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Linearized Kinetic Theory of Plasma Waves

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الملخص

ظهر وضوح صارخ لأوجه القصور في الوصف البسيط للموائع في علاقة التشتت التي حُصِلَ عليها في وجود التأثيرات الحرارية. في هذا العمل، وُضِّحَ بأن استبدال وصف السوائل بأدوات النظرية الحركية يؤدي إلى معالجة هذه القيود. تكون دالة التوزيع f_α خطية ويُوصَف نظام الحركة باستعمال معادلة "Vlasov– Poisson"، نُوقِشَت المصطلحات الثلاثة التي تتحكم في تطور دالة التوزيع المضطرب وتأثيراتها الفيزيائية.

الكلمات المفتاحية: التشتت والاضطراب والتخميد وعدم الاستقرار.

Abstract

A striking illustration of the shortcomings of the simple fluid description appeared in the dispersion relation obtained in the presence of thermal effects. In this work, it is shown that the substitution of the fluid description with the tools of the kinetic theory leads to solving these limitations. The distribution function f_α is linearized and the Vlasov–Poisson kinetic system is described. The three terms that control the evolution of the perturbed distribution function and their physical effects are discussed.

Keywords: dispersion, perturbation, damping, instabilities.

1. INTRODUCTION

In the framework of the fluid theory, the relation has been established that the properties of the various waves propagating in the plasma have been obtained by solving Maxwell's equations in which the anisotropic, time and space dispersive relation between the current and the electromagnetic field has been injected. However, for the problem at hand, this approach has limitations; some of them have been encountered in the fluid theory [1,2] . For instance, the individuality of particles is not taken into account, which has the consequence that the plasma fluid can exchange energy with the wave only if a resonance relation such as ω=k ⃗·υ ⃗_(th,α) is verified. The υ_(th,) α, is an averaged velocity for the species α. The given relation clearly does not allow the selection of a class of particles with a given velocity. Realistically, however, we expect that something will happen for particles having velocity υ ⃗ such as ω=k ⃗·υ ⃗, regardless of the details of the distribution function f_α. Another striking illustration of the shortcomings of the simple fluid description appeared in the dispersion relation obtained in the presence of thermal effects. In this work, we will show that the substitution of the fluid description with the tools of the kinetic theory leads to curing these limitations [3,4] . In Section 2, the distribution function f_α is linearized and the Vlasov–Poisson kinetic system is described. The three terms that control the evolution of the perturbed distribution function and their physical effects are discussed. In Section 3, it is possible to construct a hierarchy of approximations to answer and learn many interesting physics in the process; the technique of the initial value problem is outlined. Finally, in Section 4, the dielectric function is calculated using the Landau contour to deal with the Langmuir waves.

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2. LINEARIZED KINETIC THEORY

Formally, what we are embarking on is an attempt to set up a mean-field theory, separating slow (large-scale) and fast (small-scale) parts of the distribution function, [4]:

$$\boldsymbol{f(r,v,t)=f(\epsilon r,v,\epsilon t)+\hat{f}(r,v,t)} \tag{1}$$

Where ϵ is some small parameter characterizing the scale separation between fast and slow variation (note that this separation need not be the same for spatial and time scales. For simplicity, the spatial dependence of the equilibrium distribution will be dropped altogether and considered homogeneous systems:

$$\boldsymbol{f_0=f(r,v,t)} \tag{2}$$

Which also means $\vec{E}_0$ (there is no equilibrium electric field). Equivalently, all our considerations are restricted to scales much smaller than the characteristic system size. This equilibrium distribution can be defined as the average of the exact distribution over the volume of space that we are considering and over time scales intermediate between the fast and the slow ones:

$$\boldsymbol{f_0(v,t)=\langle f(r,v,t)\rangle=\frac{1}{\Delta t}\int_{t-\Delta t/2}^{t+\Delta t/2}dt'\int\frac{d^3r}{V}\,f(r,v,t')} \tag{3}$$

Where $\omega^{-1}\ll\Delta t\ll t_{eq}$, where t_{eq} represents the equilibrium time scale. In the collisionless limit, the kinetic equation for a neutral gas:

$$\boldsymbol{\frac{\partial f_\alpha}{\partial t}+\vec{v}\cdot\vec{\nabla}f_\alpha=0} \tag{4}$$

Simply describes particles with some initial distribution individualistically flying in straight lines along their initial directions of travel. In contrast, for a plasma, even the collisionless kinetics (and, indeed, especially the collisionless—or weakly collisional— kinetics) is interesting and nontrivial because the particles, via the average properties of their distribution—charge densities and currents,—collectively modify $\vec{E}$ and $\vec{B}$, which then act on individual particles and thus modify f_α, etc. [5]. Thus, we shall henceforth consider a simplified kinetic system, called the Vlasov– Poisson system:

