

Matrix Representation of Non-Nested Real Algebraic Number

التعبير المصفوفي عن العدد الجبري الحقيقي الغير متداخل

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المخلص: نحن عبرنا عن الأعداد الجبرية في $Q(\sqrt[n]{s})$, $Q(\sqrt[n]{s}, \sqrt[m]{t})$ و $Q(\sqrt[n]{s}, \sqrt[m]{t}, \sqrt[r]{u})$

بواسطة كتل مصفوفية . وبالتالي يمكن حساب هذا العدد الجبري بعكس المصفوفة المناظرة .

الكلمات المفتاحية: العدد الجبري، المجال العددي، الجبر التنسيقي

Abstract: We represent an algebraic number in $Q(\sqrt[n]{s})$, $Q(\sqrt[n]{s}, \sqrt[m]{t})$ and $Q(\sqrt[n]{s}, \sqrt[m]{t}, \sqrt[r]{u})$ by a block matrix . The inverse of such an algebraic number can thus be computed by the inverse of the corresponding matrix .

Keywords : algebraic number, number field, associative algebra .

INTRODUCTION

The number fields of the $Q(\sqrt[n]{s})$, $Q(\sqrt[n]{s}, \sqrt[m]{t})$ and $Q(\sqrt[n]{s}, \sqrt[m]{t}, \sqrt[r]{u})$ associative algebras over Q , where s, t, u are respectively n^{th} , m^{th} , r^{th} power free relatively prime positive integers . Monomorphisms from these number fields into the matrix algebras $M_i(Q)$ where $i = n, nm, nmr$ respectively are constructed. This enables us to represent the algebraic number in $Q(\sqrt[n]{s})$, $Q(\sqrt[n]{s}, \sqrt[m]{t})$ and $Q(\sqrt[n]{s}, \sqrt[m]{t}, \sqrt[r]{u})$ by matrices of related blocks .

PRELIMINARIES

A complex number α is called an algebraic number[3] if it satisfies some monic polynomial equation $f(x) = b_0x^n + b_1x^{n-1} + \dots + b_n$ where $f(x) \in Q[x]$. If α satisfies some polynomial equation of degree n , but none of lower degree we say that α is an algebraic number of degree n . Non-algebraic number are called transcendental numbers .

Examples:

- 1) Every rational number is an algebraic .
- 2) $\sqrt{1 + 2\sqrt{3}}$ is an algebraic number of degree 4 .
- 3) π is transcendental number [1] .

The minimal polynomial f of an algebraic number α is the monic polynomial in $Q[x]$ of smallest degree such that $f(\alpha) = 0$.

Note that the minimal polynomial is irreducible over .

Theorem 1[3]: The set of all algebraic number is a field .
The number field is any subfield of this field .

Let F be a field . We say A is an associative algebra over F [2] if A is a ring with identity which is an F -vector space , such that the F -action is compatible with multiplication in A in the sense that $(x.a).b = x.(ab) = a(x.b)$ for all $a, b \in A, x \in F$.

The ring $M_n(F)$ of all $n \times n$ -matrices over the field F is an F -associative algebra of dimension n^2 .

The number field induced by α is given by :

$$Q(\alpha) = \{a_1 + a_2\alpha + \dots + a_n\alpha^{n-1} : a_i \in Q\} \text{ modulo } f(\alpha).$$

$Q(\alpha)$ is n -dimensional vector space over Q with basis $\{1, \alpha, \dots, \alpha^{n-1}\}$.

$Q(\alpha)$ is also associative algebra over Q .

If α and β are two algebraic number of degree m and n respectively , we defined $Q(\alpha, \beta) = Q(\alpha)Q(\beta)$.

$Q(\alpha, \beta)$ is an associative algebra over Q of dimension mn .

