



On n -adic Algebra

Nabila M. Bennour *, Kahtan H. Alzubaidy

Departement of Mathematics, University of Benghazi, Libya,

Highlights

- **n -adic algebra is a generalization of monadic algebra by extending the number of quantifiers from 1 to n .**
- **The method used in deduction of monadic algebra is generalized to the case of n -adic algebra with respect to terms.**

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*Address of correspondence:

E-mail address: n.bennour@yahoo.com
 N. M. Bennour

ABSTRACT

This paper studies n -adic algebras with terms. Deduction is also done in these algebras.

1. Introduction

n -adic algebra is a generalization of monadic algebra by extending the number of quantifiers from 1 to n . Our main results are given in theorems 12, 13. The method used in (Alzubaidy, 2007) is generalized to the case of n -adic algebra. A Boolean algebra is an algebraic structure $B = \langle A, \vee, \wedge, ', 0, 1 \rangle$ in which $\vee, \wedge$ are binary operations on A , $'$ is a unary operation, while 0 and 1 are nullary operations (distinguished elements of A), which satisfy the following conditions: For arbitrary $a, b, c \in A$

- $a \vee b = b \vee a, a \wedge b = b \wedge a$
- $(a \vee b) \vee c = a \vee (b \vee c), (a \wedge b) \wedge c = a \wedge (b \wedge c)$
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- $a \vee (a \wedge b) = a, a \wedge (a \vee b) = a$
- $a \vee a' = 1, a \wedge a' = 0.$

A partial order $\leq$ on a Boolean algebra is defined by $a \leq b$, if $a \wedge b = a$ or $a \vee b = b$.

A Boolean algebra B is said to be complete if any subset A of B has infimum and supremum in B .

Suppose that B is a complete Boolean algebra. An existential operator on B is a mapping $\exists: B \rightarrow B$ such that:

- $\exists(0) = 0$
- $a \leq \exists(a)$ for any $a \in B$
- $\exists(a \wedge \exists(b)) = \exists(a) \wedge \exists(b)$ for any $a, b \in B$

$(B, \exists)$ is called monadic algebra.

2. n -adic algebras

An n -adic algebra $(n \geq 0)(B, \exists_1, \dots, \exists_n)$ consists of a complete Boolean algebra B and n mappings $\exists_i: B \rightarrow B; 1 \leq i \leq n$ called existential operators such that:

- $\exists_i(0) = 0$
- $a \leq \exists_i(a)$ for any $a \in B$
- $\exists_i((a) \wedge \exists_i(b)) = \exists_i(a) \wedge \exists_i(b)$ for any $a, b \in B$
- $\exists_i \exists_j = \exists_j \exists_i.$

Note that 0- n -adic algebra is just Boolean algebra B and 1- n -adic algebra is the monadic algebra $(B, \exists)$. Obviously, the n -adic algebra is a locally finite polyadic algebra in the sense given in (Alzubaidy and Bennour, 2012).

The universal operator $\forall_i: B \rightarrow B; 1 \leq i \leq n$ is defined by $\forall_i(a) = (\exists_i(a'))'$, for any $a \in B$.

Proposition 1. (Alzubaidy, 2007)

- $\forall_i(0) = 0$
- $a \leq \forall_i(a)$ for any $a \in B$
- $\forall_i((a) \wedge \forall_i(b)) = \forall_i(a) \wedge \forall_i(b)$ for any $a, b \in B$
- $\forall_i \forall_j = \forall_j \forall_i.$

Proposition 2. (Alzubaidy, 2007)

$\exists_i \forall_j = \forall_j \exists_i$ if $i \neq j$.

Suppose Q_i is an existential operator or universal operator. By the previous proposition, we have

Corollary 3.

$Q_1 Q_2 \dots Q_n \leq \forall \forall \dots \exists \exists \dots \exists.$

2.1 Functional n -adic Algebra

Let X be a nonempty set and B a Boolean algebra. Suppose that

$B^{X^n} = \{p | p: X^n \rightarrow B \text{ is a function}\}$, where X^n is the Cartesian product of n copies of X .

For $p, q \in B^{X^n}$ define $p \wedge q, p \vee q, p'$ and $0, 1$ pointwise as follows:

$$(p \wedge q)(x_1, \dots, x_n) = p(x_1, \dots, x_n) \wedge q(x_1, \dots, x_n),$$

$$(p \vee q)(x_1, \dots, x_n) = p(x_1, \dots, x_n) \vee q(x_1, \dots, x_n), p'(x_1, \dots, x_n) = (p(x_1, \dots, x_n))', 0(x_1, \dots, x_n) = 0 \text{ and } 1(x_1, \dots, x_n) = 1 \text{ for any } (x_1, \dots, x_n) \in X^n.$$

Proposition 4. (Alzubaidy, 2007)

$(B^{X^n}, \vee, \wedge, ', 0, 1)$ is an Boolean algebra.