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\alpha - \frac{q_\alpha}{m_\alpha} (\vec{\nabla}\varphi) \cdot \vec{\nabla}_v f_\alpha = 0 \quad (5)$$

$$-\nabla^2 \varphi = 4\pi \sum_\alpha q_\alpha \int d^3v\, f_\alpha \quad (6)$$

Formally, considering collisionless plasma would appear to be legitimate as long as the collision frequency is small compared to the characteristic frequencies of any other evolution that might be going on. It is convenient to work in Fourier space:

$$\varphi(r,t) = \sum_k e^{i\vec{k}\cdot\vec{r}} \varphi_k(t) \; f(r,v,t) = f_0(v,t) + \sum_k e^{i\vec{k}\cdot\vec{r}} \hat{f}_k(v,t \quad (7)$$

Then the Poisson Eq. (6) becomes

$$\varphi_k = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3v\, \hat{f}_{k\alpha} \quad (8)$$

The Vlasov Eq. (5) can be written for k = 0 (i.e., the spatial average of the equation) as;

$$\frac{\partial f_0}{\partial t} + \frac{\partial \hat{f}_{k=0}}{\partial t} = -\frac{q}{m} \sum_k \varphi_{-k}\; i\vec{k} \cdot \frac{\partial \hat{f}_k}{\partial v} \quad (9)$$

Where we can replace $\varphi_{-k} = \varphi_k^*$ because $\varphi(r,t)$ is a real field. Averaging over time according to Eq. (3) eliminates fast variation and gives

$$\frac{\partial f_0}{\partial t} = -\frac{q}{m} \sum_k \langle \varphi_k^*\, i\vec{k} \cdot \frac{\partial \hat{f}_k}{\partial v} \rangle \quad (10)$$

The right-hand side of Eq.(10) describes the slow evolution of the equilibrium (mean) distribution due to the effect of fluctuations. In practice, the main question is often how the equilibrium evolves and so we need a closed equation for the evolution of f_0. This should be obtainable at least in principle because the fluctuating fields appearing in the right-hand side of Eq.(10) themselves depend on f_0. Indeed, writing the Vlasov equation Eq. (5) for the $k \neq 0$ modes, we find the following evolution equation for the fluctuations,[4]:

$$\frac{\partial \hat{f}_k}{\partial v} + i\vec{k} \cdot \vec{v} \hat{f}_k = \frac{q}{m} \varphi_k\, i\vec{k} \cdot \frac{\partial \hat{f}_k}{\partial v} + \frac{q}{m} \sum_{k'} \varphi_{k'}\; i\vec{k}' \cdot \frac{\partial \hat{f}_{k-k'}}{\partial v} \quad (11)$$

The three terms that control the evolution of the perturbed distribution function in Eq.(11) represent the three physical effects that we shall focus on in this work. The second term on the left-hand side describes the free streaming motion of particles. It gives rise to the phenomenon of phase mixing and in its interplay with plasma waves, leading to Landau damping and kinetic instabilities. The first term on the right-hand side contains the interaction of the electric-field perturbations (waves) with the equilibrium particle distribution. The second term on the right-hand side captures the nonlinear interactions between the fluctuating fields and the perturbed distribution—it is negligible only when fluctuation amplitudes are small enough (which, they rarely are) and is responsible for plasma turbulence and other nonlinear phenomena [6]. The programme for determining the slow evolution of the equilibrium is “simple”: solve Eq.(11) together with Eq.(8), calculate the correlation function of the fluctuations, $\langle \varphi_k^* \hat{f}_k \rangle$, as a functional of f_0, and use it to close Eq.(10); then proceed to solve the latter. Obviously, this is impossible to do in most cases. However, it is possible to construct a hierarchy of approximations to the answer and learn many interesting physics in the process [4,7] .