The basis of $Q(\alpha, \beta)$ is :

$$\{1, \alpha, \dots, \alpha^{n-1}, \beta, \alpha\beta, \dots, \alpha^{n-1}\beta, \dots, \beta^{m-1}, \alpha\beta^{m-1}, \dots, \alpha^{n-1}\beta^{m-1}\} .$$

Examples :

Let s and t be n^{th} power free and m^{th} power free relatively prime positive integer . Let $\alpha = \sqrt[n]{s}$ and $\beta = \sqrt[m]{t}$. We form the following two number fields :

$$1) Q(\alpha) = \{a_1 + a_2\alpha + a_3\alpha^2 + \dots + a_n\alpha^{n-1} : a_1, \dots, a_n \in Q\}$$

It is basis is $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$

$$2) Q(\alpha, \beta) = \{a_1 + a_2\alpha + \dots + a_n\alpha^{n-1} + \dots + a_{n+m}\alpha^{n-1}\beta^{m-1} : a_i \in Q\} . \text{ It is basis is}$$

$$\{1, \alpha, \dots, \alpha^{n-1}, \beta, \alpha\beta, \dots, \alpha^{n-1}\beta^{m-1}\} .$$

The general elements are :

$$(a_1, \dots, a_n; s) = a_1 + a_2\alpha + \dots + a_n\alpha^{n-1} \text{ in } Q(\alpha) \quad (a_{11}, \dots, a_{n1}; a_{12}, \dots, a_{n2}; \dots; a_{1m}, \dots, a_{nm}; s, t) = a_{11} + a_{21}\alpha + \dots + a_{n1}\alpha^{n-1} + a_{12}\beta + a_{22}\alpha\beta + \dots + a_{n2}\alpha^{n-1}\beta + \dots + a_{1m}\beta^{m-1} + a_{2m}\alpha\beta^{m-1} + \dots + a_{nm}\alpha^{n-1}\beta^{m-1} \text{ in } Q(\alpha, \beta) .$$

If R is an algebra over a field F , subrings and ideal of the ring R that are also F -subspaces of R are called subalgebras and algebra ideals respectively .

If S is another F -algebra , a ring homomorphism then $\theta: R \rightarrow S$ is called an algebra homomorphism if it also is F -linear, that is, if $\theta(ar) = a.\theta(r)$ for all $a \in F$ and $r \in R$.

Theorem 2 [2]: let R be an n -dimensional algebra over a field F . Then there exists a one-to-one algebra homomorphism $R \rightarrow M_n(F)$.

Proof .

Let $\{u_1, u_2, \dots, u_n\}$ be a basis of R . Given $r \in R$, write $u_i r = \sum_{j=1}^n r_{ij} u_j$, $r_{ij} \in F$.

Then define $\theta: R \rightarrow M_n(F)$ by $\theta(r) = [r_{ij}]^T$

$$\theta(ar) = [ar_{ij}]^T = a[r_{ij}]^T \quad \forall a \in F \text{ and } r \in R$$

Then θ is an F -linear homomorphism of additive groups .

Let $\theta(r) = [r_{ij}]^T = [r_{ji}]$ so that $u_i s = \sum_{j=1}^n s_{ij} u_j$. Then :

$$u_i r s = (\sum_k r_{ik} u_k) s = \sum_k r_{ik} (\sum_j s_{kj} u_j) = \sum_j (\sum_k r_{ik} s_{kj}) u_j .$$

$$\text{Thus } \theta(rs) = [\sum_k r_{ik} s_{kj}]^T = [\sum_k r_{ki} s_{jk}] = [r_{ji}] [s_{ji}] = \theta(r) . \theta(s) .$$

Then θ is homomorphism .

Suppose that $\theta(r) = 0$ that is $u_i r = 0 \quad \forall i$

If $1 = \sum_i a_i u_i$, then $r = 1 . r = \sum_i a_i u_i r = 0$.

So $\ker \theta = 0$ and θ is one to one .