Proposition 5. (Alzubaidy, 2007)

$(B^{X^n}, \exists)$ is a monadic algebra.

Existential and universal operators $\exists_i, \forall_i$ on the functional Boolean algebra B^{X^n} are defined as follows:

$$\text{for each } p \in B^{X^n}, \exists_i(p)(x_1, \dots, x_n) = \sup_i \{p(x_1, \dots, x_n) : (x_1, \dots, x_n) \in X^n\} \text{ and}$$

$$\forall_i(p)(x_1, \dots, x_n) = \inf_i \{p(x_1, \dots, x_n) : (x_1, \dots, x_n) \in X^n\}.$$

Proposition 6.

$(B^{X^n}, \exists_i)$ is a n -adic algebra, $1 \leq i \leq n$

Proof

$$\text{i) } \exists_i(0)(x_1, \dots, x_n) = \sup_i \{0(x_1, \dots, x_n) : (x_1, \dots, x_n) \in X^n\} =$$

$$0(x_1, \dots, x_n) = 0. \text{ Then } \exists_i(0) = 0$$

$$\text{ii) Let } p \in B^{X^n}, \exists_i(p)(x_1, \dots, x_n) = \sup_i \{p(x_1, \dots, x_n) : (x_1, \dots, x_n) \in X^n\} \geq p(x_1, \dots, x_n). \text{ Then } \exists_i(p) \geq p$$

$$\text{iii) Let } p, q \in B^{X^n}, \exists_i(p \wedge \exists_i(q))(x_1, \dots, x_n) = \sup_i \{p \wedge$$

$$\exists_i(q)(x_1, \dots, x_n) : (x_1, \dots, x_n) \in X^n\} = \sup_i \left\{ (p)(x_1, \dots, x_n) \wedge \sup_i \{q(x_1, \dots, x_n) : (x_1, \dots, x_n) \in X^n\} \right\} = \sup_i$$

$$\{p(x_1, \dots, x_n) : (x_1, \dots, x_n) \in X^n\} \wedge \sup_i \{q(x_1, \dots, x_n) : (x_1, \dots, x_n) \in X^n\} = \exists_i(p)(x_1, \dots, x_n) \wedge \exists_i(q)(x_1, \dots, x_n). \text{ Then } \exists_i(p \wedge \exists_i(q)) =$$

$$\exists_i(p) \wedge \exists_i(q)$$

$$\text{iv) } \exists_i(p)(x_1, \dots, x_n) \exists_j(p)(x_1, \dots, x_n) = \sup_i \{p(x_1, \dots, x_n)\} \sup_j$$

$$\{p(x_1, \dots, x_n)\} = \sup_j \{p(x_1, \dots, x_n)\} \sup_i \{p(x_1, \dots, x_n)\} =$$

$$\exists_j(p)(x_1, \dots, x_n) \exists_i(p)(x_1, \dots, x_n). \text{ Then } \exists_i \exists_j = \exists_j \exists_i.$$

A more general case is the locally finite polyadic algebra. This is given in (Alzubaidy, 2007).

3. Terms

Let X be a set of variables and $\mathcal{F}$ be a type of algebra. The set $T(X)$ of terms of type $\mathcal{F}$ over X is the smallest set such that: i) $X \cup \mathcal{F}_0 \subseteq T(X)$

ii) If $t_1, \dots, t_n \in T(X)$ and $f \in \mathcal{F}_n$ then $f(t_1, \dots, t_n) \in T(X)$ (Burris and Shakappanavar, 1981).

For $t \in T(X)$ we often write t as $t(x_1, \dots, x_n)$ to indicate that the variables occurring in t are among $x_1, \dots, x_n$.

A term t is n -ary if the number of variables appearing explicitly is less than or equal n .

The term t define a function $t_A: A^n \rightarrow A$ as $t_A(a_1, \dots, a_n) = t(x_i/a_i)$ for $a_1, \dots, a_n \in A$, where A is an algebra of type $\mathcal{F}$.

