Generalized $(\alpha\eta)$-contractions in extended b_2-metric spaces with Application

د. عبدالحميد سالم المبروك. أستاذ الرياضيات المساعد. كلية العلوم. جامعة بنغازي.
د. سعاد إبراهيم المقصبي. أستاذ الرياضيات المشارك. كلية العلوم. جامعة بنغازي.

الخلاصة: في هذا البحث نقدم بعض نظريات النقطة الثابتة للتناقض $(\alpha\eta)$ للدوال الذاتية في الفضاء المتري الموسع-b2. ثم نستعرض تطبيق على المعادلات التكاملية لتوضيح النتائج التي تم الحصول عليها. نتائجنا تعمل على تعميم وتحسين بعض نتائج النقطة الثابتة لـ (مريم ف. س ، أفراح أ. ن ،2020).

Abstract:
In this paper we present some $(\alpha\eta)$-contraction fixed point theorems for self-mapping in the setting of an extended b_2-metric space. An Application to integral equations is provided to illustrate the obtained results. Our results generalize and extend some results of Maryam F. S, Afrah A. N, (2020).
Keywords: Extended b_2-metric space, Fredholm integral equation, Fixed point, $(\alpha\eta)$-contraction mappings.

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1. Introduction

Studies of metric fixed point have been an important research area for the last many years. There are many generalizations in the literature about the concept of metric spaces. For example, the notion of a 2-metric space was instituted in Cähler, S. (1963), and then many fixed-point results were obtained Ahmed. M. A, (2009). Moreover, the notion of a b-metric space was first introduced in Czerwik, S. (1993), similarly, several fixed-point results were also obtained (Alqahtani, B., et al., (2018), Afshari. H, et al., (2018)). Later, Zead Mustafa (Mustafa, Z., et al., 2014) introduced a new type of generalized metric space called $b2$-metric space, as a generalization of both 2-metric space and b-metric space. Then, Kamran et al., (2017) have dealt with an extended b-metric space and obtained unique fixed-point results. Recently, the concept of an extended b_2-metric space has been introduced in Elmabrok. S. A and Alkaleeli. R. S, (2018), as a generalization of both b_2-metric space and extended b-metric space. Similarly, several fixed point results were also obtained for mappings defined on these kind of spaces, Alkaleeli. R. S and Dabnoun. M. N, (2021).
On the other hand, Samet, B., et al. (2012) introduced a new concept of α-ψ-contractive mapping and established various fixed point theorems for such mappings in complete metric spaces. Afterwards Hussain, N.; Salimi, P. (2014), Modified the notions of α-ψ –contractive mappings and established certain fixed point theorems. After that, many researchers worked on the α-η -contraction mapping in different settings; for examples, see (Nashine. H. K and Samet. B, (2011). Ahmad. J. et al., (2015), Hussain. N. et al., (2015)) and the references therein. Recently, the concept of a 2-α-η-admissible mapping which extends the notion of α-admissible mapping with respect to η on 2-metric spaces has been introduced in Fathollahi. S., et al. (2014). and then many fixed-point results were obtained Jalal Shahkoohi, R., Bagheri, Z. (2019).
Motivated by the work done in Maryam F. S, Afrah A. N, (2020), we study some results for generalized $(\alpha\eta)$-contraction fixed point theorems on the class of an extended b_2-metric space. Then we present an application to integral equations on these spaces.
In the supplement, the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ stand for the sets of real, positive real and positive integers, respectively. Moreover, we will denote to the symbols as $\mathbb{R}_0^+ = \mathbb{R}^+ \cup\{0\}$ and $\mathbb{N}_0 = \mathbb{N} \cup\{0\}$.

2. Preliminaries

In 2017, Kamran. T, et al, (2017), introduced the notion of extended b-metric spaces.

