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On Matrix Representations of Octonions

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الملخص:

الأعداد الرباعية الحقيقية $\mathbb{H}$ على $\mathbb{R}$ جبر قسمة تنسيقي هي قابلة للتمثيل بمصفوفات رباعية حقيقية. الأعداد الثمانية الحقيقية $\mathbb{O}$ على $\mathbb{R}$ جبر قسمة غير تنسيقي وبذلك لا يمكن تمثيلها خطيا بمصفوفات حقيقية كما في حالة $\mathbb{H}$. النقطة الأساسية في هذا البحث هي تمثيل الأعداد الثمانية بزوج مرتب من المصفوفات الرباعية الحقيقية.

الكلمات المفتاحية:

الأعداد الرباعية، الأعداد الثمانية، جبر قسمة غير تنسيقي، مصفوفات حقيقية، التمثيل الخطي.

Abstract

The real Quaternions $\mathbb{H}$ over $\mathbb{R}$ is an associative division algebra.They can be linearly represented by 4×4 real matrices. The real Octonions $\mathbb{O}$ over $\mathbb{R}$ is a non –associative division algebra. They can't be represented linearly by matrices as in the case of $\mathbb{H}$. The point of this paper is to represent Octonions linearly by ordered pairs of two 4×4 matrices.

Keywords: hypercomplex; quaternions; octonions; associative; alternative and division algebras

1. INTRODUCTION

Hypercomplex numbers of importance are of dimensions 1,2,4 and 8. They are respectively the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions (Hamiltonians) $\mathbb{H}$, and the Octonions (Cayley numbers) $\mathbb{O}$. The basis of $\mathbb{H}$ is $\{1, i, j, k\}$ where $i^2 = j^2 = k^2 = ijk = -1$. For $q = a + bi + cj + dk \ in\ \mathbb{H}$, the conjugate and norm of q are respectively defined by $q^* = a - bi - cj - dk$ and $\|q\|_{\mathbb{H}} = q^* q = a^2 + b^2 + c^2 + d^2$.Thus we have

Lemma 1 .
$(i) q^{**} = q \qquad (ii) qq^* = q^* q \qquad (iii)(q_1 q_2)^* = q_2{}^* q_1{}^*$

$(iv)\|rq\|_{\mathbb{H}} = |r| \|q\|_{\mathbb{H}} \qquad (v)\|q_1 q_2\|_{\mathbb{H}} = \|q_1\|_{\mathbb{H}} \|q_2\|_{\mathbb{H}}$.

$\mathbb{R}, \mathbb{C}, and\ \mathbb{H}\ are\ associative\ algebras\ over\ \mathbb{R}$. Therefore they can be represented linearly by matrices as follows [1].

$$\mathbb{R}: r \mapsto [r]$$

$$\mathbb{C}: \ a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\mathbb{H}: \ a + bi + cj + dk \mapsto \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

Also, $\mathbb{H}$ can be represented by 2×2-complex matrices as follows :

$$z + wj \mapsto \begin{bmatrix} z & -w^* \\ w & z^* \end{bmatrix}$$

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$\mathbb{O}$ is a non-associative algebra over $\mathbb{R}$ (see theorem 7). Therefore $\mathbb{O}$ cannot be linearly represented by the same way above. However, there are different approaches to this problem. They produce different results [2, 3]. The point of this paper is to represent octonions linearly by ordered pairs of two matrices.

2. CONSTRUCTION

We use the Cayley-Dikson theorem [4] to construct hypercomplex numbers.

Let Γ be the set of all hypercomplex numbers of dimension n (n=1,2, or 4). The next set of hypercomplex numbers Γ' of dimension 2n is constructed as follows

$$\Gamma' = \{(\alpha, \beta): \alpha, \beta \in \Gamma\}$$

Addition and scalar multiplication on Γ' are defined by

$(\alpha, \beta) + (\gamma, \delta) = (\alpha + \gamma, \beta + \delta)$, and

$r(\alpha, \beta) = (r\alpha, r\beta)$ where $r \in \mathbb{R}$

Theorem 2. Γ' is a vector space over $\mathbb{R}$ of dimension 2n

Proof.