3. HIERARCHY OF APPROXIMATIONS

3.1 Linear Theory Waves

Consider first infinitesimal perturbations of the equilibrium. All nonlinear terms can then be ignored, Eq. (10) turns into $f_0 = const$ and Eq.(11) becomes;

$$\frac{\partial \hat{f}_k}{\partial t} + i\vec{k} \cdot \vec{v} \hat{f}_k = \frac{q}{m} \varphi_k\, i\vec{k} \cdot \frac{\partial \hat{f}_0}{\partial v} \quad (12)$$

Eq. (12) is the linearized kinetic equation. Solving this together with Eq. (8) allows one to find oscillating and growing or decaying perturbations of a particular equilibrium f_0. The theory for doing this is very well developed and contains some of the core ideas that give plasma physics its intellectual shape. Physically, the linear solutions will describe what happens over the short term, that is on times t such that

$$\omega^{-1} \ll \Delta t \ll t_{eq} \text{ or } t_{nl} \quad (13)$$

where ω is the characteristic frequency of the perturbations, t_{eq} is the time after which the equilibrium starts getting modified by the perturbations via Eq. (10) (which depends on the amplitude to which they can grow; if perturbations do grow, i.e., the equilibrium is unstable, they can modify the equilibrium by this mechanism so as to render it stable), and t_{nl} is the time at which perturbation amplitudes become large enough for nonlinear interactions between individual modes to matter (the second term on the right-hand side of Eq.(11); if perturbations grow, they can saturate by this mechanism), [4,8] .

We are concerned with the linearized Vlasov–Poisson system, Eq.(12) and Eq. (8):

$$\frac{\partial \hat{f}_k}{\partial t} + i\vec{k} \cdot \vec{v} \hat{f}_k = \frac{q}{m} \varphi_k\, i\vec{k} \cdot \frac{\partial \hat{f}_0}{\partial v} \quad (14)$$

$$\varphi_k = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3v\, \hat{f}_{k\alpha} \quad (15)$$

For compactness of notation, we have dropped both the species index α and the wave number k in the subscripts. We will discover that electrostatic perturbations in a plasma described by Eq. (14) and Eq.(15) oscillate can pass their energy to particles (damp) or even grow, sucking energy from the particles. We will also discover that it is useful to comprehend some complex analyses.

3.2. Initial-Value Problem

We shall follow Landau’s original paper [4, 8] in considering an initial-value problem—because, as we will see, perturbations can be damped or grow, so it is not appropriate to think of them over $t \in (-\infty, +\infty)$. So we look for $\hat{f}(v,t)$ satisfying Eq.(14) with the initial condition:

$$\hat{f}(v, t=0) = g(v) \quad (16)$$

It is, therefore, appropriate to use the Laplace transform to solve Eq. (14):

$$\mathcal{L}\hat{f}(p) = \int_0^\infty e^{-pt} \hat{f}(t) \quad (17)$$

It is a mathematical certainty that if there exists a real number $\sigma > 0$ such that

$$\left|\hat{f}(t)\right| < e^{\sigma t} \quad t \to \infty \tag{18}$$

Then the integral Eq. (17) exists (i.e., is finite) for all values of p such that $Re\, p > \sigma$. The inverse Laplace transform, giving us back our distribution function as a function of time, is:

$$\hat{f}(t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \mathcal{L}\hat{f}(p) \tag{19}$$

Where the integral in the complex plane is along a straight line parallel to the imaginary axis and intersecting the real axis at $Re\, p = \sigma$, see (Fig.1a). Since we expect to be able to recover our desired time-dependent function $\hat{f}(v,t)$ from its Laplace transform, it is worth knowing the latter. To find it, we Laplace transform Eq. (14):

$$\int_0^\infty dt e^{-pt}\frac{\partial \hat{f}}{\partial t} \to \left[e^{-pt}\hat{f}\right]_0^\infty + p\int_0^\infty dt e^{-pt}\hat{f} \to -g + p\mathcal{L}\hat{f}$$

$$= -i\vec{k}\cdot\vec{v}\mathcal{L}\hat{f} + i\frac{q}{m}\mathcal{L}\varphi\vec{k}\cdot\frac{\partial f_0}{\partial v} \tag{20}$$

Rewriting Eq. (20) we find the following expression:

$$\mathcal{L}\hat{f}(p) = \frac{1}{(p+i\vec{k}\cdot\vec{v})}\left(i\frac{q}{m}\mathcal{L}\varphi\vec{k}\cdot\frac{\partial f_0}{\partial v} + g\right) \tag{21}$$

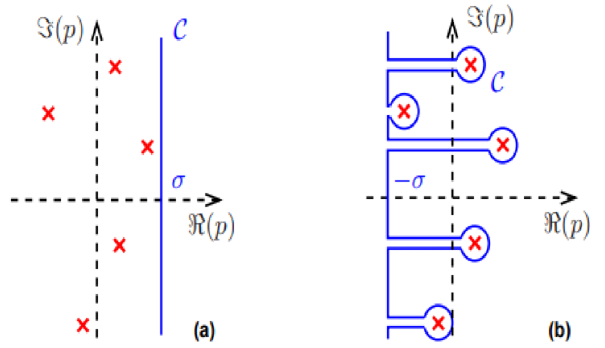

Figure (1): Layout of the complex-p plane: (a) Original contour. (b) Deformation of the integration contour following Landau's method. The poles are denoted by crosses. $\hat{f}(p)$, is analytic for $Re\, p > \sigma$. At $Re\, p < \sigma$, $\hat{f}(p)$ may have singularities (poles).

