The Scientific Journal of University of Benghazi

SJUOB (2024) 37 (1) Applied Sciences: 28 – 36

Altwaty et al

http://journals.uob.edu.ly/sjuob

The Sardar sub-equation technique for obtaining some optical solitons of cubic nonlinear Schrödinger equation involving beta derivatives with Kerr law nonlinearity

Abdulmalik A. Altwaty [1]* - Jaffalah J. Amhalhil [2] - Ahmed El Sakori [1] - Ngla F. Meriki [1]

1 Department of Mathematics, Faculty of Arts & Science, University of Benghazi, Al Marj, Libya.

2 Department of Mathematics, Faculty of Education, Sirte University, Libya.

Received: 14 / 03 / 2024; Accepted: 10 / 05 / 2024

ABSTRACT

This study investigates new optical and chirped optical solitons for the space-time fractional cubic nonlinear Schrödinger equation using the Sardar sub-equation technique in the presence of Kerr law nonlinearity. The solutions are expressed in terms of hyperbolic and trigonometric functions, revealing a diverse range of behaviors within the system. The identified optical and chirped optical soliton types include dark, bright, kink, and periodic, showcasing a rich spectrum of phenomena. Representing soliton solutions using 2D and 3D graphs with varying parameters leads to a better understanding of their formation and characteristics. The findings contribute to the comprehension of nonlinear dynamics, offering insights into phenomena relevant to nonlinear optics, quantum mechanics, and condensed matter physics.

KEYWORDS: Cubic nonlinear Schrödinger equation, Kerr law nonlinearity, beta derivative, optical solitons, Sardar sub, equation technique.

1. INTRODUCTION

Nonlinear models have been used to explain a variety of real-world occurrences, revealing significant information in the process. Advanced classes of differential equations that produce better outcomes are represented by fractional nonlinear evolution equations. Due to their important uses, these equations aid in the illustration of complex physical events and draw a large number of researchers to work in this subject. The nonlinear Schrödinger equation is an essential component of fractional nonlinear evolution equations and is applied in quantum mechanics, biology, optical fiber, plasma physics, fluid mechanics, electricity, shallow water wave phenomena, heat flow, finance, and fractal dynamics.

The intricate interplay between the nonlinear and dispersive components of solitons within a medium has revealed that their wave-like structure remains preserved during propagation. The soliton solutions originating from Fractional Nonlinear Evolution Equations (FNLEEs) offer a wide array of practical and commercial advantages across numerous industries. In the realm of fiber optic technology [1-3], these soliton solutions contribute to enhancing data transmission speeds and reliability, crucial for meeting the growing demands of high-speed communication networks.

***Correspondence**: Abdulmalik A. Altwaty

abdulmalik.altwaty@uob.edu.ly

Within the telecommunications sector [4], FNLEE solitons play a pivotal role in ensuring seamless connectivity and efficient signal transmission. In signal processing applications [5], these solutions aid in the precise analysis and manipulation of data signals, facilitating advancements in fields such as digital communications and information processing. Moreover, in image processing [6], FNLEE solitons are utilized for advanced image enhancement techniques, enabling clearer and more detailed visual representation. System identification benefits from the application of soliton solutions by providing accurate modeling tools for complex systems, aiding in predictive analysis and control. Water treatment processes leverage FNLEE solitons for optimized purification methods, enhancing the efficiency of wastewater treatment and desalination processes. In the realm of plasma physics, these solutions assist in understanding and controlling plasma behavior, essential for various applications ranging from fusion research to materials processing. Medical device sterilization procedures benefit from the use of soliton solutions, ensuring effective and reliable sterilization processes in healthcare settings. Furthermore, in the field of chemistry, FNLEE solitons offer valuable insights into intricate chemical reactions and phenomena, aiding in the development of novel materials and pharmaceuticals. These diverse applications underscore the versatility and impact of soliton solutions derived from FNLEEs in addressing challenges and driving innovation across a broad spectrum of industries and disciplines.

Various dynamic approaches have been introduced and implemented in the literature to solve nonlinear fractional differential equations (NFDES) and obtain analytical traveling wave solutions, for example, the fractional differential transform method [7], the fractional modified Kudryashov method [8], the generalized differential transform method [9], the fractional finite difference method [10-12], the fractional finite element method [13-15], the fractional boundary element method [16-18], the fractional radial basis function method [19-21], the fractional homotopy analysis method [22,23], the fractional homotopy perturbation transform method [24,25].