Γ is a vector space over $\mathbb{R}$ of dimension n.

Let $u_1, u_2, \ldots, u_n$ *be the basis of* Γ.

Γ' is a vector space with respect to the addition and scalar multiplication just defined. The basis of Γ' is $(u_1, 0), (u_2, 0), \ldots, (u_n, 0), (0, u_1), (0, u_2), \ldots, (0, u_n)$. Therefore the dimension of Γ' is 2n.

Corollary 3 : $\Gamma' \cong \Gamma \oplus \Gamma$ as vector space .

We introduced the conjugate and multiplication on Γ' in terms of conjugate and multiplication of Γ as follows :

$$(\alpha, \beta)^* = (\alpha^*, -\beta) \ \text{ and}$$

$$(\alpha,\beta)(\gamma,\delta)=(\alpha\gamma-\delta^*\beta,\delta\alpha+\beta\gamma^*).$$

Lemma 4 . $(i)\ \phi^{**}=\phi\ (ii)\ (\phi\psi)^*=\psi^*\phi^*$, for any $\phi,\psi\in\Gamma'$

Proof .

Let
$\phi,\psi\in\Gamma'$. $Then\ \phi=(\alpha,\beta)$,
$\psi=(\gamma,\delta)\ where\ \alpha,\beta,\gamma,\delta\in\Gamma$

(i) $\phi^*=(\alpha,\beta)^*=(\alpha^*,-\beta)$

$\phi^{**}=(\alpha^*,-\beta)^*=(\alpha^{**},\beta)=(\alpha,\beta)=\phi$

(ii) $(\phi\psi)=(\alpha,\beta)(\gamma,\delta)=(\alpha\gamma-\delta^*\beta,\delta\alpha+\beta\gamma^*)$

$(\phi\psi)^*=(\alpha\gamma-\delta^*\beta,\delta\alpha+\beta\gamma^*)^*$

$=(\gamma^*\alpha^*-\beta^*\delta,-\beta\gamma^*-\delta\alpha)=(\gamma^*,-\delta)(\alpha^*,-\beta)$
$=\psi^*\phi^*$

The norm of a hyper complex number θ is defined by

$\|\theta\|_{\Gamma'}=\theta^*\theta$.

Lemma 5 . $\|\phi\psi\|_{\Gamma'}=\|\phi\|_{\Gamma'}\|\psi\|_{\Gamma'}$

Proof .

$\|\phi\psi\|_{\Gamma'}=(\phi\psi)^*(\phi\psi)=(\psi^*\phi^*)(\phi\psi)\ =\psi^*(\phi^*\phi)\psi$

$=\psi^*\|\phi\|_{\Gamma'}\psi\ =\|\phi\|_{\Gamma'}\psi^*\psi=\|\phi\|_{\Gamma'}\|\psi\|_{\Gamma'}$

Lemma 6 . $\|(\alpha,\beta)\|_{\Gamma'}=\|\alpha\|_{\Gamma}+\|\beta\|_{\Gamma}$

Proof .

$$\|(\alpha,\beta)\|_{\Gamma'}=(\alpha,\beta)^*(\alpha,\beta)=(\alpha^*,-\beta)(\alpha,\beta)$$
$$=(\alpha^*\alpha+\beta^*\beta,\beta\alpha^*-\beta\alpha^*)$$
$$=(\alpha^*\alpha+\beta^*\beta,0)=\|\alpha\|_{\Gamma}+\|\beta\|_{\Gamma}$$

Octonions $\mathbb{O}$

$\mathbb{O}=\{(q_1,q_2):q_1,q_2\in\mathbb{H}\}$, and $\|(q_1,q_2)\|_{\mathbb{O}}=\|q_1\|_{\mathbb{H}}+\|q_2\|_{\mathbb{H}}$.

