The Scientific Journal of University of Benghazi

http://journals.uob.edu.ly/sjuob

Existence and Uniqueness Solution for a Semimartingale Stochastic Integral Equation

Hanan Salem Abd Alhafid [1*]

[1]*Department of Mathematics, College of Education, University of Benghazi, Benghazi, Libya*

Received 22 / 11 / 2022; Accepted 21 / 03 / 2023

الملخص

هذه الورقة تقوم بإيضاح إيجاد الحل الوحيد لمعادلة **semimartingale**عشوائية تكاملية:

$$X(t,\omega) = H(t,\omega) + \int_{(0,t]} F\big(s,\omega,X(\omega)\big)dY(s,\omega) \qquad (2.1)$$

باستخدام نظرية الوجود والوحدانية في معالجة martingale، وذلك باستخدام مفهوم التقارب لمتتالية كوشي $\{X_n\}$ للمتغير العشوائي X حيث $X \in \mathcal{L}_2$ ، ممكن إيجاد تقارب لمتتالية كوشي للتكامل العشوائي $X_n \cdot M$ ، حيث M تكون a square-integrable cadlag martingale على الفضاء العشوائي (Ω, F, P) و $Y = \lim_{n\to\infty} X_n \cdot M = X \cdot M = I(X).$

حيث $Y \in \mathcal{M}_2$ فضاء martingles ، وبعض الفرضيات المهمة

i- الدالة F المعرفة من الفضاء العشوائي $R_+ \times \Omega \times D_{R^d}[0,\infty)$ الى $R^{d\times m}$ تحقق a spatial Lipschitz condition: لكل $0 < T < \infty$ يوجد $L(T)$ بحيث

$$|F(t,\omega,\eta) - F(t,\omega,\xi)| \le L(T)\cdot \sup_{s\in(0,t]} |\eta(s) - \xi(s)|$$

لكل $(t,\omega) \in [0,T] \times \Omega$ و $\eta, \xi \in D_{R^d}[0,\infty)$.

ii-لأي متغير عشوائي X يوجد دالة توقف مقيدة $v_k \to \infty$ بحيث $1_{(0,v_k)}(t)F(t,X)$ مقيدة لأي k .

الكلمات المفتاحية: Semimartingale, Stochastic Integral Equation, Lipschitz Condition, Stopped Process

Abstract

This paper studied existence and uniqueness of a solution for a semimartingale stochastic integral equation

$$X(t,\omega) = H(t,\omega) + \int_{(0,t]} F\big(s,\omega,X(\omega)\big)dY(s,\omega) \qquad (2.1),$$

by using Existence and Uniqueness Theorem on the martingale process. Using the concept of convergence Cauchy sequence $\{X_n\}$ to a cadlag process X, where $X \in \mathcal{L}_2$, we can find a convergence Cauchy sequence $\{X_n \cdot M\}$ to a cadlag process Y on the space $\mathcal{M}_2$ of martingales, where M is a square-integrable cadlag martingale on a probability space (Ω, F, P), as

$$Y = \lim_{n\to\infty} X_n \cdot M = X \cdot M = I(X).$$

And some important assumptions are

i. F is a map from the space $R_+ \times \Omega \times D_{R^d}[0,\infty)$ into the space $R^{d\times m}$ of $d \times m$-matrices. F satisfies a spatial Lipschitz condition uniformly in the other variables: for each $0 < T < \infty$ there exists a finite constant $L(T)$ such that this holds for $(t,\omega) \in [0,T] \times \Omega$ and all $\eta, \xi \in D_{R^d}[0,\infty)$: $|F(t,\omega,\eta) - F(t,\omega,\xi)| \le L(T)\cdot \sup_{s\in(0,t]} |\eta(s) - \xi(s)|$. **ii**. Given any adapted R^d-valued cadlag process X on Ω, the function $(t,\omega) \to F\big(t,\omega,X(\omega)\big)$ is a predictable process, and there exist stopping times $v_k \to \infty$ such that $1_{(0,v_k)}(t)F(t,X)$ is bounded for each k.

Keywords: A Semimartingale Process, Stochastic Integral Equation, Lipschitz Condition, Stopped Process.