In this paper, we will utilize for the first time the fractional cubic NLFSE with Kerr law nonlinearity by using Sardar sub-equation technique. The fractional cubic NLFSE with the Kerr law nonlinearity is stated as [26, 27].

$$iD_t^{\alpha}u + a_1 D_t^{\alpha} D_x^{\alpha} u + a_2 D_x^{2\alpha} u + a_3 |u|^2 u = 0. \quad (1)$$

The wave profile with complex values is represented by u(x,t). Here a_1, a_2, a_3 are real coefficients, and (0≤α≤1). The cubic fractional nonlinear Schrödinger equation, which includes beta derivatives in both space and time, is utilized to represent various nonlinear optical phenomena. For instance, it can be applied to model the behavior of solitons in optical fibers, showcasing their unique propagation characteristics and stability in the presence of nonlinear effects.

The structure of this article is outlined as follows: Section 2 discusses the characteristics of the beta derivative. The methodology of the proposed approach is detailed in Section 3. A mathematical analysis is presented in Section 4. The article is concluded in Section 5.

2. DEFINITION OF BETA DERIVATIVE AND ITS PROPERTIES

In recent years, researchers have introduced various definitions of fractional derivatives, including the Riemann-Liouville, modified Riemann-Liouville, Caputo, Caputo-Fabrizio, conformable fractional derivative, and Atangana-Baleanu derivatives. These fractional derivatives often deviate from the familiar properties of classical calculus, such as the chain rule, the Leibniz rule, and the derivative of a constant being zero. To address this, Atangana and colleagues proposed a novel and significant definition known as the beta derivative. This beta derivative adheres to the fundamental properties of classical calculus, marking a crucial advancement in the field of fractional calculus.

Definition 1: Given α∈R and a function h=h(x) defined on the interval [α,∞)→R, the beta derivative of order α with respect to x is formally defined as follows [28]:

$$D_x^{\alpha}(h(x)) = Lim_{\varepsilon \to 0} \frac{h\left(x+\varepsilon\left(x+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right)-h(x)}{\varepsilon}, \quad (2)$$

whereΓ is the gamma function and D_x^α h(x)=d/dx h(x) for α=1.Given that h(x)and g(x) are α -order differentiable for x>0, and a and b are real constants, the beta derivative exhibits the following properties:

1. $D_x^{\alpha}(a\,h(x) + b\,g(x)) = a\,D_x^{\alpha}h(x) + b\,D_x^{\alpha}g(x)$.
2. $D_x^{\alpha}(k) = 0$, *wherek is a constant.*
3. $D_x^{\alpha}(h(x)\,g(x)) = h(x)D_x^{\alpha}g(x) + g(x)D_x^{\alpha}h(x)$.
4. $D_x^{\alpha}\left(\frac{h(x)}{g(x)}\right) = \frac{g(x)D_x^{\alpha}(h(x)) - h(x)D_x^{\alpha}(g(x))}{g^2(x)}$.
   5-$D_x^{\alpha}(h^{-1}(x)) = -\frac{D_x^{\alpha}(h(x))}{h^2(x)}$.
5. 6-$D_x^{\alpha}h(x) = \left(x + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\frac{dh(x)}{dx}$.

Utilizing these properties of the beta derivative, fractional differential equations can be effectively reduced to ordinary differential equations. Notably, the beta derivative has demonstrated a comprehensive fulfillment of properties equivalent to those observed with integer-order derivatives without encountering any discernible limitations thus far.

3. ALGORITHM OF THE SARDAR SUB-EQUATION TECHNIQUE

In this section, we introduced the Sardar sub-equation technique as a powerful tool. This method empowers us to derive new and extensive analytical solutions for the model (1). By leveraging this technique, we can convert fractional partial differential equations into ordinary differential equations, streamlining the computational procedure. The algorithm outlining this process is detailed below:

Step 1*: Let*

$$H(u, u_x, u_t, D_t^{\alpha}u, D_x^{\alpha}u, D_{tt}^{2\alpha}u, D_{xt}^{2\alpha}u, D_{xx}^{2\alpha}u, \dots) = 0, \quad (3)$$

Where $\mathcal{F}$ is a polynomial of $u(x,t)$, and D_t^{α} is a fractional derivative of α-order. Consider the wave transformation $u = \varphi(\xi)e^{i(\emptyset(\xi)-\theta t)}$, where$\xi = \frac{1}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} - \frac{v}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}$ and v, ω are respectively velocity and wave number. According to the definition provided in reference [29], we have:

$$\delta w = -\frac{\partial(\emptyset(\xi) - \omega\, t)}{\partial t} = \emptyset(\xi)'$$

Using the considerable transformation, Eq. (3) reduced to the following ordinary differential equation:

$$\mathcal{H}(u, u', u'', u''', \dots \dots,) = 0 \qquad (4)$$

***Step 2**: the exact solution of Eq. (4) given as*

$$u(\xi) = \sum_{l=0}^{N} b_l M^l(\xi); \;\; b_l \neq 0, \qquad (5)$$

Where $b_l (0 \le l \le N)$ are constant coefficients to be evaluated later and $M(\xi)$ satisfies the following ordinary differential equation,

$$M'(\xi) = \sqrt{\mu + \gamma M(\xi)^2 + M(\xi)^4}, \qquad (6)$$

Where μ and $\gamma$are real constants. The general solutions of Equation (6) are given as.