The Laplace transform of the potential, $\mathcal{L}\varphi(p)$, itself depends on $\hat{f}(p)$ by Eq. (14):

$$\mathcal{L}\varphi(p) = \int_0^\infty dt e^{-pt}\varphi(t) = \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int d^3v \mathcal{L}\hat{f}_\alpha(p)$$

$$= \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int d^3v \frac{1}{(p+i\vec{k}\cdot\vec{v})}\left(i\frac{q_\alpha}{m_\alpha}\mathcal{L}\varphi(p)\vec{k}\cdot\frac{\partial f_{\alpha 0}}{\partial v} + g_\alpha\right) \tag{22}$$

This is an algebraic equation for $\mathcal{L}\varphi(p)$. Collecting terms, we get

$$\left[1 + \sum_\alpha i\frac{4\pi q_\alpha^2}{k^2 m_\alpha}\int d^3v \frac{1}{(p+i\vec{k}\cdot\vec{v})}\vec{k}\cdot\frac{\partial f_{\alpha 0}}{\partial v}\right]\mathcal{L}\varphi(p) \equiv \epsilon(p,\vec{k})\mathcal{L}\varphi(p)$$

$$= \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int d^3v \frac{g_\alpha}{(p+i\vec{k}\cdot\vec{v})} \tag{23}$$

Where $\epsilon(p,\vec{k})$ is called the dielectric function because it encodes all the self-consistent charge-density perturbations that plasma sets up in response to an electric field. This is going to be an important function, written as:

$$\epsilon(p,\vec{k}) \equiv 1 + \sum_\alpha i\frac{4\pi q_\alpha^2}{k^2 m_\alpha}\int d^3v \frac{1}{(p+i\vec{k}\cdot\vec{v})}\vec{k}\cdot\frac{\partial f_{\alpha 0}}{\partial v} \tag{24}$$

Where the plasma frequency of species α is defined by:

$$\omega_{pe} = \sqrt{\frac{4\pi e^2 n_e}{m_e}} \tag{25}$$

The solution of Eq.(23) is:

$$\mathcal{L}\varphi(p) = \frac{4\pi}{k^2\epsilon(p,\vec{k})}\sum_\alpha q_\alpha \int d^3v \frac{g_\alpha}{(p+i\vec{k}\cdot\vec{v})} \tag{26}$$

To calculate $\varphi(t)$, we need to inverse-Laplace-transform $\mathcal{L}\varphi(p)$: similar to Eq.(19):

$$\varphi(t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt}\mathcal{L}\varphi(p) \tag{27}$$

Recall that $\mathcal{L}\hat{f}(p)$ and, therefore, $\mathcal{L}\varphi(p)$ only exists (i.e., is finite) for $Re\, p > \sigma$, whereas at $Re\, p < \sigma$, it can have singularities, i.e., poles-let us call them p_i, indexed by i. If we analytically continue $\mathcal{L}\varphi(p)$ everywhere to $Re\, p < \sigma$ except those poles, the result must have the form:

$$\mathcal{L}\varphi(p) = \sum_i \frac{a_i}{p-p_i} + A(p) \tag{28}$$

Where a_i are some coefficients (residues) and $A(p)$ is the analytic part of the solution. The integration contour in Eq.(27) can be shifted to $Re\, p \to -\infty$ but with the condition that it cannot cross the poles, as shown in Fig. (1b) (this is proven by making a closed loop out of the old and the new contours, joining them at $\pm i\infty$, and noting that this loop encloses no poles). Then the contributions to the integral from the vertical segments of the contour are exponentially small, the contributions from the segments leading towards and away from the poles cancel, and the contributions from the circles around the poles can, by Cauchy's formula, be expressed in terms of the poles and residues:

$$\varphi(t) = \sum_i a_i e^{p_i t} \tag{29}$$

Thus, in the long-time limit, perturbations of the potential will evolve $\propto e^{p_i t}$, where p_i, are poles of $\mathcal{L}\varphi(p)$. In general, $p_i = -i\omega_i + \gamma_i$, where ω_i is a real frequency (giving wavelike behavior of perturbations), $\gamma_i < 0$ represents damping and $\gamma_i > 0$ growth of the perturbations (instability).