***Correspondence**: Hanan Salem Abd Alhafid

hanan_e_m74@yahoo.com

©2023 University of Benghazi. All rights reserved. ISSN:Online **2790-1637**, Print **2790-1629**; National Library of Libya, Legal number : **154/2018**

1. INTRODUCTION

Integral equations form one of the most useful tools in many branches of pure analysis, such as functional analysis and stochastic calculus see [1,2,3,4]. Stochastic integral equations are extremely important in the study of many physical phenomena in life sciences and engineering [5,6,7]. There are currently two basic versions of stochastic integral equations being studied by probabilists and mathematical statisticians, namely, those integral equations involving Ito integral and those which can be formed as probabilistic analogues of classical deterministic integral equations whose formulation involves the usual Lipschitz conditions of the variables.

Several papers have appeared on the problem of the existence of solutions of stochastic integral equations and are discussed in [8,9,10,11,12]. In this paper, we will prove an existence and uniqueness for the semimartingale stochastic integral equation by the Existence and Uniqueness theorem.

2. PRELIMINARIES

Let (Ω, F, P) be a probability space with a filtration $\{F_t\}$. We consider

$$X(t,\omega) = H(t,\omega) + \int_{(0,t]} F\big(s,\omega,X(\omega)\big)dY(s,\omega) \qquad (2.1)$$

Where Y is a given R^m-valued cadlag semimartingale, H is a given R^d-valued adapted cadlag process, and X is the unknown R^d-valued process. The coefficient F is a $d \times m$-matrix valued function of its arguments. For the coefficient F, we make these assumptions.

i. F is a map from the space $R_+ \times \Omega \times D_{R^d}[0,\infty)$ into the space $R^{d\times m}$ of $d \times m$-matrices. F satisfies a spatial Lipschitz condition uniformly in the other variables: for each $0 < T < \infty$ there exists a finite constant $L(T)$ such that this holds for $(t,\omega) \in [0,T] \times \Omega$ and all $\eta, \xi \in D_{R^d}[0,\infty)$: $|F(t,\omega,\eta) - F(t,\omega,\xi)| \le L(T) \cdot \sup_{s\in(0,t]} |\eta(s) - \xi(s)|$.

ii. Given any adapted R^d-valued cadlag process X on Ω, the function $(t,\omega) \to F\big(t,\omega,X(\omega)\big)$ is a predictable process, and there exist stopping times $v_k \to \infty$ such that $1_{(0,v_k)}(t)F(t,X)$ is bounded for each integer k.

Definition 2.1.[13] If X is the unknown R^d-valued process on a probability space (Ω, F, P) which is finite with probability one, w.p.1, then its distribution function is $F(t) = P(\omega: X(\omega) \le t)$. This gives us the convenient expressions $f(\omega: X(\omega) \in h) = \int_h dF(t)$, for any Borel set h of F.

Definition 2.2.[1] A cadlag process Y is a semimartingale if it can be written as $Y_t = Y_0 + M_t + V_t$ where M is a cadlag local martingale, V is a cadlag FV process, and $M_0 = V_0 = 0$.

Lemma 2.3.[2] Assume F satisfies assumptions (i),(ii). Suppose there exists a path $\bar{\xi} \in D_{R^d}[0,\infty)$ such that for all $T < \infty$, $c(T) = \sup_{t\in(0,T],\omega\in\Omega} \left|F\big(t,\omega,\bar{\xi}\big)\right| < \infty$. Then for any adapted R^d-valued cadlag process, X there exists stopping times $v_k \to \infty$ such that $1_{(0,v_k)}(t)F(t,X)$ is bounded for each integer k.

Definition 2.4.[4] For predictable processes X, we define L_2 norm over the set $[0,T] \times \Omega$ under the measure μM by

$$= \|X\|_{\mu M,T} \left(\int_{[0,T]\times\Omega} |X|^2 d\mu M\right)^{1/2} = \left(E\int_{[0,T]\times\Omega} |X(t,\omega)|^2 d[M]_t(\omega)\right)^{1/2}$$

Let $\mathcal{L}_2 = \mathcal{L}_2(M,P)$ denote the collection of predictable processes X such that $\|X\|_{\mu M,T} < \infty$ for all $T < \infty$. A metric on $\mathcal{L}_2$ is defined by $d_{\mathcal{L}_2}(X,Y) = \|X - Y\|_{\mathcal{L}_2}$ where $\|X\|_{\mathcal{L}_2} = \sum_{k=1}^{\infty} 2^{-k}(1 \wedge \|X\|_{\mu M,T})$. $\mathcal{L}_2$ is not an L_2 space, but instead a local L_2 space of sorts.