The set $T(X)$ can be transformed into an algebra (Burris and Shakappanavar, 1981). The term algebra $T(X)$ of type $\mathcal{F}$ over X has as its universe the set $T(X)$ and the fundamental operations satisfy:

$$f^{T(X)}: (t_1, \dots, t_n) \mapsto f(t_1, \dots, t_n) \text{ for } f \in \mathcal{F}_n \text{ and } t_i \in T(X), 1 \leq i \leq n.$$

Now consider the functional n -adic algebra $(B^{X^n}, \exists_i)$. A term t defines a function $t_*: B^{X^n} \rightarrow B^{X^n}$ as follows:

$$t_*(p)(x) = p(t_1(x), \dots, t_n(x)), \text{ for any } x \in X \text{ and } p \in B^{X^n} \text{ where } t = (t_1, \dots, t_n) \text{ as functions.}$$

$\text{End}(B^{X^n})$ is the set of an endomorphisms from B^{X^n} into itself.

Proposition 7.

$$\text{i) } t_* \in \text{End}(B^{X^n}) \quad \text{ii) } \exists_i t_* = t_* \exists_i$$

Proof

$$\text{i) } 1) t_*(p \vee q)(x) = (p \vee q)(t_1(x), \dots, t_n(x)) = p(t_1(x), \dots, t_n(x)) \vee q(t_1(x), \dots, t_n(x)) = t_*(p)(x) \vee t_*(q)(x).$$

$$\text{Then } t_*(p \vee q) = t_*(p) \vee t_*(q).$$

$$2) t_*(p \wedge q)(x) = (p \wedge q)(t_1(x), \dots, t_n(x)) = p(t_1(x), \dots, t_n(x)) \wedge q(t_1(x), \dots, t_n(x)) = t_*(p)(x) \wedge t_*(q)(x).$$

$$\text{Then } t_*(p \wedge q) = t_*(p) \wedge t_*(q).$$

$$3) (t_*(p)(x))' = (p(t_1(x), \dots, t_n(x)))' = p'(t_1(x), \dots, t_n(x)) = t_*(p')(x). \text{ Then } (t_*(p))' = t_*(p)'$$

$$4) t_*(0)(x) = 0(t_1(x), \dots, t_n(x)) = 0 = 0(x). \text{ Then } t_*(0) = 0.$$

$$5) t_*(1)(x) = 1(t_1(x), \dots, t_n(x)) = 1 = 1(x). \text{ Then } t_*(1) = 1.$$

$$\text{ii) } t_* \exists_i(p)(x) = t_* \left(\sup_i \{p(x)\} \right) = \sup_i \{p(t_1(x), \dots, t_n(x))\} = \sup_i \{t_*(p)(x)\} = \exists_i t_*(p)(x). \text{ Then } \exists_i t_* = t_* \exists_i.$$

4. Deduction

4.1 n -adic ideals and filters

A subset U of a Boolean algebra B is called a Boolean ideal if

$$\text{i) } 0 \in U$$

$$\text{ii) } a \vee b \in U \text{ for any } a, b \in U$$

$$\text{iii) If } a \in U \text{ and } b \leq a \text{ then } b \in U.$$

A subset F of a Boolean algebra B is called a Boolean filter if

$$\text{i) } 1 \in F$$

$$\text{ii) } a \wedge b \in F \text{ for any } a, b \in F$$

$$\text{iii) If } a \in F \text{ and } b \geq a \text{ then } b \in F.$$

A subset U of an n -adic algebra B is called an n -adic ideal of B if

$$\text{i) } U \text{ is a Boolean ideal}$$

$$\text{ii) } \exists_i(U)(a) \in U, a \in U \text{ for } 1 \leq i \leq n.$$

A subset F of an n -adic algebra B is called an n -adic filter of B if

- i) F is a Boolean filter
- ii) $\forall_i(F)(a) \in F, a \in F$ for $1 \leq i \leq n$.

Thus we have the following two propositions.

Proposition 8.

There is a one to one correspondence between ideals and filters.

Proposition 9.

The set of all n -adic ideals and the set of all n -adic filters are closed under the arbitrary intersection.

Let B be an n -adic algebra and $\Gamma \subseteq B$. Let $U(\Gamma)$ denote the least n -adic ideal containing Γ and $F(\Gamma)$ denote the least n -adic filter containing Γ . We say that $U(\Gamma)$ and $F(\Gamma)$ are generating by Γ .

Proposition 10.