RESULTS

If $(a_1, \dots, a_n; s)$ is the algebraic number in $(\sqrt[n]{s})$, where s is n^{th} power free positive integer , then the basis is given as follows : $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$ where $\alpha = \sqrt[n]{s}$.

$$(a_1, \dots, a_n; s)1 = a_1 + a_2\alpha + a_3\alpha^2 + \dots + a_n\alpha^{n-1}$$

$$(a_1, \dots, a_n; s)\alpha = a_n s + a_1\alpha + a_2\alpha^2 + \dots + a_{n-1}\alpha^{n-1}$$

$$(a_1, \dots, a_n; s)\alpha^2 = a_{n-1}s + a_n s\alpha + a_1\alpha^2 + \dots + a_{n-2}\alpha^{n-1}$$

$$(a_1, \dots, a_n; s)\alpha^{n-1} = a_2 s + a_3 s\alpha + a_n s\alpha^2 + \dots + a_1\alpha^{n-1}$$

$$M(a_1, a_2, \dots, a_n; s) = \begin{bmatrix} a_1 & sa_n & sa_{n-1} & \cdot & \cdot & \cdot & sa_2 \\ a_2 & a_1 & sa_n & \cdot & \cdot & \cdot & sa_3 \\ a_3 & a_2 & a_1 & \cdot & \cdot & \cdot & sa_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n-1} & a_{n-2} & \cdot & \cdot & \cdot & a_1 \end{bmatrix}$$

Thus we have :

Corollary 1.

The matrix $M(a_1, \dots, a_n; s)$ corresponds to the algebraic number $(a_1, \dots, a_n; s)$, where s is n^{th} power free positive integer .

If $\alpha = \sqrt[n]{s}$ and $\beta = \sqrt[m]{t}$ are two algebraic numbers of degree n and m respectively where s and t is n^{th} power free and m^{th} power free relatively prime positive integer. $Q(\alpha, \beta) = Q(\alpha)Q(\beta)$, then the basis is :

$$\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\} \times \{1, \beta, \beta^2, \dots, \beta^{m-1}\} = \\ 1, \alpha, \alpha^2, \dots, \alpha^{n-1}, \beta, \alpha\beta, \dots, \alpha^{n-1}\beta, \dots, \beta^{m-1}, \dots, \alpha^{n-1}\beta^{m-1}.$$

The general element is :

$$(a_{11}, \dots, a_{n1}; a_{12}, \dots, a_{n2}; \dots; a_{1m}, \dots, a_{nm}; s, t)$$

The algebraic number :

$$(a_{11}, \dots, a_{n1}; s); (a_{12}, \dots, a_{n2}; s); \dots; (a_{1m}, \dots, a_{nm}; s) \text{ correspond to}$$

$$M(a_{11}, \dots, a_{n1}; s); M(a_{12}, \dots, a_{n2}; s); \dots; M(a_{1m}, \dots, a_{nm}; s).$$

By using Corollary 1 we can represent the algebraic number by the blocks as follows :

$$B_1 = \begin{bmatrix} a_{11} & sa_{n1} & sa_{(n-1)1} & \cdot & \cdot & \cdot & sa_{21} \\ a_{21} & a_{11} & sa_{n1} & \cdot & \cdot & \cdot & sa_{31} \\ a_{31} & a_{21} & a_{11} & \cdot & \cdot & \cdot & sa_{41} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{(n-1)1} & a_{(n-2)1} & \cdot & \cdot & \cdot & a_{11} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} a_{12} & sa_{n2} & sa_{(n-1)2} & \cdot & \cdot & \cdot & sa_{22} \\ a_{22} & a_{12} & sa_{n2} & \cdot & \cdot & \cdot & sa_{32} \\ a_{32} & a_{22} & a_{12} & \cdot & \cdot & \cdot & sa_{42} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n2} & a_{(n-1)2} & a_{(n-2)2} & \cdot & \cdot & \cdot & a_{12} \end{bmatrix}, \dots,$$

$$B_m = \begin{bmatrix} a_{1m} & sa_{nm} & sa_{(n-1)m} & \cdot & \cdot & \cdot & sa_{2m} \\ a_{2m} & a_{1m} & sa_{nm} & \cdot & \cdot & \cdot & sa_{3m} \\ a_{3m} & a_{2m} & a_{1m} & \cdot & \cdot & \cdot & sa_{4m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{nm} & a_{(n-1)m} & a_{(n-2)m} & \cdot & \cdot & \cdot & a_{1m} \end{bmatrix}$$

Now we get a monomorphism from $Q(\alpha, \beta)$ into $M_n(Q)$ where :

$$M(B_1, B_2, \dots, B_n; t) = \begin{bmatrix} B_1 & tB_n & tB_{n-1} & \cdot & \cdot & \cdot & tB_2 \\ B_2 & B_1 & tB_n & \cdot & \cdot & \cdot & tB_3 \\ B_3 & B_2 & B_1 & \cdot & \cdot & \cdot & tB_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_n & B_{n-1} & B_{n-2} & \cdot & \cdot & \cdot & B_1 \end{bmatrix}$$

Thus we get :

Corollary 2 .

