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Existence and unique of the mild solution of stochastic integro differential equation

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الملخص:

هذه الورقة تقوم بتوظيف التقريب المتتالي لإظهار وجود الحل المبسط لمعادلة تفاضلية تكاملية عشوائية مع W(t standard Brownian motion).

$$\frac{dx(t)}{dt} - h(t)x(t) = \int_0^t B(t,s)x(s)dw(s), \qquad (1.1)$$

باستخدام النظرية (2.1) لإيجاد الحل المبسط

$$X(t) = \psi + \int_0^t a(s,X(s))ds + \int_0^t b(s,X(s))dW(s) \text{ For } 0 \le t \le T.$$

ويتم ذلك بفرض المعادلة التفاضلية العشوائية (2.1) التي تملك الحل (2.2) [1] وبالتعويض من (2.2) في (2.1) نحصل على:

$$V(t) = \int_0^t B(t,s)Q(s)\, x_0 dw(s) + \int_0^t \int_0^s B(t,s)Q(s,\tau)V(\tau)d\tau dw(s)$$

وإثبات وجوده بالتقريب المتتالي, وهو ما يكافئ التكامل العشوائي في المعادلة (1.1) عندها نستطيع القول إن:

$$X(t) = \psi + \int_0^t a(s,X(s))ds + \int_0^t b(s,X(s))dW(s) \text{ For } 0 \le t \le T.$$

هو الحل المبسط للمعادلة التفاضلية التكاملية العشوائية .

أيضا استخدمنا Gronwall inequality لضمان وحدانية الحل.

والتوصية باستخدام طريقة التقريب المتتالي لإظهار وجود الحل المبسط لمعادلة تفاضلية تكاملية عشوائية وذلك بفرض معادلتها التفاضلية العشوائية وحلها [2] وبعض التعويضات, أيضا استخدام Gronwall inequality لضمان وحدانية الحل.

الكلمات المفتاحية:

المعادلة التفاضلية العشوائية، طريقة التقريب المتتالي، وحدانية الحل، مساحة الاحتمال، نظرية التفرد.

Abstract

This paper is devoted to show the existence of the mild solution of the stochastic integro-differential equation by employing the successive approximation with standard Brownian motion W(t).

$$\frac{dx(t)}{dt} - h(t)x(t) = \int_0^t B(t,s)x(s)dw(s), \qquad (1.1)$$

Using theorem (2.1), to find the mild solution

$$X(t) = \psi + \int_0^t a(s,X(s))ds + \int_0^t b(s,X(s))dW(s) \text{ For } 0 \le t \le T.$$

This is done by applying the stochastic differential equation (2.1) and that provides the solution (2.2) [2] and by compensation (2.2) for (2.1) we get:

$$V(t) = \int_0^t B(t,s)Q(s)\, x_0 dw(s) + \int_0^t \int_0^s B(t,s)Q(s,\tau)V(\tau)d\tau dw(s)$$

It proves its existence by the successive approximation and its equivalent integral stochastic in equation (1.1), then we can say:

$$X(t) = \psi + \int_0^t a(s,X(s))ds + \int_0^t b(s,X(s))dW(s) \text{ For } 0 \le t \le T.$$

To find the mild solution of the stochastic integro-differential equation, we used the descriptive and experimental approach. Also, we used Gronwall inequality to ensure the uniqueness of the solution.

Keywords: Standard Brownian motion W(t), stochastic differential equation, the successive approximation, the mild solution, stochastic integr- differential equation.

1. INTRODUCTION

Many differential equations that were developed to describe physical phenomena have ignored stochastic effects because of the solution difficulty. Deterministic models can often be improved by including stochastic effects; however, a more detailed study of the properties of solutions to stochastic differential equations (SDE's) is needed nowadays. These

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stochastic differential equations occur naturally in many fields of mechanics, mathematical physics and physics and mathematical finance. They also rise as representation formulas for the solutions of integral equations according to their boundary conditions. Integral equations also form one of the most useful tools in many branches of pure analysis, such as a functional analysis and stochastic calculus[2, 3,4].

