2\theta}}{\Gamma(2\theta+1)},
\end{aligned}$$

$$\begin{aligned}
\xi_3 &= \Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{2\,\vartheta\vartheta}-\left(1-4\vartheta^2\right)\xi_2\right]\right]\\
&= \Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\frac{e^{\vartheta^2}\tau^{2\theta}}{\Gamma(2\theta+1)}\right]\right]\\
&= e^{\vartheta^2}\,\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega^{3\theta+1}(s)}\right]\\
&= \frac{e^{\vartheta^2}\tau^{3\theta}}{\Gamma(3\theta+1)},
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
\xi_{j+1} &= \Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{j\,\vartheta\vartheta}-\left(1-4\vartheta^2\right)\xi_j\right]\right]\\
&= \frac{e^{\vartheta^2}\tau^{j\theta}}{\Gamma(j\theta+1)}.
\end{aligned}$$

....(36)

Thus, the solution of Eq. (36) is given as

$$\begin{aligned}\xi(\vartheta,\tau) &= \xi_0(\vartheta,\tau)+\xi_1(\vartheta,\tau)+\xi_2(\vartheta,\tau)+\cdots\\ &= e^{\vartheta^2}\left(1+\frac{\tau^{\theta}}{\Gamma(\theta+1)}+\frac{\tau^{2\theta}}{\Gamma(2\theta+1)}+\frac{\tau^{3\theta}}{\Gamma(3\theta+1)}+\cdots\frac{\tau^{j\theta}}{\Gamma(j\theta+1)}+\cdots\right)\\ &= e^{\vartheta^2}\sum_{j=0}^{\infty}\frac{\tau^{j\theta}}{\Gamma(j\theta+1)}\\ &= e^{\vartheta^2}\mathrm{E}_{\theta}\left(\tau^{\theta}\right).\end{aligned} \quad \text{........(37)}$$

If $\theta=1$ is substituted into Eq. (37), the exact solution is obtained

$$\xi(\vartheta,\tau)=e^{\vartheta^2}\mathrm{E}(\tau)=e^{\vartheta^2+\tau}.$$

This is entirely consistent with the findings in Reference [18-19].

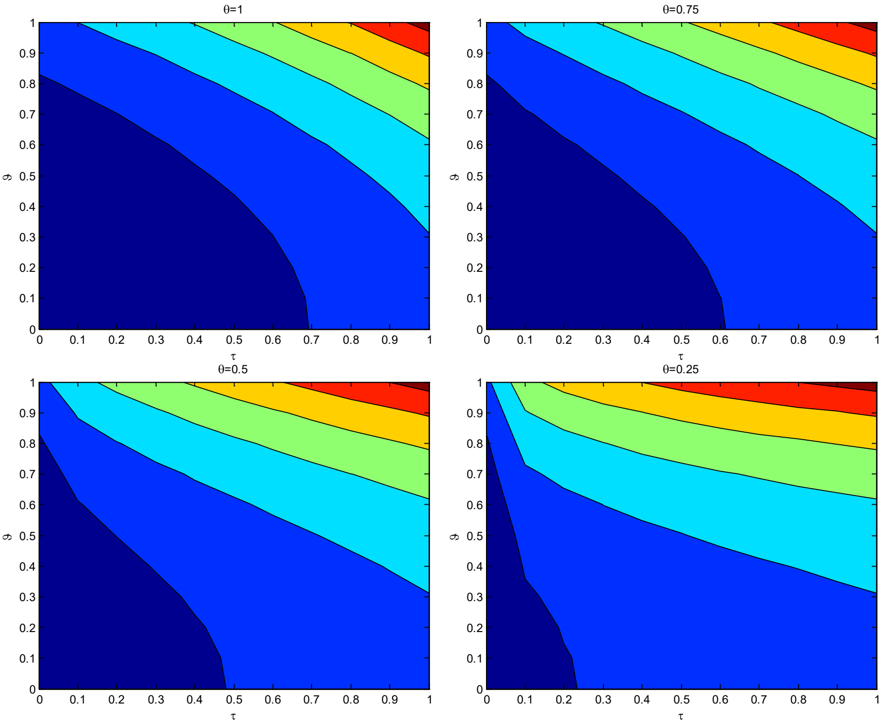

Fig 3. The three-dimensional contours solution of $\xi(\vartheta,\tau)$.