Theorem 7. $\mathbb{O}$ is a non-commutative , a non-associative and alternative division algebra over $\mathbb{R}$ of dimension 8 .

Proof .

Let $(i,j),(k,i)\in\ \mathbb{O}$

$(i,j)(k,i)=(ik+ij,-1-jk)=(-j+k,-1-i)$

$(k,i)(i,j)=(ki+ji,jk+1)=(j-k,i+1)$

Then $\mathbb{O}$ is a non-commutative .

Let $(i,j),(k,i),(j,k)\in\ \mathbb{O}$

$\big((i,j)(k,i)\big)(j,k)=(-j+k,-1-i)(j,k)=(1-i-k+j,i-1+j+k)$

$(i,j)\big((k,i)(j,k)\big)=(i,j)(-i+j,-1-k)=(1+k+j+i,-i-j-k+1)$

Then $\mathbb{O}$ is non-associative .

Let $\phi=(q_1,q_2)$, $\psi=(q_3,q_4)\in\ \mathbb{O}$

$(\phi\psi)\psi=\big((q_1,q_2)(q_3,q_4)\big)(q_3,q_4)=$

$(q_1q_3-q_4^*q_2,q_4q_1+q_2q_3^*)(q_3,q_4)$

$=$
$\big((q_1q_3)q_3-(q_4^*q_2)q_3-q_4^*(q_4q_1)-q_4^*(q_2q_3^*),q_4(q_1q_3)-q_4(q_4^*q_2)+(q_4q_1)q_3^*+(q_2q_3^*)q_3^*\big)$

$=$
$\big(q_1(q_3q_3)-(q_4^*q_4)q_1-\ q_4^*(q_2q_3)-q_4^*(q_2q_3^*),(q_4q_1)q_3+(q_4q_1)q_3^*-(q_4q_4^*)q_2+q_2(q_3^*q_3^*)\big)$

$=$
$(q_1(q_3q_3)-q_1(q_4^*q_4)-(q_4^*q_2)q_3-(q_4^*q_2)q_3^*,q_4q_1(q_3+q_3^*)-q_2(q_4q_4^*)+q_2(q_3^*q_3^*)$

$=(q_1(q_3q_3-q_4^*q_4)-q_4^*q_2(q_3+q_3^*),q_4q_1(q_3+q_3^*)+q_2(q_3^*q_3^*-q_4q_4^*))$

$=(q_1(q_3q_3-q_4^*q_4)-(q_3+q_3^*)q_4^*q_2,q_4(q_3+q_3^*)q_1+q_2(q_3^*q_3^*-q_4q_4^*))$

$=(q_1(q_3q_3-q_4^*q_4)-(q_4q_3^*+q_4q_3)^*q_2,(q_4q_3+q_4q_3^*)q_1+q_2(q_3q_3-q_4^*q_4)^*)=(q_1,q_2)(q_3q_3-q_4^*q_4,q_4q_3+q_4q_3^*)\ =\phi(\psi\psi)$

$(\phi\phi)\psi=\big((q_1,q_2)(q_1,q_2)\big)(q_3,q_4)=(q_1q_1-q_2^*q_2,q_2q_1+q_2q_1^*)(q_3,q_4)$

$=$
$((q_1q_1)q_3-(q_2^*q_2)q_3-q_4^*(q_2q_1)-q_4^*(q_2q_1^*),q_4(q_1q_1)-q_4(q_2^*q_2)+(q_2q_1)q_3^*+(q_2q_1^*)q_3^*)$

$=((q_1q_1)q_3-q_4^*q_2(q_1+q_1^*)-(q_2^*q_2)q_3,q_4(q_1q_1)+q_2(q_1+q_1^*)q_3^*-q_4(q_2^*q_2))$

$=((q_1q_1)q_3-(q_1+q_1^*)q_4^*q_2-(q_2^*q_2)q_3,(q_4q_1)q_1+q_2q_3^*(q_1+q_1^*)-q_4(q_2q_2^*))$