The transition from ordinary differential equation (ODE) to (SDE) takes place by incorporating random elements in the differential equation. Randomness can be included in the initial value problem; alternatively, the function that describes the physical system can be a random function.

This randomness is suitable for describing the rapidly fluctuating random phenomena and can be modeled by a Brownian motion or Wiener process see[5,6,7,8,9,10,11].

Now consider the stochastic integro-differential equation

$$\frac{dx(t)}{dt} - h(t)x(t) = \int_0^t B(t,s)x(s)dw(s), \qquad (1.1)$$

with the initial condition x(0)=x_0,

for each t∈J=[0,T], x is unknown function, W(t) is a standard Browning motion defined over the filtered probability space (Ω,F,P). And h is a bounded Borel function from [0,∞)×R^d→R^d to R^d. It is supposed that {B(t,s),0≤s≤t≤T} are family of bounded operators, and B(t,s)g is continuous on 0≤t≤T , 0≤s≤t≤T for every g∈H, where H a Hilbert space then ‖B(t,s)g‖≤K>0.

1.1. Definition A probability space is a measured space with total mass. The space Ω is called the sample space, and its sample points ω∈Ω are the elementary outcomes of the experiment, F is a σ-algebra of Ω. The measurable sets in F are called events. P is a probability measure. X:Ω→S with values in some measurable space S. A random variables X is a measurable function.

1.2. Lemma See1 If A_nis a sequence of independent events and if $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A_n i.o.) = 0$, where $P(A_n i.o.) = \lim_{n\to\infty} P(\cup_{n=1}^{\infty} A_n)$.

1.3. Definition If X is a random variable on (Ω, F, P) which is finite w.p.1 then its distribution function is $F(t) = P(\omega: X(\omega) \le t)$. this gives us the convenient expressions $f(\omega: X(\omega) \in h) = \int_h dF(t)$, for any Borel set h of F.

1.4. Definition Let W(t) is a standard Browning motion in R^m with respect to a filtration $\{F_t\}$. Let ψ be $\mathbb{R}^d$- valued F_0-measurable random variable. Often ψ is a nonrandom point x_0. The assumption of F_0-measurability implies that ψ is independent of the Browning motion. Then an equation

$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t), \quad X(0) = \psi,$ is called Itô stochastic differential equation or in integral form

$X(t) = \psi + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dW(s)$, where $a(t,x), b(t,x)$ are Borel measurable functions of $(t,x) \in [0,\infty) \times \mathbb{R}^d$.

1.5. Definition let $0 < t_1 < t_2 < \cdots < t_n$ be a partition of the interval $[0,T]$ and let $g(W(t),t) = g(t)$ is a continuous function in $[0,T]$. The stochastic integral $\int_0^T g(t)\,dW(t)$ is defined by

$$I(g) = \int_0^T g(t)\,dW(t) = \lim_{n\to\infty} g(t_j)(W(t_{j+1}) - W(t_j)).$$

1.6. Theorem consider the equation

$$X(t) = H(t) + \int_{(0,t]} F(s,X)dW(s) \qquad (1.2),$$

Where W is a given $\mathbb{R}^m$-valued cadlag semimartingle $H$is agiven $\mathbb{R}^d$-valued process. The coefficient F is a $d \times m$-matrix valued function of its arguments.

- If F is a map from the space $\mathbb{R}_+ \times \Omega \times O_{\mathbb{R}^d}[0,\infty)$ into the space $\mathbb{R}^{d\times m}$ of $d \times m$ matrices. F satisfies a spatial Lipschitz condition uniformly in the other variables (i.e) ∃ a finite constant L s.t $|F(t,\eta) - F(t,\xi)| \le L\, sup|\eta(s) - \xi(s)|$ for all $t \in \mathbb{R}_+$ and $\eta, \xi \in O_{\mathbb{R}^d}[0,\infty)$ where $O_{\mathbb{R}^d}$ be an open subset of $\mathbb{R}^d$.
- Given any adapted $\mathbb{R}^d$-valued cadlag process X on Ω, the function $(\omega) \mapsto F(T, \omega, X(\omega))$ is a predictable process, and there exist stopping times $\tau_k \nearrow \infty$ such that $1_{(0,\tau_k)}(t)F(t,X)$ is bounded for each k.
  Then there exists a unique cadlag process $X(t): 0 \le t < \infty$ adapted to F_t that satisfies equation (1.2).