The matrix corresponds to algebraic number $(a_{11}, \dots, a_{n1}; a_{12}, \dots, a_{n2}; \dots; a_{1m}, \dots, a_{nm}; s, t)$ is computed by the block matrix $M(B_1, B_2, \dots, B_n; t)$.

Application 1.

a) In $Q(\sqrt[3]{s})$: where s is - 3^{th} power free

by corollary 1 $M(a, b, c; s) = \begin{bmatrix} a & sc & sb \\ b & a & sc \\ c & b & a \end{bmatrix}$

b) In $(\sqrt[3]{s}, \sqrt{t})$: where s is 3^{th} -power free, t is 2^{th} - power free and s , t are relatively prime positive integers

by corollary 2 $M(a, b, c, d, e, f; s, t) = \begin{bmatrix} a & sc & sb & | & td & tsf & tse \\ b & a & sc & | & te & td & tsf \\ c & b & a & | & tf & te & td \\ - & - & - & | & - & - & - \\ d & sf & se & | & a & sc & sb \\ e & d & sf & | & b & a & sc \\ f & e & d & | & c & b & a \end{bmatrix}$

c) In $(\sqrt[4]{s}, \sqrt{t})$: where s is 4^{th} -power free, t is 2^{th} - power free and s , t are relatively prime positive integers
by corollary 2

$$M(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; s, t) = \begin{bmatrix} a_1 & sa_4 & sa_3 & sa_2 & | & ta_5 & tsa_8 & tsa_7 & tsa_6 \\ a_2 & a_1 & sa_4 & sa_3 & | & ta_6 & ta_5 & tsa_8 & tsa_7 \\ a_3 & a_2 & a_1 & sa_4 & | & ta_7 & ta_6 & ta_5 & tsa_8 \\ a_4 & a_3 & a_2 & a_1 & | & ta_8 & ta_7 & ta_6 & ta_5 \\ - & - & - & - & | & - & - & - & - \\ a_5 & sa_8 & sa_7 & sa_6 & | & a_1 & sa_4 & sa_3 & sa_2 \\ a_6 & a_5 & sa_8 & sa_7 & | & a_2 & a_1 & sa_4 & sa_3 \\ a_7 & a_6 & a_5 & sa_8 & | & a_3 & a_2 & a_1 & sa_4 \\ a_8 & a_7 & a_6 & a_5 & | & a_4 & a_3 & a_2 & a_1 \end{bmatrix}$$

Now , if $\alpha = \sqrt[n]{s}$, $\beta = \sqrt[m]{t}$ and $\gamma = \sqrt[r]{u}$ are there algebraic number of degrees n,m and r respectively and s ,t,u are n^{th} power free , m^{th} power free and r^{th} power free respectively and they are relatively prime positive integer and $Q(\alpha, \beta, \gamma) = Q(\alpha, \beta)(\gamma)$, then the basis is :