Definition 2.1 Let X be a nonempty set and $\theta: X \times X \to [1, \infty)$ be a mapping. A function $d_\theta: X \times X \longrightarrow [0, \infty)$ is an extended b-metric on X if for all $x, y, z \in X$, the following conditions hold:

1) $d_\theta(x, y) = 0 \Leftrightarrow x = y$,
2) $d_\theta(x, y) = d_\theta(y, x)$,
3) $d_\theta(x, y) \leq \theta(x, y)[d_\theta(x, z) + d_\theta(z, x)]$ for all $x, y, z, a \in X$.

Then d_θ is called an extended b-metric on X and the pair (X, d_θ) is called an extended b-metric space.

Definition 2.2 Samet, B., et al. (2012). Let X be a non-empty set and $\alpha: X \times X \to \mathbb{R}_0^+$ be a mapping. A mapping $T: X \to X$ is called an α-admissible, if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1, \qquad (2.1)$$

for all $x, y \in X$.

Hussain, N.; Salimi, P (2014), extended the above notion of α-admissible mapping as follows.

Definition 2.3 Let T be a self-mapping on X and $\alpha, \eta: X \times X \to [0, \infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty), \qquad (2.2)$$

for all $x, y \in X$.

Definition 2.4 Maryam, F., et al. (2020). Let, (X, d_θ) be an extended b-metric space and $T: X \to X$. Then T is said to be generalized $(\alpha\eta)EB$-contraction if there exists two functions $\alpha, \eta: X \times X \to [0, \infty)$ and $k \in [0, 1)$ such that:

$$\alpha(x, Tx)\, \alpha(y, Ty) \geq \eta(x, Tx)\, \eta(y, T\, y),$$

implies that,

$$d_\theta(Tx, Ty) \leq k \max\{d_\theta(x, y), \min\{d_\theta(x, Tx)\}, d_\theta(y, Ty)\}, \qquad (2.3)$$

for all $x, y \in X$.

Elmabrok. S. A and Alkaleeli. R. S, (2018), initiated the concept of extended b_2-metric spaces as follow:

Definition 2.5 Let X be a nonempty set and $\theta: X \times X \times X \to [1, \infty)$ be a mapping. A function $d_\theta: X \times X \times X \longrightarrow [0, \infty)$ is an extended b_2-metric on X, if the following conditions hold:

1) For every pair of distinct points $x, y \in X$,there exists a point $z \in X$ such that $d_\theta(x, y, z) \neq 0$,
2) If at least two of three points x, y, z are the same, then $d_\theta(x, y, z) = 0$,
3) The symmetry: $d_\theta(x, y, z) = d_\theta(x, z, y) = d_\theta(y, x, z) = d_\theta(y, z, x) = d_\theta(z, x, y) = d_\theta(z, y, x)$,
4) The rectangle inequality: $d_\theta(x, y, z) \leq \theta(x, y, z)[d_\theta(x, y, a) + d_\theta(y, z, a) + d_\theta(z, x, a)]$, for all $x, y, z, a \in X$.

Then d_θ is called an extended b_2-metric on X and the pair (X, d_θ) is called an extended b_2-metric space.

Note that the class of an extended b_2-metric space is larger than b_2-metric space, since for $\theta(x, y, z) = s \geq 1$, we obtain the definition of a b_2-metric space. Furthermore, if $\theta(x, y, z) = s = 1$, the b_2-metric reduces to a 2-metric.

Example 2.6 Elmabrok. S. A and Alkaleeli. R. S, (2018).

Let $X = [0,1]$. Define $\theta: X \times X \times X \longrightarrow [1, \infty)$ by

$\theta(x, y, z) = \frac{1+x+y+z}{x+y+z}$, for all $x, y, z \in X$. And $d_\theta: X \times X \times X \longrightarrow [0, \infty)$ by

$$d_{\theta}(x,y,z)=\begin{cases}\dfrac{1}{xyz} & \text{if } x,y,z\in(0,1] \text{ and } x\neq y\neq z,\\ 0 & \text{if } x,y,z\in[0,1] \text{ and at least two of } x,y \text{ and } z \text{ are equal},\\ \dfrac{1}{xy} & \text{if } x,y\in(0,1] \text{ and } z=0.\end{cases}$$

Then, (X,d_{θ}) is an extended b_2-metric space.

