University of Benghazi … Faculty of Education
Journal of Faculty Education … The nineteen number…December 2

# Existence, Uniqueness, and Asymptotic Behavior of Solutions to a Nonlinear Volterra Integro-Differential Equation with Time-Dependent Delay

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## التحليل الدقيق لمعادلة فولتيرا التكاملية–التفاضلية غير الخطية ذات التأخير المتغيّر زمنياً: معايير الوجود، الوحدانية، والتقارب التقاربي

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## Abstract

This paper presents a comprehensive analysis of the asymptotic behavior and stability properties of a novel class of nonlinear Volterra integro-differential equations featuring time-varying delays. The model under investigation incorporates a linear instantaneous dissipation term coupled with a nonlinear integral memory component characterized by a saturation-type response. The nonlinearity is defined by a function $\phi(\zeta)$ satisfying specific growth and boundedness conditions. We establish the existence and uniqueness of solutions using a constructive framework based on Krasnoselskii's fixed point theorem. Subsequently, we employ the Razumikhin method to derive explicit, computationally verifiable stability criteria. Our main theoretical contribution lies in formulating concrete stability conditions that substantially extend and generalize previous results for similar equation structures. The practical relevance of our theoretical framework is demonstrated through applications to biological systems, such as population dynamics, and engineered systems, including artificial neural networks.



**Key Words**

**Nonlinear Volterra Integro-Differential Equation, Time-Varying Retardation, Asymptotic Convergence, Krasnoselskii's Fixed Point Theorem, Razumikhin Technique**



الملخص:

يقدّم هذا البحث تحليلاً دقيقاً للسلوك التقاربي وخصائص الاستقرار لفئة جديدة من معادلات فولتيرا التكاملية–التفاضلية غير الخطية التي تتضمن تأخيراً غير ثابت.

يحتوي النموذج الرياضي قيد الدراسة على حد تبديد لحظي خطي إلى جانب مكوّن ذاكرة تكاملي غير خطي يتميز باستجابة من نوع التشبّع. الشكل الخاص لهذه اللاخطية يمثَّل بدالة نرمز لها بـ63φ(ζ)، التي تحقق مجموعة من قيود النمو والتقيي، ولإثبات وجود ووحدانية الحلول، نعتمد إطاراً إنشائياً قائماً على مبرهنة نقطة التثبيت لكراسنوسلسكي، أمّا لدراسة الاستقرار بعمق، فنستخدم تقنية رازوميخين.

المساهمة النظرية الأساسية لهذا العمل تتمثل في اشتقاق معايير صريحة وقابلة للتحقق حسابياً للاستقرار، وهي نتيجة توسّع وتعمّم بصورة ملحوظة النتائج السابقة المتعلقة بهياكل مشابهة من المعادلات في الأدبيات العلمية، كما يجري توضيح الأهمية التطبيقية للنتائج النظرية من خلال أمثلة عملية مرتبطة بأنظمة بيولوجية معقدة مثل ديناميكيات التجمعات السكانية، وأنظمة هندسية مصممة مثل الشبكات العصبية الاصطناعية.

يقدّم هذا العمل إطاراً منهجياً مبتكراً لتحليل هذه الفئة من المعادلات، بما يعزّز قابلية المعالجة التحليلية مع الحفاظ على درجة عالية من الصلة بالظواهر البيولوجية والهندسية الواقعية.



الكلمات المفتاحية : **معادلة فولتيرا التفاضلية التكاملية غير الخطية ، تأخير زمني متغير ، التقارب التقاربي ، نظرية النقطة الثابتة لكراسنوسيلسكي ، تقنية رازومخين**

## 1. Introduction

Volterra integro-differential equations with delays represent a fundamental class of mathematical structures within the broader theory of functional differential equations. Since Vito Volterra's pioneering work on population dynamics [15], these formulations have proven indispensable across numerous modeling paradigms. These equations excel at describing systems where the instantaneous rate of change depends on both the current state and the system's historical evolution, with the integral memory term capturing this temporal dependence. The analytical challenges posed by such models, particularly those incorporating time-dependent delays and complex nonlinearities, have motivated sustained research efforts over the past century [4, 5].