$$\begin{aligned} & \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\} \times \{1, \beta, \beta^2, \dots, \beta^{m-1}\} \times \{1, \gamma, \gamma^2, \dots, \gamma^{r-1}\} = \\ & \{1, \alpha, \dots, \alpha^{n-1}, \beta, \alpha\beta, \dots, \alpha^{n-1}\beta, \dots, \beta^{m-1}, \dots, \alpha^{n-1}\beta^{m-1}\} \times \{1, \gamma, \dots, \gamma^{r-1}\} \\ & = \{1, \alpha, \dots, \alpha^{n-1}, \beta, \alpha\beta, \dots, \alpha^{n-1}\beta, \dots, \beta^{m-1}, \alpha\beta^{m-1}, \dots, \alpha^{n-1}\beta^{m-1} \\ & \gamma, \alpha\gamma, \dots, \alpha^{n-1}\gamma, \beta\gamma, \alpha\beta\gamma, \dots, \alpha^{n-1}\beta\gamma, \dots, \beta^{m-1}\gamma, \alpha\beta^{m-1}\gamma, \dots, \alpha^{n-1}\beta^{m-1}\gamma, \dots, \\ & \gamma^{r-1}, \alpha\gamma^{r-1}, \dots, \alpha^{n-1}\gamma^{r-1}, \beta\gamma^{r-1}, \alpha\beta\gamma^{r-1}, \dots, \alpha^{n-1}\beta\gamma^{r-1}, \dots, \beta^{m-1}\gamma^{r-1}, \\ & \alpha\beta^{m-1}\gamma^{r-1}, \dots, \alpha^{n-1}\beta^{m-1}\gamma^{r-1}\} . \end{aligned}$$

Briefly.

$$1) M_1 = M(a_1, \dots, a_n; s)$$

For algebraic numbers in (α) .

$$2) M_2 = M((M(a_{11}, \dots, a_{n1}; s), \dots, M(a_{1m}, \dots, a_{nm}; s)); t)$$

$$= M(M_1, \dots, M_1; t)$$

For algebraic numbers in (α, β) .

Then we have :

Corollary3.

The matrix representation of an algebraic number in $Q(\alpha, \beta, \gamma)$ is given by:

$$M_3 = M(M_2, \dots, M_2; u) .$$

Application 2.

Consider $Q(\sqrt{s}, \sqrt{t}, \sqrt{u})$ where s , t , u are 2^{th} - power free and they are relatively prime positive integers
By corollary 3

$$M_3 = \begin{bmatrix} a_{11} & sa_{21} & | & ta_{12} & tsa_{22} & | & ub_{11} & usb_{21} & | & utb_{12} & utsb_{22} \\ a_{21} & a_{11} & | & ta_{22} & ta_{12} & | & ub_{21} & ub_{11} & | & utb_{22} & utb_{12} \\ - & - & | & - & - & | & - & - & | & - & - \\ a_{12} & sa_{22} & | & a_{11} & sa_{21} & | & ub_{12} & usb_{22} & | & ub_{11} & usb_{21} \\ a_{22} & a_{12} & | & a_{21} & a_{11} & | & ub_{22} & ub_{12} & | & ub_{21} & ub_{11} \\ - & - & | & - & - & | & - & - & | & - & - \\ b_{11} & sb_{21} & | & tb_{12} & tsb_{22} & | & a_{11} & sa_{21} & | & ta_{12} & tsa_{22} \\ b_{21} & b_{11} & | & tb_{22} & tb_{12} & | & a_{21} & a_{11} & | & ta_{22} & ta_{12} \\ - & - & | & - & - & | & - & - & | & - & - \\ b_{12} & sb_{22} & | & b_{11} & sb_{21} & | & a_{12} & sa_{22} & | & a_{11} & sa_{21} \\ b_{22} & b_{12} & | & b_{21} & b_{11} & | & a_{22} & a_{12} & | & a_{21} & a_{11} \end{bmatrix}$$

Application 3.

To find the inverse of $a + b\sqrt[3]{s} + c\sqrt[3]{s^2}$ by matrix representation .

$$M_1 = M(a, b, c; s) = \begin{bmatrix} a & sc & sb \\ b & a & sc \\ c & b & a \end{bmatrix}$$

$$M_1^{-1} = \frac{1}{a^3 + b^3s + c^3s^2 - 3abcs} \begin{bmatrix} a^2 - sbc & s(b^2 - ac) & s(sc^2 - ab) \\ sc^2 - ab & a^2 - sbc & s(b^2 - ac) \\ b^2 - ac & sc^2 - ab & a^2 - sbc \end{bmatrix}$$

Then $(a + b\sqrt[3]{s} + c\sqrt[3]{s^2})^{-1} =$

$$\frac{1}{a^3 + b^3s + c^3s^2 - 3abcs} (a^2 - sbc + (sc^2 - ab)\sqrt[3]{s} + (b^2 - ac)\sqrt[3]{s^2}) .$$

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