Note that we need not be particularly interested in what a_i's are because, if we set up an initial perturbation with a given k and then wait long enough, only the fastest growing or, failing growth, the slowest-damped mode will survive, with all others having exponentially small amplitudes. Thus, a typical outcome of the linear theory is $\varphi(t)$ oscillating at some frequency and growing or decaying at some unique rate. Since this is a solution of a linear equation, the coefficient in front of the exponential can be scaled arbitrarily and so does not matter. Going back to Eq.(26), we realize that the poles of $\mathcal{L}\varphi(p)$ are zeros of the dielectric function, **[8]** :

$$\epsilon(p_i,\vec{k}) = 0 \quad \Rightarrow \quad p_i = p_i(k) = -i\omega_i(k) + \gamma_i(k) \tag{30}$$

To find the wave frequencies ω_i and the damping/growth rates γ_i, we must solve this equation, which is called the plasma dispersion relation.

4. CALCULATING THE DIELECTRIC FUNCTION (Landau Prescription):

Thus in order to be able to solve $\epsilon(p,\vec{k}) = 0$, we must learn how to calculate $\epsilon(p,\vec{k})$ for any given p and $\vec{k}$. The function $\mathcal{L}\varphi(p)$ given by Eq. (26), must be analytically continued to the entire complex plane from the area where its analyticity was guaranteed ($Re\, p > \sigma$), [4]. In order to do it, we must learn how to calculate the velocity integral in Eq. (24)—if we want $\epsilon(p,\vec{k})$ and, therefore, its zeros p_i—and also how to calculate the similar integral in Eq.(26) containing g_α. First of all, let us turn these integrals into a $1D$ form. Given k, we can always choose the z-axis to be along $k \rightarrow i\vec{k}\cdot\vec{v} = ikv_z$. Thus

$$\int d^3v \frac{1}{(p+i\vec{k}\cdot\vec{v})}\vec{k}\cdot\frac{\partial f_0}{\partial v}$$

$$= \int dv_z \frac{1}{p+ikv_z} k \frac{\partial}{\partial v_z}\int dv_x \int dv_y\, f_0(v_x, v_y, v_z\}$$

$$= \int dv_z \frac{1}{p+ikv_z} k \frac{\partial}{\partial v_z}\mathcal{F}(v_z) = -i\int_{-\infty}^{+\infty} dv_z \frac{\mathcal{F}'(v_z)}{v_z - ip/k} \qquad (31)$$

Assuming, reasonably, that $\mathcal{F}'(v_z)$ is a nice (analytic) function everywhere, we conclude that the integrand in Eq. (31) has one pole, $v_z = ip/k$. When $Re\, p > \sigma > 0$, this pole is harmless because, in the complex plane associated with the v_z variable, it lies above the integration contour, which is the real axis, $v_z \in (-\infty, +\infty)$. We can think of analytically continuing the above integral to $Re\, p < \sigma$ as moving the pole $v_z = ip/k$ down, towards and below the real axis. As long as $Re\, p > 0$, this can be done with impunity, in the sense that the pole stays above the integration contour, and so the analytic continuation is simply the same integral Eq.(31), still along the real axis.

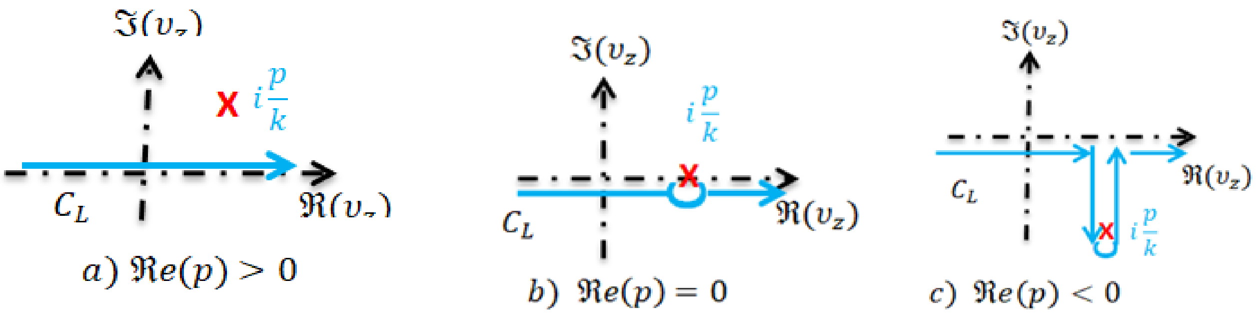

Figure (2): The Landau prescription for the contour of integration in Eq.(31).