2. THE MILD SOLUTION:

2.1. Theorem Let $W(t)$ be a standard Browning motion in $\mathbb{R}^m$ with respect to a right continuous filtration $F(t)$ and ψ on $\mathbb{R}^d$-valued. Fix $0 < T < \infty$, assume the function $a: [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $b: [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$, satisfy Lipschitz condition $|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \le L|x-y|$, and bound $|a(t,x)| + |b(t,x)| \le L(1+|x|)$ for constant L and all $0 \le t \le T$, $x, y \in \mathbb{R}^d$ then there exists a unique continuous process that is adapted to and satisfies :

$$X(t) = \psi + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dW(s)$$

For $0 \le t \le T$.

2.2. The solution

Now to find the solution of eq. (1.1) we assume the stochastic differential equation

$$\frac{dx(t)}{dt} - h(t)x(t)\; V(t) \qquad (2.1)$$

Has a solution

$$x(t) = Q(t)\,x_0 + \int_0^t Q(t,s)V(s)ds \qquad (2.2)$$

Where Q(t) is a bounded operator2. And substitute (2.1), (2.2) in (1.1), we get

$$V(t) = \int_0^t B(t,s)Q(s)\,x_0 dw(s) + \int_0^t \int_0^s B(t,s)Q(s,\tau)V(\tau)d\tau dw(s) \quad (2.3)$$

3. THE EXISTENCE AND UNIQUENESS THEOREM :

In this section, we prove the existence and uniqueness of a solution (2.3), which provided the solution of eq. (1.1).

3.1. Definition The integral of X with respect to P is called the expection of

X, and written as $E[X] = \int_{\Omega} X(\omega) dP(\omega)$, while

$E[X; A] = \int_{A} X(\omega) dP(\omega)$, Where A any event.

3.2. Proposition If $f \geq 0$ then $E[f(X)] = \int f(X) P_X dX$.

If X, Y are independent and f, g are Borel measurable functions, then $f(X)$ and $g(Y)$ are independent.

If X, Y and $X\,Y$ are integrable then $E[X\,Y] = (E[X])(E[Y])$.

The characteristic function of random variable X is Fourier transform of distribution

$\int e^{iuX} P_X(dX) = E\left[e^{iuX}\right]$, if X, Y are independent then $E\left[e^{i(uX+vY)}\right] = E\left[e^{iuX}\right] E\left[e^{ivY}\right]$, and the invers also hold.

3.3. Definition A sequence of random variables $\{X_n(\omega)\}$ converges in the mean square to

$\{X(\omega)\}$ if $\lim_{n\to\infty} \mathrm{E}(|X_n - X|^2) = 0,\ E(X^2) < \infty$.

Now we will use the method of successive approximation to find the existing of eq. (2.3) we get:

$$V_{k+1}(t) = \int_0^t B(t,s) Q(s)\, x_0 dw(s) + \int_0^t \int_0^s B(t,s) Q(s,\tau) V_k(\tau) d\tau dw(s), \qquad (3.1)$$

Then

$$V_{k+1}(t) - V_k(t) = \int_0^t \int_0^s B(t,s) Q(s,\tau)[V_k(\tau) - V_{k-1}(\tau)] d\tau dw(s),$$

Hence

$$\|V_{k+1}(t) - V_k(t)\|^2 = \left\| \int_0^t \int_0^s B(t,s) Q(s,\tau)[V_k(\tau) - V_{k-1}(\tau)] d\tau dw(s) \right\|^2,$$

$$E\|V_{k+1}(t) - V_k(t)\|^2 \leq \int_0^t \int_0^s \|B(t,s)\|^2 \|Q(s,\tau)\|^2\, E\|V_k(\tau) - V_{k-1}(\tau)\|^2 d\tau ds,$$

Or

$$E\|V_{k+1}(t) - V_k(t)\|^2 \leq K^2 C^2 \int_0^t \int_0^s E\|V_k(\tau) - V_{k-1}(\tau)\|^2 d\tau ds. \qquad (3.2)$$

3.4. Theorem If the series $\sum_{k=0}^{n} E\|V_{k+1}(t) - V_k(t)\|^2$ uniformly converges on [0,T], (i.e)

E‖V_(k+1) (t)-V_k (t)‖^2≤L^k t^2k/((2k)!) then solution (2.3) exist.