The systematic investigation of stability properties in nonlinear dynamical systems with time delays constitutes a cornerstone of modern control theory and dynamical systems analysis. Early contributions by researchers such as Burton [4, 5] established foundational analytical frameworks for studying stability and periodic behavior in both ordinary and functional differential systems. Later, Hale and Verduyn Lunel [9] systematically organized the mathematical machinery for functional differential equations, presenting sophisticated techniques that continue to influence contemporary research directions.

Within this theoretical landscape, questions regarding the existence and uniqueness of solutions represent essential prerequisites for any meaningful stability analysis. The established methodology in this field requires first demonstrating the existence of a solution for given initial conditions, then proving its uniqueness, before undertaking a detailed characterization of its stability properties. Recent methodological advances, exemplified by the work of Makhzoum and colleagues [1, 3, 2, 12, 11], have highlighted the efficacy of fixed-point approaches-particularly Krasnoselskii's theorem-in identifying periodic solutions and examining stability characteristics in nonlinear neutral differential equations with functional delays.

Parallel developments in the analysis of periodic solutions for various classes of nonlinear differential equations have further enriched this field. The works of Eze and collaborators [6, 7, 8, 14] have demonstrated sophisticated techniques for establishing existence, uniqueness, and stability of periodic solutions in equations with damping, resonance effects, and elastic coefficients. These contributions, along with related research on Duffingtype-equations [13] and almost periodic solutions on time scales [10], provide valuable methodological insights that inform our current approach.

The present study introduces and analyzes an innovative class of Volterra integro-differential equations characterized by a distinctive combination of a linear, time-varying dissipation term, a nonlinear integral operator with a specific saturation function, and a time-dependent delay. To our knowledge, this particular synthesis of elements has not been thoroughly investigated in the existing literature. Importantly, our model differs fundamentally from the class of neutral differential equations with delays studied by Makhzoum and colleagues [12, 11], where the delay affects both the state variable and its derivative. Our focus on a pure integro-differential structure necessitates a specialized analytical approach tailored to its specific characteristics.

The primary objective of this work is to establish a definitive and quantitatively verifiable stability criterion expressed as

$$\delta_0 > \frac{\kappa_1 \Psi}{1-\rho}$$

This inequality represents a significant departure from the predominantly abstract or existential stability results common in the contemporary literature. Unlike previous studies that often yield abstract conditions based on operator theory, our approach produces a tangible inequality that can be directly evaluated using the system's physical or engineering parameters. This result provides practical, quantitative guidelines for parameter selection in applied models, effectively bridging theoretical abstraction with engineering application.

Our contributions are multifaceted: we introduce and analyze a novel nonlinear Volterra integro-differential equation framework with non-constant delay and specific saturation nonlinearity; we provide detailed proofs for solution existence and uniqueness based on Krasnoselskii's fixed point theorem; we conduct a rigorous stability analysis using the Razumikhin technique adapted to this equation class; our stability criteria are explicit and readily verifiable, representing a substantial improvement over previous results; and we demonstrate the practical applicability of our theoretical advances through applications to population dynamics and neural network theory.

The remainder of this paper is organized as follows: Section 2 outlines the fundamental mathematical concepts and assumptions. Section 3 presents the existence and uniqueness proofs. Section 4 details the analysis of asymptotic stability. Section 5 provides illustrative applications. Finally, Section 6 summarizes our findings and discusses potential directions for future research.

## 2 Preliminaries

This section establishes the foundational definitions, underlying assumptions, and mathematical framework that support our subsequent analysis.

### 2.1 Essential Definitions and Mathematical Properties

**Definition 2.1 (Saturating Nonlinearity).** We consider a function $\phi: \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(\zeta) = \frac{\zeta}{1 + |\zeta|^{\nu}}, \nu > 1.$$

This functional form is commonly known as a saturating nonlinearity and possesses several key properties:

1. The function $\phi$ is globally Lipschitz continuous with constant $L_{\phi}$.

2 . For all $\zeta \in \mathbb{R}$, the inequality $|\phi(\zeta)| \leq |\zeta|$ holds.
3. The function $\phi$ exhibits odd symmetry: $\phi(-\zeta) = -\phi(\zeta)$.
4. The function $\phi$ is bounded, with $|\phi(\zeta)| \leq M_{\phi}$ for all $\zeta \in \mathbb{R}$.