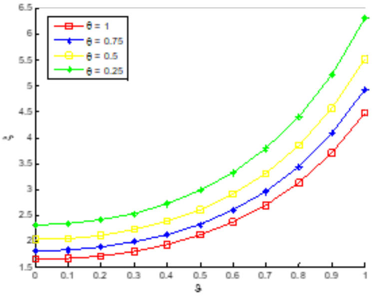

Fig 4. The two-dimensional graphical visual of $\xi(\vartheta,\,\tau)$ with $\tau=0.5$ at multiple values of θ.

Example 3: Consider the following time-fractional Cauchy reaction–diffusion equation in Caputo operator sense.

$$ {}_{0}^{C}D_{\tau}^{\theta}\,\xi(\vartheta,\tau)=\xi_{\vartheta\vartheta}(\vartheta,\tau)-\xi_{\vartheta}(\vartheta,\tau)+\xi_{\vartheta\vartheta}(\vartheta,\tau)\;\xi(\vartheta,\tau)-\xi^{2}(\vartheta,\tau)+\xi(\vartheta,\tau),\quad 0<\theta\le 1 \qquad \text{............(38)} $$

Subject to the initial condition

$$ \xi(\vartheta,0)=e^{\vartheta}. \qquad \text{.............(39)} $$

The Jafari transform is applied to Eq. (38), and the initial condition (39) is utilized to obtain

$$ \Upsilon\left[\xi(\vartheta,\tau)\right]=\frac{\varpi(s)}{\omega(s)}\left(e^{\vartheta}\right)+\omega^{-\theta}(s)\,\Upsilon\left[\begin{array}{c}\xi_{\vartheta\vartheta}(\vartheta,\tau)-\xi_{\vartheta}(\vartheta,\tau)+\xi_{\vartheta\vartheta}(\vartheta,\tau)\;\xi(\vartheta,\tau)\\-\xi^{2}(\vartheta,\tau)+\xi(\vartheta,\tau)\end{array}\right]. \qquad \text{...........(40)} $$

The inverse Jafari transform is applied to Eq. (40), and the result is obtained

$$ \xi(\vartheta,\tau)=\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega(s)}\left(e^{\vartheta}\right)+\omega^{-\theta}(s)\,\Upsilon\left[\begin{array}{c}\xi_{\vartheta\vartheta}(\vartheta,\tau)-\xi_{\vartheta}(\vartheta,\tau)+\xi_{\vartheta\vartheta}(\vartheta,\tau)\;\xi(\vartheta,\tau)\\-\xi^{2}(\vartheta,\tau)+\xi(\vartheta,\tau)\end{array}\right]\right]. \qquad \text{.....(41)} $$

5.Based on the NJTIM, the result is obtained

$$ \begin{cases} f(\vartheta,\tau)=\xi(\vartheta,0)=\Upsilon^{-1}\left[\dfrac{\varpi(s)}{\omega(s)}\left(e^{\vartheta}\right)\right],\\ L\left[\xi(\vartheta,\tau)\right]=\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{\vartheta\vartheta}(\vartheta,\tau)-\xi_{\vartheta}(\vartheta,\tau)+\xi(\vartheta,\tau)\right]\right],\\ N\left[\xi(\vartheta,\tau)\right]=\Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{\vartheta\vartheta}(\vartheta,\tau)\xi(\vartheta,\tau)-\xi^{2}(\vartheta,\tau)\right]\right]. \end{cases} \qquad \text{...............(42)} $$

The decomposition of the nonlinear term $N\left[\xi_{\vartheta\vartheta}(\vartheta,\tau)\xi(\vartheta,\tau)-\xi^{2}(\vartheta,\tau)\right]$ by a new iterative method as follows

$$A_0 = \xi_{0\vartheta\vartheta}\, \xi_0 - \xi_0^{\,2},$$
$$A_1 = \xi_{1\vartheta\vartheta}(\vartheta, \tau)\, \xi_1 - \xi_1^{\,2},$$
$$A_2 = \xi_{2\vartheta\vartheta}(\vartheta, \tau)\, \xi_2 - \xi_2^{\,2},$$
$$\vdots$$
$$A_j = \xi_{j\vartheta\vartheta}(\vartheta, \tau)\, \xi_j - \xi_j^{\,2}, \quad j = 0,1,2,\dots \;\text{......}(43)$$