Proof. The proof by induction. Set V_0 (t)=0. First at k=1 we have from eq. (3.2)

$$E\|V_2(t) - V_1(t)\|^2 \leq K^2 C^2 \int_0^t \int_0^s E\|V_1(\tau)\|^2 d\tau ds, \qquad (3.3)$$

Using eq.(3.1), we get

$$\|V_1(t)\|^2 = \left\| \int_0^t B(t,s) Q(s)\, x_0 dw(s) \right\|^2,$$

$$E\,\|V_1(t)\|^2 = \int_0^t \|B(t,s)\|^2 \|Q(s)\|^2 E\|\, x_0\|^2 ds,$$

$$E\,\|V_1(t)\|^2 = K^2 C^2 E\|\, x_0\|^2 t = M, \qquad (3.4)$$

Where 〖M=max┬(t∈[0,T])〗⁡〖{K^2 C^2 E‖ 〖 x〗_0 ‖^2 t}.〗

Now substitute (3.4) in eq. (3.3) we get

E‖V_2 (t)-V_1 (t)‖^2≤K^2 C^2 M t^2/2≤L t^2/2! .

Where L=K^2 C^2 M.

At k=n, we assume that E‖V_(n+1) (t)-V_n (t)‖^2≤L^n t^2n/(2n)! .

Now at k=n+1, from eq.(3.2)

$$E\|V_{n+2}(t) - V_{n+1}(t)\|^2 \leq K^2 C^2 \int_0^t \int_0^s E\|V_{n+1}(\tau) - V_n(\tau)\|^2 d\tau ds,$$

$$E\|V_{n+2}(t) - V_{n+1}(t)\|^2 \leq K^2 C^2 \int_0^t \int_0^s L^n \frac{\tau^{2n}}{(2n)!} d\tau ds,$$

E‖V_(n+2) (t)-V_(n+1) (t)‖^2≤L^(n+1) 1/(2(n+1))! t^2(n+1) ,

Where L^(n+1)=K^2 C^2 L^n. Thus the inequality is true for all value of n.

3.5. Gronwall inequalities See[3] assume that the continuous function

$x, y: [0,T] \to [0,\infty)$ and $K > 0$ satisfy $x(t) \leq K + \int_0^t y(s) x(s) ds$ for all, $t \in [0,T]$. Then the usual Gronwall inequality is $x(t) \leq K\, exp\left(\int_0^t y(s) ds\right)$.

3.6. Uniqueness theorem Let Let $V(t), V^*(t)$ are two solutions of equation (2.3), then

$$V(t) - V^*(t) = \int_0^t \int_0^s B(t,s) Q(s,\tau)\big(V(\tau) - V^*(\tau)\big) d\tau dw(s),$$

$$\|V(t) - V^*(t)\|^2 = \left\| \int_0^t \int_0^s B(t,s) Q(s,\tau)\big(V(\tau) - V^*(\tau)\big) d\tau dw(s) \right\|^2,$$

$$\boldsymbol{E\|V(t) - V^*(t)\|^2 \leq \int_0^t \int_0^s \|B(t,s)\|^2 \|Q(s,\tau)\|^2 E\|V(\tau) - V^*(\tau)\|^2 d\tau ds,}$$

Or

$$E\|V(t) - V^*(t)\|^2 \leq K^2 C^2 \int_0^t \int_0^s E\|V(\tau) - V^*(\tau)\|^2 d\tau ds,$$

Then by Gronwall inequality $E\|V(t) - V^*(t)\|^2 \to 0$.

4. CONCLUSION

The paper concluded that the mild solution of the stochastic integr- differential equation has been found by using successive approximation method. Also some of compensation of a solution of its stochastic differential equation was reached by using Gronwall inequality to guarantee a unique solution.

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