Definition 2.2 (Initial Function Specifications). For the integro-differential equation (1), initial conditions are specified on the interval $[\chi(\tau_0), \tau_0]$, where

$$\chi(\tau_0) = \inf\{\tau - \sigma(\tau): \tau \geq \tau_0\}.$$

The initial function $\psi$ must be continuous on this interval, i.e., $\psi \in C([\chi(\tau_0), \tau_0], \mathbb{R})$.

### 2.2 Model Formulation and Motivation

The structure of Equation (1) is motivated by its applicability in modeling systems with memory and saturation effects across various scientific domains. We briefly outline its derivation in the contexts of population dynamics and neural networks to illustrate its relevance.

Population Dynamics: Consider a population with density $\zeta(\tau)$. The term $-\gamma(\tau)\zeta(\tau)$ can represent an instantaneous mortality or harvesting rate. The integral term models a regulated, delayed growth rate. In many species, reproduction depends on population levels in the recent past due to gestation periods or resource regeneration. The kernel $\kappa(\tau,\varepsilon)$ weights the influence of past populations, while the saturation nonlinearity $\phi(\zeta)$ prevents unbounded population growth, reflecting finite resource availability in natural systems.

Artificial Neural Networks: In firing-rate models, $\zeta(\tau)$ represents the average membrane potential or firing rate of a neuronal population. The linear term $-\gamma(\tau)\zeta(\tau)$ models the inherent leakage current driving the potential toward its resting state. The integral term represents synaptic input from other neurons, incorporating transmission delays via the kernel $\kappa(\tau,\varepsilon)$. The function $\phi(\zeta)$ acts as an activation function (e.g., a sigmoid), ensuring the neuron's firing rate remains within physiologically plausible bounds.

Thus, Equation (1) provides a general framework capturing essential features like instantaneous dissipation, distributed delayed feedback, and saturating nonlinearities common to many complex biological and engineered systems.

### 2.3. Fundamental Assumptions

For a coherent development of our arguments, we formalize the fundamental assumptions underlying our analysis.

(H1). The function $\gamma: [\tau_0,\infty) \to \mathbb{R}$ is continuous, and there exists $\gamma_1 > 0$ such that $|\gamma(\tau)| \leq \gamma_1$ for all $\tau \geq \tau_0$.

(H2). The delay function $\sigma: [\tau_0,\infty) \to \mathbb{R}$ is continuous and satisfies $0 \leq \sigma(\tau) \leq \Psi$ for all $\tau \geq \tau_0$, with constant $\Psi > 0$.

(H3). The kernel function $\kappa: [\tau_0,\infty) \times [\chi(\tau_0),\infty) \to \mathbb{R}$ is continuous and bounded, with $|\kappa(\tau,\varepsilon)| \leq \kappa_1$ for all $\tau \geq \tau_0$ and $\varepsilon \leq \tau$.

For stability analysis, we strengthen these assumptions as follows:
Assumption 2.4 (A1). There exist positive constants $\delta_0$ and $\delta_1$ such that $0 < \delta_0 \leq \gamma(\tau) \leq \delta_1 < \infty$ for all $\tau \geq \tau_0$.

(A2). The kernel function $\kappa(\tau,\varepsilon)$ is continuous and bounded, with $|\kappa(\tau,\varepsilon)| \leq \kappa_1$ for all $\tau \geq \tau_0$ and $\varepsilon \leq \tau$.

(A3). The delay function $\sigma(\tau)$ is differentiable and strictly positive, with $0 \leq \sigma'(\tau) \leq \rho < 1$ for all $\tau \geq \tau_0$. This condition ensures that $\tau - \sigma(\tau)$ is strictly increasing in $\tau$.

**Remark 2.1.** The condition $0 \leq \sigma'(\tau) \leq \rho < 1$ in Assumption (A3) is crucial. It guarantees that the delayed argument $\tau - \sigma(\tau)$ increases monotonically with $\tau$, a prerequisite for applying the Razumikhin technique in our stability analysis.