However, if the pole moves so far down that $Re\, p = 0$ or $Re\, p < 0$, we must deform the contour of integration in such a way as to keep the pole always above it, as shown in Fig. (2). This is called the Landau prescription and the contour thus deformed is called the Landau contour, C_L. Let us prove that this is indeed an analytic continuation, i.e., that the integral Eq. (31) adjusted to be along C_L, is an analytic function for all values of p. Let us cut the Landau contour at $v_z = \pm R$ and close it in the upper half-plane with a semicircle C_R of radius $R > \sigma/k$ see, Fig. (.3). Then, with integration running along the truncated C_L and counterclockwise along C_R, we get, by Cauchy's formula, [4,9] :

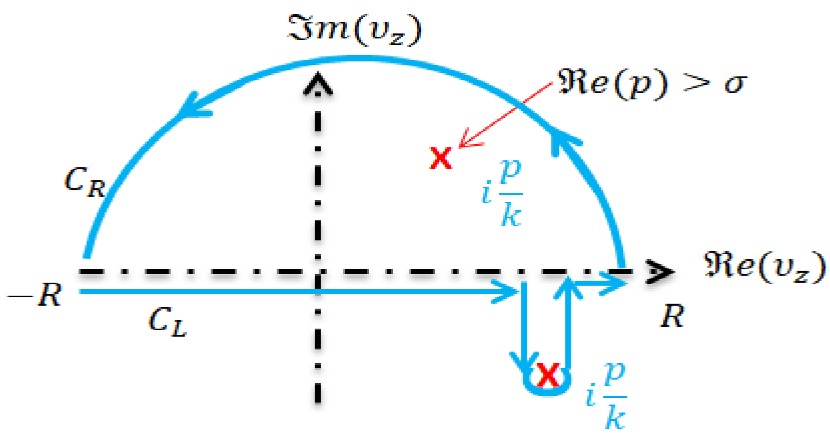

Figure (3):. Proof of Landau's prescription, see Eq. (32).

$$\int_{C_L} dv_z \frac{\mathcal{F}'(v_z)}{v_z - ip/k} + \int_{C_R} dv_z \frac{\mathcal{F}'(v_z)}{v_z - ip/k} = 2\pi i\, \mathcal{F}'(ip/k) \quad (32)$$

Since analyticity is guaranteed for $Re\, p > \sigma$, the integral along C_R is analytic. The righthand side is also analytic, by assumption. Therefore, the integral along C_L is analytic— this is the integral along the Landau contour if we take $R \to \infty$. With the Landau prescription, our integral is calculated as follows:

$$\int_{C_L} dv_z \frac{\mathcal{F}'(v_z)}{v_z - \frac{ip}{k}} = \begin{cases} \int_{-\infty}^{+\infty} dv_z \frac{\mathcal{F}'(v_z)}{v_z - \frac{ip}{k}}, & Re\, p > 0 \\ \mathcal{P}\int_{-\infty}^{+\infty} dv_z \frac{\mathcal{F}'(v_z)}{v_z - ip/k} + i\pi \mathcal{F}'(ip/k), & Re\, p = 0 \\ \int_{-\infty}^{+\infty} dv_z \frac{\mathcal{F}'(v_z)}{v_z - ip/k} + i2\pi\mathcal{F}'(ip/k)\Big\}, & Re\, p < 0 \end{cases} \quad (33)$$

Where the integrals are again over the real axis and the imaginary bits come from the contour making a half (when $Re\, p = 0$) or a full (when $Re\, p < 0$) circle around the pole. In the case of $Re\, p = 0$, or $ip = \omega$, the integral along the real axis is formally divergent and so we take its principal value, defined as

$$\mathcal{P}\int_{-\infty}^{+\infty} dv_z \frac{\mathcal{F}'(v_z)}{v_z - \omega/k} = \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{\frac{\omega}{k}-\epsilon} dv_z \frac{\mathcal{F}'(v_z)}{v_z - \omega/k} + \int_{\frac{\omega}{k}+\epsilon}^{+\infty} dv_z \frac{\mathcal{F}'(v_z)}{v_z - \omega/k} \right\} \quad (34)$$

The difference between Eq.(34) and the usual Lévesque definition of an integral is that the latter would be:

$$\int_{-\infty}^{+\infty} dv_z \frac{\mathcal{F}'(v_z)}{v_z - \omega/k} = \left\{ \lim_{\epsilon_1 \to 0} \int_{-\infty}^{\frac{\omega}{k}-\epsilon_1} dv_z \frac{\mathcal{F}'(v_z)}{v_z - \omega/k} + \lim_{\epsilon_2 \to 0} \int_{\frac{\omega}{k}+\epsilon_2}^{+\infty} dv_z \frac{\mathcal{F}'(v_z)}{v_z - \omega/k} \right\} \quad (35)$$

And this, with, in general, $\epsilon_1 \neq \epsilon_2$, diverges logarithmically, whereas in Eq.(34), the divergences neatly cancel.