Equations (17), (43) and (42) can be substituted into Eq. (16) to produce

$$\sum_{j=0}^{\infty} \xi_j(\vartheta, \tau) = \Upsilon^{-1}\left[\frac{\varpi(s)}{\omega(s)}\left(e^{\vartheta}\right)\right] + \Upsilon^{-1}\left[\omega^{-\theta}\, \Upsilon\left[\sum_{j=0}^{\infty}\left[\xi_{j\vartheta\vartheta}(\vartheta, \tau) - \xi_{j\vartheta}(\vartheta, \tau) + \xi_j(\vartheta, \tau) + A_j\right]\right]\right]. \;\text{......}(44)$$

The following findings are acquired by iteration:

$$\xi_0 = e^{\vartheta^2},$$

$$\begin{aligned}
\xi_1 &= \Upsilon^{-1}\left[\omega^{-\theta}(s)\, \Upsilon\left[\xi_{0\vartheta\vartheta}(\vartheta, \tau) - \xi_{0\vartheta}(\vartheta, \tau) + \xi_0(\vartheta, \tau) + A_0\right]\right] \\
&= \Upsilon^{-1}\left[\omega^{-\theta}(s)\, \Upsilon\left[e^{\vartheta}\right]\right] \\
&= e^{\vartheta}\, \Upsilon^{-1}\left[\frac{\varpi(s)}{\omega^{\theta+1}(s)}\right] \\
&= \frac{e^{\vartheta}\tau^{\theta}}{\Gamma(\theta+1)},
\end{aligned}$$

$$\begin{aligned}
\xi_2 &= \Upsilon^{-1}\left[\omega^{-\theta}(s)\, \Upsilon\left[\xi_{1\vartheta\vartheta}(\vartheta, \tau) - \xi_{1\vartheta}(\vartheta, \tau) + \xi_1(\vartheta, \tau) + A_1\right]\right] \\
&= \Upsilon^{-1}\left[\omega^{-\theta}(s)\, \Upsilon\left[\frac{e^{\vartheta}\tau^{\theta}}{\Gamma(\theta+1)}\right]\right] \\
&= \frac{e^{\vartheta}\tau^{2\theta}}{\Gamma(2\theta+1)},
\end{aligned}$$

$$
\begin{aligned}
\xi_3 &= \Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{2\,\vartheta\vartheta}(\vartheta,\,\tau)-\xi_{2\,\vartheta}(\vartheta,\,\tau)+\xi_{2}(\vartheta,\,\tau)+A_{2}\right]\right]\\
&= \Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\frac{e^{\vartheta}\tau^{2\theta}}{\Gamma(2\theta+1)}\right]\right]\\
&= e^{\vartheta}\,\Upsilon^{-1}\left[\frac{\varpi(s)}{\omega^{3\theta+1}(s)}\right]\\
&= \frac{e^{\vartheta}\tau^{3\theta}}{\Gamma(3\theta+1)},\\
&\vdots\\
\xi_{j+1} &= \Upsilon^{-1}\left[\omega^{-\theta}(s)\,\Upsilon\left[\xi_{j\,\vartheta\vartheta}(\vartheta,\,\tau)-\xi_{j\,\vartheta}(\vartheta,\,\tau)+\xi_{j}(\vartheta,\,\tau)+A_{j}\right]\right]\\
&= \frac{e^{\vartheta}\tau^{j\theta}}{\Gamma(j\,\theta+1)}. \qquad \text{.................................. (45)}
\end{aligned}
$$

Thus, the solution of Eq. (45) is given as

$$
\begin{aligned}
\xi(\vartheta,\,\tau) &= \xi_0(\vartheta,\,\tau)+\xi_1(\vartheta,\,\tau)+\xi_2(\vartheta,\,\tau)+\cdots\\
&= e^{\vartheta}\left(1+\frac{\tau^{\theta}}{\Gamma(\theta+1)}+\frac{\tau^{2\theta}}{\Gamma(2\theta+1)}+\frac{\tau^{3\theta}}{\Gamma(3\theta+1)}+\cdots+\frac{\tau^{j\theta}}{\Gamma(j\,\theta+1)}+\cdots\right)\\
&= e^{\vartheta}\sum_{j=0}^{\infty}\frac{\tau^{j\theta}}{\Gamma(j\,\theta+1)}\\
&= e^{\vartheta}\,\mathrm{E}_{\theta}\left(\tau^{\theta}\right). \qquad \text{.....(46)}
\end{aligned}
$$

If $\theta=1$ is substituted into Eq. (47), the exact solution is obtained

$$\xi(\vartheta,\,\tau)=e^{\vartheta}\,\mathrm{E}(\tau)=e^{\vartheta+\tau}.$$

This is entirely consistent with the findings in Reference [19, 23].