$=$
$(q_1(q_1q_3)-q_1(q_4^*q_2)+q_1^*(q_4^*q_2)-q_3(q_2^*q_2),(q_4q_1)q_1+(q_2q_3^*)q_1+q_2(q_3^*q_1^*)-q_2(q_2^*q_4))$

$=(q_1(q_1q_3-q_4^*q_2)-(q_4q_1+q_2q_3^*)^*q_2,(q_4q_1+q_2q_3^*)q_1+q_2(q_1q_3-q_4^*q_2)^*)$

$=(q_1,q_2)(q_1q_3-q_4^*q_2,q_4q_1+q_2q_3^*)$
$=(q_1,q_2)\big((q_1,q_2)(q_3,q_4)\big)=\phi(\phi\psi)$

Then $\mathbb{O}$ is alternative algebra .

Let $(q_1,q_2)\neq 0\ in$. $\mathbb{O}$

Then $(q_1,q_2)^{-1}=\frac{(q_1,q_2)^*}{\|(q_1,q_2)\|_{\mathbb{O}}}=\frac{(q_1^*,-q_2)}{\|q_1\|_{\mathbb{H}}+\|q_2\|_{\mathbb{H}}}$

Thus $\mathbb{O}$ is division algebra over $\mathbb{R}$.

3. LINEAR REPRESENTATIONS

Let $\mu g: \mathbb{H} \longrightarrow M_4(\mathbb{R})$ be the algebra monomorphism of the matrix representation of $\mathbb{H}$.

Then $\mu_{\mathbb{H}}(a+bi+cj+dk) = \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$

Define $\mu_{\mathbb{O}}: \mathbb{O} \longrightarrow M_4(\mathbb{R}) \oplus M_4(\mathbb{R})$ be given by

$$\mu_{\mathbb{O}}(q_1, q_2) = (\mu_{\mathbb{H}}(q_1), \mu_{\mathbb{H}}(q_2)).$$

Theorem 8 . $\mu_{\mathbb{O}}$ is an algebra monomorphism.

Proof .

Let (q_1,q_2) , $(q_3,q_4) \in \mathbb{O}$ $and\ a \in \mathbb{R}$.

$\mu_{\mathbb{O}}\left(a(q_1,q_2)\right) = \mu_{\mathbb{O}}(aq_1,aq_2) = (\mu_{\mathbb{H}}(aq_1), \mu_{\mathbb{H}}(aq_2))$

$= (a\,\mu_{\mathbb{H}}(q_1), a\,\mu_{\mathbb{H}}(q_2))$ since $\mu_{\mathbb{H}}$ monomorphism

$= a\,(\mu_{\mathbb{H}}(q_1), \mu_{\mathbb{H}}(q_2)) = a\,\mu_{\mathbb{O}}\left((q_1,q_2)\right)$.

$\mu_{\mathbb{O}}\left((q_1,q_2)+(q_3,q_4)\right) = \mu_{\mathbb{O}}(q_1+q_3, q_2+q_4)$

$= (\mu_{\mathbb{H}}(q_1+q_3), \mu_{\mathbb{H}}(q_2+q_4))$

$=(\mu_{\mathbb{H}}(q_1)+\mu_{\mathbb{H}}(q_3), \mu_{\mathbb{H}}(q_2)+\mu_{\mathbb{H}}(q_4))$ since $\mu_{\mathbb{H}}$ monomorphism

$=(\mu_{\mathbb{H}}(q_1), \mu_{\mathbb{H}}(q_2)) + (\mu_{\mathbb{H}}(q_3), \mu_{\mathbb{H}}(q_4)) = \mu_{\mathbb{O}}(q_1,q_2) + \mu_{\mathbb{O}}(q_3,q_4)$

$\mu_{\mathbb{O}}\left((q_1,q_2)(q_3,q_4)\right) = \mu_{\mathbb{O}}(q_1q_3 - q_4^*q_2, q_4q_1 + q_2q_3^*)$