The $Re\, p = 0$ case in Eq. (33):

$$\int_{C_L} dv_z \frac{\mathcal{F}'(v_z)}{v_z - \frac{\omega}{k}} = \mathcal{P}\int_{-\infty}^{+\infty} dv_z \frac{\mathcal{F}'(v_z)}{v_z - \omega/k} + i\pi\mathcal{F}'(\omega/k) \quad (36)$$

Which tends to be of most use in analytical theory, is a particular instance of Plemelj's formula: for a real ζ and a well-behaved function f (no poles on or near the real axis)

$$\lim_{\epsilon \to +0} \int_{-\infty}^{+\infty} dx \frac{f(x)}{x - \zeta \pm i\epsilon} = \mathcal{P}\int_{-\infty}^{+\infty} dx \frac{f(x)}{x - \zeta} + i\pi f(\zeta) \quad (37)$$

Also sometimes written as

$$\lim_{\epsilon \to +0} \frac{1}{x - \zeta \pm i\epsilon} = \mathcal{P}\frac{1}{x - \zeta} \pm i\pi\delta(x - \zeta) \quad (38)$$

Finally, armed with Landau's prescription, we are ready to calculate. The dielectric function Eq. (24) becomes

$$\epsilon(p, \vec{k}) \equiv 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n^2} \int_{C_L} dv_z \frac{\mathcal{F}_\alpha'(v_z)}{v_z - ip/k} \quad (39)$$

And, analogously, our Laplace-transformed solution Eq. (26) becomes

$$\mathcal{L}\varphi(p) = \frac{4\pi i}{k^3 \epsilon(p, \vec{k})} \sum_\alpha q_\alpha \int dv_z \frac{G_\alpha(v_z)}{(v_z + ip/k)} \quad (40)$$

Where $G_\alpha(v_z) = \int dv_x \int dv_y g_\alpha(v_x, v_y, v_z)$

4.1. Solving the Dispersion Relation

The limit of slow damping and growth is a particularly analytically solvable and physically interesting case is one in which, for $p = -i\omega + \gamma$, $\gamma \ll \omega$ and $\gamma \ll k v_{th\alpha}$ that is the case of the damping or growth time of the waves being longer than either their period or the time particles take to cross them. In this limit, the dispersion relation Eq. (30), [4, 9] is:

$$\epsilon(p, \vec{k}) \approx \epsilon(-i\omega, \vec{k}) + i\gamma \frac{\partial}{\partial\omega}\epsilon(-i\omega, \vec{k}) = 0 \quad (41)$$

Setting the real part of Eq. (41) to zero gives the equation for the real frequency:

$$Re\, \epsilon(-i\omega, k) = 0 \quad (42)$$

Setting the imaginary part of Eq. (41) to zero gives us the damping/growth rate in terms of the real frequency:

$$\gamma = Im\epsilon(-i\omega, \vec{k}) \left[\frac{\partial}{\partial\omega} Re\epsilon(-i\omega, \vec{k})\right]^{-1} \quad (43)$$

Hence, we need only $\epsilon(p, \vec{k})$ with $p = -i\omega$. Using Eq. (36), we get

$$Re\epsilon = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n^2} \int_{C_L} dv_z \frac{\mathcal{F}_\alpha'(v_z)}{v_z - \omega/k} \quad (44)$$

$$Im\epsilon = -\sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \frac{\pi}{n^2} \mathcal{F}_\alpha'\left(\frac{\omega}{k}\right) \quad (45)$$

Let us consider two-species plasma, consisting of electrons and a single species of ions. There will be two interesting limits:

- "High-frequency" electron waves: $\omega \gg k v_{th,}$ where $v_{th,e} = \sqrt{2T_e/m_e}$ is the "thermal speed" of the electrons; this limit will give us Langmuir waves, slowly damped or growing.
- "Low-frequency" ion waves: a particularly tractable limit will be that of "hot" electrons and "cold" ions, viz., $v_{th,e} \gg \omega \gg v_{th,}$ i, where $v_{th,i} = \sqrt{2T_i/m_i}$ is the "thermal speed" of the ions; this limit will give us the sound "ion-acoustic waves", which also can be damped or growing.