## 3. Main Results

### 3.1. Establishing Solution Existence and Uniqueness

Before undertaking stability analysis, we must first rigorously demonstrate the existence of a unique solution to the proposed nonlinear Volterra integrodifferential equation with time-varying delay. We consider the equation:

$$\frac{d\zeta}{d\tau} = -\gamma(\tau)\zeta(\tau) + \int_{\tau-\sigma(\tau)}^{\tau} \kappa(\tau,\varepsilon)\phi(\zeta(\varepsilon))d\varepsilon, \tau \geq \tau_0 \#(1)$$

with initial condition:

$$\zeta(\tau) = \psi(\tau), \tau \in [\chi(\tau_0), \tau_0] \#(2)$$

where $\chi(\tau_0) = \inf\{\tau - \sigma(\tau): \tau \geq \tau_0\}$, and $\psi \in C([\chi(\tau_0), \tau_0], \mathbb{R})$ is the given continuous initial function.

**Theorem 3.1 (Existence and Uniqueness of Solution).** Under assumptions (H1)-(H3), the integro-differential equation (1) with initial condition (2) admits a unique continuous solution on $[\chi(\tau_0), \tau_0 + \chi]$ for some $\chi > 0$.

**Proof.** We reformulate the initial value problem as an equivalent integral equation. Integrating from $\tau_0$ to an arbitrary time $\tau$ yields:

$$\zeta(\tau) = \psi(\tau_0) - \int_{\tau_0}^{\tau} \gamma(\nu)\zeta(\nu)d\nu + \int_{\tau_0}^{\tau}\left(\int_{\nu-\sigma(\nu)}^{\nu} \kappa(\nu,\omega)\phi(\zeta(\omega))d\omega\right)d\nu$$

To prove our claim, we construct a fixed point problem. Let $T = \tau_0 + \chi$ for some $\chi > 0$, and consider the Banach space $X = C([\chi(\tau_0), T], \mathbb{R})$ equipped with the supremum norm $\|\cdot\|$. Define an upper bound:

$$N = \max\left\{\sup_{\tau\in[\chi(\tau_0),\tau_0]} |\psi(\tau)|, |\psi(\tau_0)| + 1\right\}.$$

Let $S \subset X$ be the closed, convex, nonempty subset:
$S = \{\zeta \in X: \zeta(\tau) = \psi(\tau)$ for $\tau \in [\chi(\tau_0), \tau_0]$, and $|\zeta(\tau) - \psi(\tau_0)| \leq 1$ for $\tau \in [\tau_0, T]\}$.
Define operators $A$ and $B$ on $S$ :

$$(A\zeta)(\tau) = \begin{cases} \psi(\tau), & \tau \in [\chi(\tau_0), \tau_0] \\ \psi(\tau_0) - \int_{\tau_0}^{\tau} \gamma(\nu)\zeta(\nu)d\nu, & \tau \in [\tau_0, T] \end{cases}$$

$$(B\zeta)(\tau) = \begin{cases} 0, & \tau \in [\chi(\tau_0), \tau_0] \\ \int_{\tau_0}^{\tau}\left(\int_{\nu-\sigma(\nu)}^{\nu} \kappa(\nu,\omega)\phi(\zeta(\omega))d\omega\right)d\nu, & \tau \in [\tau_0, T] \end{cases}$$

Finding a solution is equivalent to finding a fixed point of $P = A + B$. We now verify that these operators satisfy Krasnoselskii's Fixed Point Theorem for sufficiently small $\chi > 0$.
(1) $A$ is a contraction: For $\zeta, \xi \in S$ and $\tau \in [\tau_0, T]$ :

$$|(A\zeta)(\tau) - (A\xi)(\tau)| = \left| -\int_{\tau_0}^{\tau} \gamma(\nu)(\zeta(\nu) - \xi(\nu))d\nu \right| \leq \gamma_1(\tau - \tau_0)\|\zeta - \xi\|$$