$= (\mu_{\mathbb{H}}(q_1q_3 - q_4^*q_2), \mu_{\mathbb{H}}(q_4q_1 + q_2q_3^*))$

$=(\mu_{\mathbb{H}}(q_1)\,\mu_{\mathbb{H}}(q_3) - \mu_{\mathbb{H}}(q_4^*)\,\mu_{\mathbb{H}}(q_2), \mu_{\mathbb{H}}(q_4)\,\mu_{\mathbb{H}}(q_1) + \mu_{\mathbb{H}}(q_2)\,\mu_{\mathbb{H}}(q_3^*))$

since $\mu_{\mathbb{H}}$ monomorphism and $\mu_{\mathbb{H}}(q_4^*) = (\mu_{\mathbb{H}}(q_4))^*$,

$\mu_{\mathbb{H}}(q_3^*) = (\mu_{\mathbb{H}}(q_3))^*$

$= (\mu_{\mathbb{H}}(q_1), \mu_{\mathbb{H}}(q_2))\,(\mu_{\mathbb{H}}(q_3), \mu_{\mathbb{H}}(q_4))$

$= \mu_{\mathbb{O}}(q_1,q_2)\,\mu_{\mathbb{O}}(q_3,q_4)$

Therefore $\mu_{\mathbb{O}}$ is homomorphism .

Suppose that $\mu_{\mathbb{O}}(q_1,q_2) = (0,0)$.

Then $(\mu_{\mathbb{H}}(q_1), \mu_{\mathbb{H}}(q_2)) = (0,0)$ and

$\mu_{\mathbb{H}}(q_1) = 0,\ \mu_{\mathbb{H}}(q_2) = 0$. Then $q_1 = 0,\ q_2 = 0$ since $\mu_{\mathbb{H}}$ monomorphism

Therefore $\mu_{\mathbb{O}}$ is an algebra monomorphism.

Corollary 9. $\mu_{\mathbb{O}}(\mathbb{O})$ is a subalgebra of $M_4(\mathbb{R}) \oplus M_4(\mathbb{R})$ of dimension 8 with basis $(1,0), (i,0), (j,0), (k,0), (0,1), (0,i), (0,j), (0,k)$.

Let $\phi \in \mathbb{O}$ $.Then\ \phi = (q_1,q_2)\,;\ q_1,q_2 \in \mathbb{H}$.

$\mu_{\mathbb{O}}: \mathbb{O} \longrightarrow (\mu_{\mathbb{H}}(q_1), \mu_{\mathbb{H}}(q_2))$

Let $\acute{\mu}_{\mathbb{O}} : \mathbb{O} \longrightarrow M_4(\mathbb{R}) \oplus M_4(\mathbb{R})$ be give by

$$\acute{\mu}_{\mathbb{O}}(\phi) = \begin{bmatrix} \mu_{\mathbb{H}}(q_1) & 0 \\ 0 & \mu_{\mathbb{H}}(q_2) \end{bmatrix}$$

Theorem 10 . $\acute{\mu}_{\mathbb{O}}$ is an algebra monomorphism

Proof .

Let $\phi = (q_1,q_2)$, $\psi = (q_3,q_4) \in \mathbb{O}$ $and\ a \in \mathbb{R}$

$$\acute{\mu}_{\mathbb{O}}(a\phi) = \acute{\mu}_{\mathbb{O}}(aq_1,aq_2) = \begin{bmatrix} \mu_{\mathbb{H}}(aq_1) & 0 \\ 0 & \mu_{\mathbb{H}}(aq_2) \end{bmatrix}$$

$$= \begin{bmatrix} a\mu_{\mathbb{H}}(q_1) & 0 \\ 0 & a\mu_{\mathbb{H}}(q_2) \end{bmatrix} = a\begin{bmatrix} \mu_{\mathbb{H}}(q_1) & 0 \\ 0 & \mu_{\mathbb{H}}(q_2) \end{bmatrix} = a\,\acute{\mu}_{\mathbb{O}}(\phi) .$$