5. CONCLUSION:

Consider the limit $\frac{\omega}{k} \gg v_{th,e}$ that is the phase velocity of the waves is much greater than the typical velocity of a particle from the "thermal bulk" of the distribution. This means that in Eq. (44), we can expand in $v_z \sim v_{th,e}$ being small compared to ω/k (higher values of v_z are cut off by the "thermal" fall-off of the equilibrium distribution function). Note that $\omega \gg k v_{th,e}$ also implies $\omega \gg k v_{th,i}$ because $\frac{v_{th,i}}{v_{th,e}} = \sqrt{\frac{T_i}{T_e}\frac{m_e}{m_i}} \ll 1$, this is as long as that $\frac{T_i}{T_e}$ is not too huge. Thus, Eq. (44) becomes:

$$Re\epsilon = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_\alpha} \frac{k}{\omega} \mathcal{P}\int dv_z \mathcal{F}_\alpha'(v_z) \left[1 + \frac{k v_z}{\omega} + \left(\frac{k v_z}{\omega}\right)^2 + \left(\frac{k v_z}{\omega}\right)^3 + \cdots\right]$$

$$= \mathbf{1} + \sum_\alpha \frac{\omega_{p\alpha}^2}{k\omega}\left[\frac{1}{n_\alpha}\int dv_z \mathcal{F}'_\alpha(v_z) - \frac{k}{\omega}\frac{1}{n_\alpha}\int dv_z \mathcal{F}_\alpha(v_z) - 2\left(\frac{k}{\omega}\right)^2 \frac{1}{n_\alpha}\int dv_z\, v_z \mathcal{F}_\alpha(v_z) - 3\left(\frac{k}{\omega}\right)^3 \frac{1}{n_\alpha}\int dv_z v_z^2\, \mathcal{F}_\alpha(v_z)\right] = \mathbf{1} + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\left[\mathbf{1} + \frac{3}{2}\frac{k^2 v_{th,\alpha}^2}{\omega^2} + \cdots\right] \quad \mathbf{(46)}$$

Where we have integrated by parts everywhere and assumed that there are no mean flows, $\langle v_z \rangle = 0$, and for the terms inside the bracket we obtain zeros for the 1st term and the 3 rd terms because the integrals are odd in v_z. The 2nd term is unity and in the last term, we used: $\langle v_z^2 \rangle = \frac{v_{th,\alpha}^2}{2}$, Which is the result in the case for a Maxwellian $\mathcal{F}_\alpha$ or, if $\mathcal{F}_\alpha$ is not a Maxwellian, can be viewed as the definition of $v_{th,\alpha}$. The ion contribution to Eq. (46) is small because $\frac{\omega_i^2}{\omega_e^2} = \frac{Zm_e}{m_i} \ll 1$ so that ions do not participate in this dynamics at all. Therefore, in the lowest order, the dispersion relation Eq. (42) becomes

$$\boldsymbol{Re\epsilon} = \mathbf{1} - \frac{\omega_{pe}^2}{\omega^2} = \mathbf{0} \quad \Rightarrow \boldsymbol{\omega^2} = \boldsymbol{\omega_{pe}^2} = \frac{4\pi e^2 n_e}{m_e} \quad \mathbf{(47)}$$

This is the Langmuir dispersion relation or plasma oscillations. We can do a little better if we retain the (small) k-dependent term in Eq. (46):

$$\boldsymbol{Re\epsilon} \approx \mathbf{1} - \frac{\omega_{pe}^2}{\omega^2}\left(\mathbf{1} + \frac{3}{2}\frac{k^2 v_{th,\alpha}^2}{\omega^2}\right) = \mathbf{0} \quad \Rightarrow \boldsymbol{\omega} \approx \boldsymbol{\omega_{pe}}\sqrt{\left(\mathbf{1} + \mathbf{3}\boldsymbol{k^2\lambda_{De}^2}\right)} \quad \mathbf{(48)}$$

Where $\lambda_{De} = \frac{v_{th,e}}{\sqrt{2}\omega_{pe}} = \sqrt{\frac{T_e}{4\pi e^2 n_e}}$ is the electron Debye length. Eq. (48) is the Bohm and Gross dispersion relation, describing an upgrade of the Langmuir oscillations to dispersive Langmuir waves, which have a non-zero group velocity (this effect is due to the electron pressure perturbation joining the electric field in providing the restoring force for the waves. Note that all this is only valid for $\omega \gg kv_{th,e}$ which we now see is equivalent to $k\lambda_{De} \ll 1$ This shows that the wavelength of the perturbation is long compared to the Debye length, λ_{De}.

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