Theorem 2.5. "Existence and uniqueness" [1,4] Let $X \in \mathcal{L}_2$. There exists a sequence $X_n \in S_2$, where S_2 is the space of predictable simple processes, such that $\|X - X_n\|_{\mathcal{L}_2} \to 0$. From the triangle inequality, it then follows that $\{X_n\}$ is a Cauchy sequence in $\mathcal{L}_2$, given $\varepsilon > 0$, choose η_0 so that $\|X - X_n\|_{\mathcal{L}_2} \le \varepsilon/2$ for $n \ge \eta_0$. Then if $m, n \ge \eta_0$, $\|X_m - X_n\|_{\mathcal{L}_2} \le \|X_m - X\|_{\mathcal{L}_2} + \|X - X_n\|_{\mathcal{L}_2} \le \varepsilon$. For the stochastic integral $X_n \cdot M$, where M is a square-integrable cadlag martingale on a probability space (Ω, F, P), the additivity of the integral, $\|X_m \cdot M - X_n \cdot M\|_{\mathcal{M}_2} \le \|(X_m - X_n) \cdot M\|_{\mathcal{M}_2} = \|X_m - X_n\|_{\mathcal{L}_2}$. Consequently, $\{X_n \cdot M\}$ is a Cauchy sequence in the space $\mathcal{M}_2$ of martingales. Then there exists a limit process $Y = \lim_{n\to\infty} X_n \cdot M$. This process is called $I(X) = X \cdot M$. Let Z_n be another sequence in S_2 that converges to X in $\mathcal{L}_2$. Then $Z_n \cdot M \to Y$, in $\mathcal{M}_2$ same as $X \cdot M$. The uniqueness of the stochastic integral hold is in a sense stronger than indistinguish ability. If W is a process that is indistinguishable from $X \cdot M$, which means that $P\{\omega: W_t(\omega) = (X \cdot M)_t(\omega)\ for\ all\ t \in R_+\} = 1$, then W also has to be regarded as the stochastic integral. This is built into the definition of $I(X)$ as the limit: if $\|X \cdot M - X_n \cdot M\|_{\mathcal{M}_2} \to 0$ and W is indistinguishable from $X \cdot M$, then also $\|W - X_n \cdot M\|_{\mathcal{M}_2} \to 0$.

Corollary 2.6.[1] Let $0 < T < \infty$. Assume $\{F_t\}$ is right continuous, Y is a cadlag semimartingale and H is an adapted process, all defined for $0 \le t \le T$. Let F satisfy assumption (i),(ii) for $(t,\omega) \in [0,T] \times \Omega$. In particular, part (ii) take this form: if X is a predictable process defined on $[0,T] \times \Omega$, then so is $F(t,X)$, and there is a non-decreasing sequence of stopping times $\{v_k\}$ such that $1_{(0,v_k)}(t)F(t,X)$ is is bounded for each integer k, and for almost every ω, $v_k = T$ for all large enough k. Then there exists a unique solution $X$to equation (2.1).

3. MAIN RESULTS

Theorem 3.1. Assume $\{F_t\}$ is complete and right-continuous. H is an adapted Rd-valued cadlag process and Y is an R^m-valued cadlag semimartingale. Assume F satisfies assumptions (i),(ii). Then there exists a unique cadlag process

$\{X(t,\omega): 0 \le t < \infty\}$ adapted to $\{F_t\}$ that satisfies equation (2.1).

Proof: For $k \in N \subset [0,\infty)$, the function $1_{\{0\le t\le k\}}(t)F(t,\omega,\eta)$ satisfies the original hypotheses. There exists a process X_k that satisfies the equation

$$X_k(t,\omega) = H^k(t,\omega) + \int_{(0,t]} 1_{[0,k]}(s)F\big(s,\omega,X_k(\omega)\big)dY^k(s,\omega) \qquad (3.1)$$

©2023 University of Benghazi. All rights reserved. ISSN:Online 2790-1637, Print 2790-1629; National Library of Libya, Legal number : 154/2018

We have $H^k(t,\omega) = H(k\wedge t,\omega)$ for a stopped process. Let $k < m$. Stopping the equation

$$X_m(t,\omega) = H^m(t,\omega) + \int_{(0,t]} 1_{[0,m]}(s)F\big(s,\omega,X_m(\omega)\big)dY^m(s,\omega)$$