$$\acute{\mu}_{\mathbb{O}}(\phi+\psi) = \acute{\mu}_{\mathbb{O}}(q_1+q_3, q_2+q_4) = \begin{bmatrix} \mu_{\mathbb{H}}(q_1+q_3) & 0 \\ 0 & \mu_{\mathbb{H}}(q_2+q_4) \end{bmatrix}$$

$$= \begin{bmatrix} \mu_{\mathbb{H}}(q_1)+\mu_{\mathbb{H}}(q_3) & 0 \\ 0 & \mu_{\mathbb{H}}(q_2)+\mu_{\mathbb{H}}(q_4) \end{bmatrix}$$

$$= \begin{bmatrix} \mu_{\mathbb{H}}(q_1) & 0 \\ 0 & \mu_{\mathbb{H}}(q_2) \end{bmatrix} + \begin{bmatrix} \mu_{\mathbb{H}}(q_3) & 0 \\ 0 & \mu_{\mathbb{H}}(q_4) \end{bmatrix}$$

$$= \acute{\mu}_{\mathbb{O}}(\phi) + \acute{\mu}_{\mathbb{O}}(\psi) .$$

$$\acute{\mu}_{\mathbb{O}}(\phi\psi) = \acute{\mu}_{\mathbb{O}}\left((q_1,q_2)(q_3,q_4)\right) = \acute{\mu}_{\mathbb{O}}(q_1q_3 - q_4^*q_2, q_4q_1 + q_2q_3^*)$$

$$= \begin{bmatrix} \mu_{\mathbb{H}}(q_1q_3 - q_4^*q_2) & 0 \\ 0 & \mu_{\mathbb{H}}(q_4q_1 - q_2q_3^*) \end{bmatrix} =$$

$$\begin{bmatrix} \mu_{\mathbb{H}}(q_1)\mu_{\mathbb{H}}(q_3) - \mu_{\mathbb{H}}(q_4^*)\mu_{\mathbb{H}}(q_2) & 0 \\ 0 & \mu_{\mathbb{H}}(q_4)\mu_{\mathbb{H}}(q_1) + \mu_{\mathbb{H}}(q_2)\mu_{\mathbb{H}}(q_3^*) \end{bmatrix}$$

$= (\mu_{\mathbb{H}}(q_1), \mu_{\mathbb{H}}(q_2))(\mu_{\mathbb{H}}(q_3), \mu_{\mathbb{H}}(q_4)) = \acute{\mu}_{\mathbb{O}}(\phi)\,\acute{\mu}_{\mathbb{O}}(\psi)$.

Then $\acute{\mu}_{\mathbb{O}}$ is homomorphism.

Suppose that $\acute{\mu}_{\mathbb{O}}(\phi) = \acute{\mu}_{\mathbb{O}}(q_1,q_2) = 0$

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Then $\begin{bmatrix} \mu_{\mathbb{H}}(q_1) & 0 \\ 0 & \mu_{\mathbb{H}}(q_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

And $\mu_{\mathbb{H}}\ (q_1) = 0, \mu_{\mathbb{H}}\ (q_2) = 0$. Therefore $q_1 = 0, q_2 = 0$

Then $\mu_{\mathbb{O}}{}'$ is an algebra monomorphism.

Our main result is therefore given as follows.

Corollary 11. The matrix representation of Octonions is given by

$$(q_1, q_2) \mapsto \begin{bmatrix} \mu_{\mathbb{H}}(q_1) & 0 \\ 0 & \mu_{\mathbb{H}}(q_2) \end{bmatrix},$$ a diagonal block matrix.

4. CONCLUSIONS

The real Octonions have been represented linearly by order pairs of two real 4×4 matrices.

Our method maybe extended to represent the real sedenions by order pairs of two real 8×8 matrices .

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