Choosing $\chi < 1/\gamma_1$ ensures $A$ is a contraction on $S$.
(2) $B$ is continuous and compact: Consider a uniformly convergent sequence $\{\zeta_m\} \subset S$ with $\zeta_m \to \zeta \in S$. For $\tau \in [\tau_0, T]$ :

$$|(B\zeta_m)(\tau) - (B\zeta)(\tau)| = \left| \int_{\tau_0}^{\tau} \int_{\nu-\sigma(\nu)}^{\nu} \kappa(\nu, \omega)[\phi(\zeta_m(\omega)) - \phi(\zeta(\omega))]d\omega d\nu \right|$$
$$\leq \kappa_1 L_\phi \Psi(T - \tau_0)\|\zeta_m - \zeta\|$$

As $m \to \infty$, $\|\zeta_m - \zeta\| \to 0$, so $\|B\zeta_m - B\zeta\| \to 0$, establishing continuity.
For compactness, we apply the Arzelà-Ascoli theorem. For any $\zeta \in S$ and $\tau \in [\tau_0, T]$ :

$$|(B\zeta)(\tau)| \leq \kappa_1 M_\phi \Psi(T - \tau_0)$$

demonstrating uniform boundedness. For equicontinuity, take $\tau_1, \tau_2 \in [\tau_0, T]$ with $\tau_1 < \tau_2$ :

$$|(B\zeta)(\tau_2) - (B\zeta)(\tau_1)| \leq \kappa_1 M_\phi \Psi |\tau_2 - \tau_1|$$

For any $\epsilon > 0$, choosing $\delta = \epsilon / (\kappa_1 M_\phi \Psi)$ ensures $|(B\zeta)(\tau_2) - (B\zeta)(\tau_1)| < \epsilon$ when $|\tau_2 - \tau_1| < \delta$, for all $\zeta \in S$. Thus $B(S)$ is equicontinuous, and by Arzelà-Ascoli, $B$ is compact.
(3) $A\zeta + B\xi \in S$ for any $\zeta, \xi \in S$: For $\tau \in [\tau_0, T]$ :

$$|(A\zeta)(\tau) + (B\xi)(\tau) - \psi(\tau_0)|$$
$$\leq N\gamma_1(\tau - \tau_0) + N\kappa_1\Psi(\tau - \tau_0) = N(\gamma_1 + \kappa_1\Psi)(\tau - \tau_0)$$

Choosing $\chi \leq \left(N(\gamma_1 + \kappa_1\Psi)\right)^{-1}$ ensures $|(A\zeta)(\tau) + (B\xi)(\tau) - \psi(\tau_0)| \leq 1$, so the operator sum maps $S$ into itself.

All conditions of Krasnoselskii's theorem are satisfied, guaranteeing a fixed point in $S$ corresponding to a solution of the integral equation.

For uniqueness, suppose $\zeta(\tau)$ and $\xi(\tau)$ are two distinct solutions on $[\chi(\tau_0), T]$. Define:

$$W(\tau) = \sup_{\varepsilon \in [\chi(\tau_0), \tau]} |\zeta(\varepsilon) - \xi(\varepsilon)|.$$

By the initial condition, $W(\tau_0) = 0$. For $\tau \in [\tau_0, T]$ :

$$|\zeta(\tau) - \xi(\tau)| \leq (\gamma_1 + \kappa_1 \Psi L_\phi) \int_{\tau_0}^{\tau} W(\nu)d\nu$$

This implies $W(\tau) \leq \left(\gamma_1 + \kappa_1 \Psi L_\phi\right) \int_{\tau_0}^{\tau} W(\nu) d\nu$. Gronwall's inequality with $W(\tau_0) = 0$ gives $W(\tau) \equiv 0$ for $\tau \in [\tau_0, T]$, proving uniqueness.

### 3.2. Asymptotic Stability Analysis

We now derive sufficient conditions guaranteeing asymptotic stability of the trivial solution using a Razumikhin-type argument.

**Theorem 3.2 (Asymptotic Stability Criterion).** Under assumptions (A1),(A3), if the inequality

$$\delta_0 > \frac{\kappa_1 \Psi}{1-\rho} \#(3)$$

holds, then the trivial solution $\zeta(\tau) = 0$ of equation (1) is asymptotically stable.

**Proof**. We first establish global existence of solutions. The saturating nonlinearity $\phi(\zeta)$ is bounded by $M_{\phi,\max}$. From equation (1), the solution derivative satisfies $|\zeta'(\tau)| \leq \delta_1 |\zeta(\tau)| + \kappa_1 \Psi M_{\phi,\max}$. Gronwall's inequality then ensures all solutions remain bounded on finite time intervals, preventing finite-time blow-up and guaranteeing global existence on $[\tau_0, \infty)$.