At time k gives the equation

$$X_m^k(t,\omega) = H^k(t,\omega) + \int_{(0,t\wedge k]} 1_{[0,m]}(s)F\big(s,\omega,X_m(\omega)\big)dY^m(s,\omega)$$

Valid for t. By proposition stopping a stochastic integral can be achieved by stopping the integrator or by cutting off the integrator with an indicator function. If we do both, we get the equation

$$X_m^k(t,\omega) = H^k(t,\omega) + \int_{(0,t]} 1_{[0,k]}(s)F\left(s,\omega,X_m^k(\omega)\right)dY^k(s,\omega)$$

Thus X_k and X_m^k satisfy the same equation, so by the Existence and Uniqueness Theorem, $X^k = X_m^k$ for $k < m$. Thus we can unambiguously define a process X by setting $X = X_k$on $[0,k]$. then for $0 \le t \le k$ we can substitute X for X_k in equation (3.1) and get the equation

$$X(t,\omega) = H^k(t,\omega) + \int_{(0,k\wedge t]} F\big(s,\omega,X(\omega)\big)dY(s,\omega)\,, 0 \le t \le k.$$

Since this holds for all k, X is a solution to the original equation (2.1).

Uniqueness similarly, if X and $\tilde{X}$ solutions of equation (2.1) then $X(k\wedge t)$ and $\tilde{X}\,(k\wedge t)$ solve equation (2.1). By the uniqueness theorem $X(k\wedge t) = \tilde{X}\,(k\wedge t)$ for all t, and since k can be taken arbitrary, $X = \tilde{X}$.

4. CONCLUSION

The paper concluded that the solution of a semimartingale stochastic equation has been found by using Existence and Uniqueness Theorem on a martingale, by using some important assumptions on a probability space (Ω, F, P).

5. REFERENCES

1. Timo Seppalarnen. Basic of stochastic Analysis, Department of Mathematics, University of Wisconsin Madison, (2003), 53706.
2. David AM. Uniqueness of solutions of stochastic diﬀ erential equations, Mathematics Subject Classification, 2000, Primary 60H10, Secondary 34F05.
3. Ruth F. Curtain AJ. Pritchard. Functional Analysis in Modern Applied Mathematics, © by Academic Press Inc. 1977, (London) LTD.
4. Michael J. Steel. Stochastic Calculus and Financial Applications, c Springer-Verlag, New York, Inc 2001.
5. Bharucha-Reid AT. Random Integral Equations, Mathematics in Science and Engineering, vol. 96, Academic Press, New York, 1972.
6. Szynal D. Wedrychowicz S. On solutions of a stochastic integral equation of the Volterra type with applications for chemotherapy, Journal of Applied Probability 25 1988, no. 2: 257–267.
7. Tsokos CP. Padgett WJ. Random Integral Equations with Applications to Life Sciences and Engineering, Mathematics in Science and Engineering, vol. 108, Academic Press, New York, 1974.
8. J. Banas J. Szynal D. We S. drychowicz, On existence, asymptotic behaviour and stability of solutions of stochastic integral equations, Stochastic Analysis and Applications 9 (1991), no. 4, 363–385.
9. H. Gacki T. Szarek T. We S.¸drychowicz, On existence, and stability of solutions of stochastic integral equations, Indian Journal of Pure and Applied Mathematics 29, 1998, no. 2: 175–189.
10. Hardiman ST. Tsokos CP. Existence and stability behavior of random solutions of a system of nonlinear random equations, Information Sciences 9, 1975, no. 4: 299–313.
11. Szynal D. We S. drychowicz, On existence and asymptotic behaviour of solutions of a nonlinear stochastic integral equation, Annali di Matematica Pura ed Applicata. Serie Quarta 1985, 142: 105–119 (1986).
12. Subramaniam, R. and Balachandran, K. and Jong Kyu Kim. Existence of random solutions of a General Class of Stochastic Functional Integral Equations, Stochastic Analysis and Applications, 2007,21(5):1189-1205.
13. Abd Alhafid HS. Existence and uniqueness of the mild solution of stochastic integro differential equation, 2020, SJUOB 33 (2) Applied Sciences: 189 – 192.

©2023 University of Benghazi. All rights reserved. ISSN:Online 2790-1637, Print 2790-1629; National Library of Libya, Legal number : 154/2018