Let B be an n -adic algebra and $\Gamma \subseteq B$. Then

- i) $U(\Gamma) = \{b \in B: b \leq x_1 \vee x_2 \vee \dots \vee x_n \text{ for some } x_1, \dots, x_n \in \Gamma\} \cup \{0\}$
- ii) $F(\Gamma) = \{b \in B: b \geq x_1 \wedge x_2 \wedge \dots \wedge x_n \text{ for some } x_1, \dots, x_n \in \Gamma\} \cup \{1\}$

Proof

- i) Let $J = \{b \in B: b \leq x_1 \vee x_2 \vee \dots \vee x_n \text{ for some } x_1, \dots, x_n \in \Gamma\} \cup \{0\}$, $0 \in J$. Let $b_1, b_2 \in J$ then $b_1 \leq x_1 \vee x_2 \vee \dots \vee x_n$ and $b_2 \leq y_1 \vee y_2 \vee \dots \vee y_m$ for some $x_i, y_i \in \Gamma$.

$b_1 \vee b_2 \leq x_1 \vee x_2 \vee \dots \vee x_n \vee y_1 \vee y_2 \vee \dots \vee y_m$. Therefore $b_1 \vee b_2 \in J$.

If $a \leq b \leq x_1 \vee x_2 \vee \dots \vee x_n$, then $a \in J$. Then J is a Boolean ideal containing Γ .

If $a \in J$, then $a \leq x_1 \vee x_2 \vee \dots \vee x_n$, $\exists_i(J)(a) \leq \exists_i(J)(x_1 \vee x_2 \vee \dots \vee x_n) = \exists_i(J)(x_1) \vee \dots \vee \exists_i(J)(x_n)$. Then $\exists_i(J)(a) \in U(\Gamma)$.

- ii) A similar argument leads to (ii).

A filter F of an n -adic algebra is called ultrafilter if F is maximal with respect to the property that $0 \notin F$.

Ultrafilters satisfy the properties of the following proposition.

Proposition 11. (Burris and Shakappanavar, 1981)

Let F be a filter of n -adic algebra, then

- i) F is an ultrafilter of B iff for any $a \in B$ exactly one of a, a' belong to F .
- ii) F is an ultrafilter of B iff $0 \notin F$ and $a \vee b \in F$ iff $a \in F$ or $b \in F$ for any $a, b \in B$.
- iii) If $a \in B - F$, then there is an ultrafilter L such that $F \subseteq L$ and $a \notin L$.

Let $\Gamma \subseteq B$, the ultrafilter containing $F(\Gamma)$ is denoted by $UF(\Gamma)$.

A mapping $\mu: B_1 \rightarrow B_2$ between two n -adic algebras is called n -adic homomorphism if

- i) μ is a Boolean homomorphism,
- ii) $\mu \exists_i = \exists_i \mu$.

Obviously, $\mu \forall_i = \forall_i \mu$.

For $\Gamma \subseteq B, b \in B$, we define the deduction $\Gamma \vdash b$ iff $b \in UF(\Gamma)$.

Theorem 12.

Let $\Gamma \subseteq B^{X^n}$

- i) $\Gamma \vdash p \wedge q$ iff $\Gamma \vdash p$ and $\Gamma \vdash q$
- ii) $\Gamma \vdash p \vee q$ iff $\Gamma \vdash p$ or $\Gamma \vdash q$
- iii) $\Gamma \vdash p$ iff $\Gamma \not\vdash p'$
- iv) $\Gamma \vdash p$ iff $\Gamma \vdash \forall_i p, 1 \leq i \leq n$
- v) $F(\Gamma) \vdash p(x_0)$ iff $F(\Gamma) \vdash \exists_i p(x), 1 \leq i \leq n$.

Proof

(i), (ii) follow from the definition of filter and (Alzubaidy, 2007, p.2)

iii) follows from proposition 10.

iv) This is by $\forall_i p(x) \leq p(x)$ and $\forall_i(F) \subseteq F$.

v) This is by the definition $\exists_i p(x) = \sup\{p(x)\}, x = (x_1, \dots, x_n)$.

Note that $t_*(UF(\Gamma)) = UF(t_*(\Gamma))$. Define $\Gamma \vdash pt$ if $pt \in F(t_*(\Gamma))$. Then theorem 12 can be generalized with respect to terms as follows:

Theorem 13.

- i) $\Gamma \vdash pt \wedge qt$ iff $\Gamma \vdash pt$ and $\Gamma \vdash qt$
- ii) $\Gamma \vdash pt \vee qt$ iff $\Gamma \vdash pt$ or $\Gamma \vdash qt$
- iii) $\Gamma \vdash pt$ iff $\Gamma \not\vdash p't$
- iv) $\Gamma \vdash p$ iff $\Gamma \vdash \forall_i pt, 1 \leq i \leq n$
- v) $F(\Gamma) \vdash pt(x_0)$ iff $F(\Gamma) \vdash \exists_i pt(x), 1 \leq i \leq n$.

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