We now employ a Razumikhin-type stability argument. Consider the Lyapunov function candidate $V(\tau) = |\zeta(\tau)|$. The Dini derivative along trajectories of (1) is:

$$\begin{aligned}
\dot{V}^{+}(\tau) &= \operatorname{sgn}(\zeta(\tau)) \frac{d\zeta}{d\tau} \\
&= \operatorname{sgn}(\zeta(\tau)) \left[ -\gamma(\tau)\zeta(\tau) + \int_{\tau-\sigma(\tau)}^{\tau} \kappa(\tau,\varepsilon)\phi(\zeta(\varepsilon)) d\varepsilon \right] \\
&\leq -\gamma(\tau)|\zeta(\tau)| + \left| \int_{\tau-\sigma(\tau)}^{\tau} \kappa(\tau,\varepsilon)\phi(\zeta(\varepsilon)) d\varepsilon \right| \\
&\leq -\delta_0 |\zeta(\tau)| + \int_{\tau-\sigma(\tau)}^{\tau} |\kappa(\tau,\varepsilon)| |\phi(\zeta(\varepsilon))| d\varepsilon \\
&\leq -\delta_0 |\zeta(\tau)| + \kappa_1 \int_{\tau-\sigma(\tau)}^{\tau} |\zeta(\varepsilon)| d\varepsilon
\end{aligned}$$

where we used assumptions (A1), (A2), and the property $|\phi(\zeta)| \leq |\zeta|$.

We now apply a Razumikhin theorem variant for equations with variable delay. Asymptotic stability can be established if there exists $q > 1$ such that whenever $|\zeta(\varepsilon)| \leq q|\zeta(\tau)|$ for all $\varepsilon \in [\tau - \sigma(\tau), \tau]$, we have $\dot{V}^{+}(\tau) < 0$. The condition $0 \leq \sigma'(\tau) \leq \rho < 1$ allows us to choose $q = 1/(1-\rho) > 1$.

Assume for $\tau \geq \tau_0$ that we are in the Razumikhin set: $|\zeta(\varepsilon)| < \frac{1}{1-\rho} |\zeta(\tau)|$ for all $\varepsilon \in [\tau - \sigma(\tau), \tau]$. Then:

$$\int_{\tau-\sigma(\tau)}^{\tau} |\zeta(\varepsilon)| d\varepsilon < \int_{\tau-\sigma(\tau)}^{\tau} \frac{1}{1-\rho} |\zeta(\tau)| d\varepsilon = \frac{\sigma(\tau)}{1-\rho} |\zeta(\tau)| \leq \frac{\Psi}{1-\rho} |\zeta(\tau)|$$

Substituting into the Dini derivative inequality:

$$\dot{V}^{+}(\tau) < -\delta_0|\zeta(\tau)| + \kappa_1 \frac{\Psi}{1-\rho}|\zeta(\tau)| = \left(-\delta_0 + \frac{\kappa_1 \Psi}{1-\rho}\right)|\zeta(\tau)|.$$

By stability condition (3), the parenthetical expression is negative, so $\dot{V}^{+}(\tau) < 0$ whenever the Razumikhin condition holds. This establishes the conditions for the Razumikhin theorem, from which we conclude asymptotic stability of the zero solution.

## 4. Applications

### 4.1. Application I: Population Dynamics with Harvesting

To demonstrate the practical applicability of our existence and uniqueness theorem in a biological context, we consider a fish population model with harvesting and delayed, saturated growth. Let $P(t)$ represent population density at time $t$. The model is:

$$\frac{dP}{dt} = -E(t)P(t) + \int_{t-\sigma}^{t} \alpha e^{-\beta(t-s)} \frac{P(s)}{1+|P(s)|^2} ds$$

with initial condition:

$$P(t) = 1, t \in [-1,0].$$

Here, $E(t) = \frac{1}{2}|\sin(t)|$ represents time-varying harvesting effort, the integral term models delayed, density-dependent growth with saturation (preventing unbounded growth at high densities), and the kernel $\alpha e^{-\beta(t-s)}$ weights the influence of past populations.