Example 2.7 Alkaleeli. R. S and Dabnoun. M. N, (2021). Let $X=\left\{0,\ 1,\ \frac{1}{2},\frac{1}{3},\ \cdots,\ \frac{1}{n},\ \cdots\right\}$. Define $\theta: X\times X\times X\longrightarrow[1,\infty)$ by

$$\theta(x,y,z)=x+y+z+3,$$

and $d_{\theta}: X\times X\times X\longrightarrow[0,\infty)$, by

$$d_{\theta}(x,y,z)=\begin{cases}1, & \text{if x, y, z are dstinct and } \left\{\frac{1}{n},\frac{1}{n+1}\right\}\subset\{x,y,z\}, \text{for some } n\geq 1\\ 0, & \text{otherwise}\end{cases}.$$

Then, (X,d_{θ}) is a complete extended b_2-metric space.

Definition 2.8 Elmabrok. S. A and Alkaleeli. R. S, (2018). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an extended b_2-metric space (X,d_{θ}).

1) A sequence $\{x_n\}$ is a Cauchy sequence if and only if $d_{\theta}(x_n,x_m,a)\to 0$, when $n,m\to\infty$. for all $a\in X$.
2) A sequence $\{x_n\}$ is convergent to $x\in X$, if for all $a\in X$, there exists $x\in X$, such that $\lim_{n\to\infty} d_{\theta}(x_n,x,a)=0$.

Definition 2.9 An extended b_2-metric space (X,d_{θ}) is called complete if every Cauchy sequence is convergent sequence.

Definition 2.10 Jalal Shahkoohi, R., Bagheri, Z. (2019), Let, (X,d) be a 2-metric space. Assume that $T: X\to X$ be a self-mapping and $\alpha: X\times X\times X\to\mathbb{R}_0^+$ be a function. We say that $T: X\to X$ is a 2-α-admissible mapping if for all $x,y,a\in X$:

$$\alpha(x,y;a)\ \geq\ 1\ \Rightarrow\ \alpha(Tx,Ty,a)\geq\ 1, \qquad (2.4)$$

Definition 2.11 Fathollahi. S., et al. (2014). Let, (X,d) be a 2-metric space. Assume that $T: X\to X$ be a self-mapping and $\alpha,\eta: X\times X\times X\to\mathbb{R}_0^+$ be a functions. We say that $T: X\to X$ is a 2-α-admissible mapping with respect to η if

$$\alpha(x,y,a)\geq\eta(x,y,a)\ \Longrightarrow\alpha(Tx,Ty,a)\geq\eta(Tx,Ty,a), \qquad (2.5)$$

for all $x,y,a\in X$.

If we take $\eta(x,y,a)=1$, then we say that T is n 2-α-admissible mapping. Also, if we take $\alpha(x,y,a)=1$, then we say that T is a 2-η-subadmissible mapping.

3. Main Results

In the sequel, we assume that an extended b_2-metric d_{θ} is continuous functional. Motivated by Maryam, F., et al. (2020), we introduce the following notion.

Definition 3.1 Let (X,d_{θ}) be an extended b_2-metric space and $T: X\to X$. Then T is said to be generalized $(\alpha\eta)Eb_2$-contractions if there exists two functions $\alpha,\eta: X\times X\times X\to[0,\infty)$ and $k\in[0,1)$ such that

$$\alpha(x,Tx,a)\,\alpha(y,Ty,a)\geq\eta(x,Tx,a)\,\eta(y,Ty,a),$$

implies that

$$d_{\theta}(Tx,Ty,a)\leq k\max\{d_{\theta}(x,y,a),\min\{d_{\theta}(x,Tx,a),d_{\theta}(y,Ty,a)\}\}. \qquad (3.1)$$