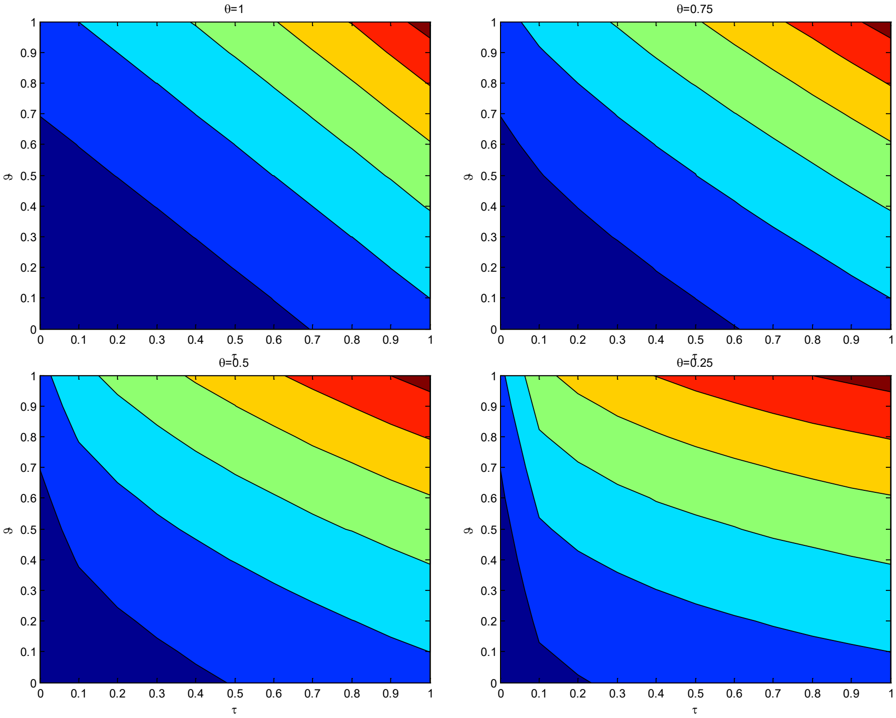

Fig 5. The three-dimensional contours solution of $\xi(\vartheta, \tau)$.

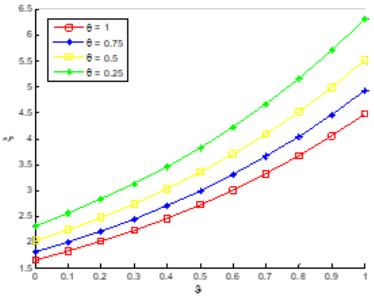

Fig 6. The two-dimensional graphical visual of $\xi(\vartheta, \tau)$ with $\tau = 0.5$ at multiple values of θ.

6. Numerical results

Figures 1, 3, and 5 illustrate the exact solutions of equations (29), (37), and (46) for various values of $\theta = 0.25, 0.5, 0.75$ and 1 for for $0 \le \vartheta \le 1$ and $0 \le \tau \le 1$, represented as three-dimensional contour graphs for the time-fractional Cauchy reaction-diffusion equation with the Caputo fractional operator. These graphs demonstrate that as the value of θ increases from fractional order, my results increasingly converge with the exact graph at $\theta = 1$. In Figures 2, 4, and 6. The exact solutions corresponding to the two-dimensional graphs have been computed $\theta = 0.25, 0.5, 0.75$ and 1 with $\tau = 0.5$ for the time-fractional Cauchy reaction-diffusion equation with the Caputo fractional operator. These figures also show that when $\theta = 0.25, 0.5, 0.75$, The obtained values are found to be very close to the exact solution at $\theta = 1$. Finally, NJTIM is a fast and efficient technique.

7. Conclusion

In this study, the NETIM approach was successfully developed to obtain an exact solution to the time-fractional Cauchy reaction-diffusion equation with the Caputo fractional derivative. Since the Jafari transform method is constrained and cannot get the series solution, The new iterative method is used to extract the iterations from the classical equation, which can lead to the exact solution in a straightforward manner. To demonstrate the efficiency and efficacy of the suggested approach, three illustrative examples were presented. It has been proven that my results are consistent and accurate in findings the exact solution. The obtained results are effective and exact, demonstrating that NETIM with the Caputo fractional derivative is accurate and reliable for solving fractional-order partial differential equations. Finally, the new method has a low computational load, which is one of its main advantages.

8. References

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