We systematically verify that all conditions of our theorem are satisfied. First, $\gamma(t) = E(t) = \frac{1}{2}|\sin(t)|$ is continuous and bounded by $|\gamma(t)| \leq \frac{1}{2}$, so we take $\gamma_1 = \frac{1}{2}$. Second, the delay function $\sigma(t) = 1$ is constant and continuous, satisfying $0 \leq \sigma(t) \leq 1$, so $\Psi = 1$. Third, for $\alpha = \frac{1}{4}$ and $\beta = 1$, the kernel $\kappa(t,s) = \alpha e^{-\beta(t-s)}$ is continuous and bounded by $|\kappa(t,s)| \leq \frac{1}{4}$, so $\kappa_1 = \frac{1}{4}$. All prerequisites are met, guaranteeing a unique continuous solution to this initial value problem on $[-1,\chi]$ for sufficiently small $\chi > 0$. This result is biologically significant, ensuring the population trajectory is uniquely determined for a finite time horizon under these modeling assumptions.

### 4.2. Stability in a Recurrent Neural Network

To exemplify the utility of our stability theorem in an engineering context, we investigate asymptotic stability of the resting state in a neuronal population model with recurrent feedback and synaptic delays. Let $x(t)$ represent the mean membrane potential. The model is:

$$\frac{dx}{dt} = -\lambda x(t) + \int_{t-\sigma}^{t} J e^{-(t-s)/\tau} \tanh(x(s)) ds$$

Here, $-\lambda x(t)$ is the leakage current, the integral term represents delayed synaptic feedback, $J$ is synaptic strength, and $\tanh(x(s))$ is a sigmoidal activation function (a saturating nonlinearity).

We verify the assumptions of Theorem 3.2. The coefficient function $\gamma(t) = \lambda = 3$ is constant and positive, so $\delta_0 = 3$. The kernel $\kappa(t,s) = Je^{-(t-s)/\tau}$ is continuous and bounded; for $J = \frac{1}{2}$ and finite $\tau$, $|\kappa(t,s)| \leq \frac{1}{2}$, so $\kappa_1 = \frac{1}{2}$. The delay function $\sigma(t) = 0.5$ is constant, so $\sigma'(t) = 0$ and $\rho = 0$.

We now check the stability criterion (3):

$$\delta_0 > \frac{\kappa_1 \Psi}{1-\rho}.$$

Substituting parameter values:

$$3 > \frac{\frac{1}{2} \times 0.5}{1-0}$$
$$3 > 0.25$$

The inequality holds, so the trivial solution $\zeta(\tau) = 0$ is asymptotically stable. This finding has significant implications for neural network design, ensuring predictable convergence to the resting state under the specified parameter conditions and preventing runaway excitation.

## 5. Conclusion

This study has provided a thorough examination of a novel nonlinear Volterra integro-differential equation with time-dependent delay. We first established solution existence and uniqueness using Krasnoselskii's fixed point theorem, aligning with contemporary approaches to functional differential equations. Subsequently, we employed the Razumikhin theorem to derive an explicit, quantifiable criterion guaranteeing asymptotic stability of the zero solution.

Our work enriches the literature on functional equations by providing detailed, rigorous analysis for a specific yet generalizable nonlinearity and structural form. The formulation of an explicit stability condition

$$\delta_0 > \frac{\kappa_1 \Psi}{1-\rho}$$

represents a substantial practical advance beyond mere existential stability results, offering definitive quantitative guidance for parameter optimization in applied modeling.

The applications to population dynamics and neural network theory demonstrate the immediate utility of our theoretical findings. In the population model, our analysis guarantees a unique population trajectory essential for ecosystem prediction. The stability criterion in the neural network model ensures bounded neuronal activity and return to baseline, a fundamental requirement for reliable information processing.

Promising future research directions include extending this analytical framework to models with neutral terms, investigating periodic solutions with periodically varying coefficients using advanced fixed-point techniques, and exploring applications in more complex biological and engineering systems with multiple time delays and intricate distributed parameters. The rigorous methodological framework developed here provides a solid foundation for these future investigations into more complex functional differential equation structures.

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