(1) If $\gamma > 0$ and $\mu = 0$ then

$$M_1^{\pm}(\xi) = \pm\sqrt{-pq\gamma}\, sech_{pq}(\sqrt{\gamma}\,\xi)$$
$$M_2^{\pm}(\xi) = \pm\sqrt{pq\gamma}\, csch_{pq}(\sqrt{\gamma}\,\xi).$$

(2) If $\gamma < 0$ and $\mu = 0$ then

$$M_3^{\pm}(\xi) = \pm\sqrt{-pq\gamma}\, sec_{pq}(\sqrt{-\gamma}\,\xi)$$
$$M_4^{\pm}(\xi) = \pm\sqrt{-pq\gamma}\, csc_{pq}(\sqrt{-\gamma}\,\xi).$$

(3) If $\gamma < 0$ and $\mu = \frac{\gamma^2}{4}$ then

$$M_5^{\pm}(\xi) = \pm\sqrt{\frac{-\gamma}{2}}\, tanh_{pq}\left(\sqrt{\frac{-\gamma}{2}}\,\xi\right),$$
$$M_6^{\pm}(\xi) = \pm\sqrt{\frac{-\gamma}{2}}\, coth_{pq}\left(\sqrt{\frac{-\gamma}{2}}\,\xi\right),$$
$$M_7^{\pm}(\xi) = \pm\sqrt{\frac{-\gamma}{2}}\left(tanh_{pq}(\sqrt{-2\gamma}\,\xi) \pm i\sqrt{pq}\, sech_{pq}(\sqrt{-2\gamma}\,\xi)\right)$$
$$M_8^{\pm}(\xi) = \pm\sqrt{\frac{-\gamma}{2}}\left(coth_{pq}(\sqrt{-2\gamma}\,\xi) \pm \sqrt{pq}\, csch_{pq}(\sqrt{-2\gamma}\,\xi)\right)$$
$$M_9^{\pm}(\xi) = \pm\sqrt{\frac{-\gamma}{8}}\left(tanh_{pq}\left(\sqrt{\frac{-\gamma}{8}}\,\xi\right) + oth_{pq}\left(\sqrt{\frac{-\gamma}{8}}\,\xi\right)\right)$$

.

(4) If $\gamma > 0$ and $\mu = \frac{\gamma^2}{4}$ then

$$M_{10}^{\pm}(\xi) = \pm\sqrt{\frac{\gamma}{2}}\, tan_{pq}\left(\sqrt{\frac{\gamma}{2}}\,\xi\right)$$
$$M_{11}^{\pm}(\xi) = \pm\sqrt{\frac{\gamma}{2}}\, cot_{pq}\left(\sqrt{\frac{\gamma}{2}}\,\xi\right)$$
$$M_{12}^{\pm}(\xi) = \pm\sqrt{\frac{\gamma}{2}}\left(tan_{pq}(\sqrt{2\gamma}\xi) \pm \sqrt{pq}\; sec_{pq}(\sqrt{2\gamma}\xi)\right).$$
$$M_{13}^{\pm}(\xi) = \pm\sqrt{\frac{\gamma}{2}}\left(cot_{pq}(\sqrt{2\gamma}\xi) \pm \sqrt{pq}\; csc_{pq}(\sqrt{2\gamma}\xi)\right)$$
$$M_{14}^{\pm}(\xi) = \pm\sqrt{\frac{\gamma}{8}}\left(tan_{pq}\left(\sqrt{\frac{\gamma}{8}}\xi\right) + cot_{pq}\left(\sqrt{\frac{\gamma}{8}}\xi\right)\right)$$

Where

$$tan_{pq}(\xi) = -i\frac{pe^{i\xi} - qe^{-i\xi}}{pe^{i\xi} + qe^{-i\xi}}, cot_{pq}(\xi) = i\frac{pe^{i\xi} + qe^{-i\xi}}{pe^{i\xi} - qe^{-i\xi}}$$

$$tanh_{pq}(\xi) = \frac{pe^{\xi} - qe^{-\xi}}{pe^{\xi} + qe^{-\xi}}, \quad coth_{pq}(\xi) = \frac{pe^{\xi} + qe^{-\xi}}{pe^{\xi} - qe^{-\xi}}$$

$$sec_{pq}(\xi) = \frac{2}{pe^{i\xi} + qe^{-i\xi}}, \quad csc_{pq}(\xi) = \frac{2i}{pe^{i\xi} - qe^{-i\xi}}$$

$$sech_{pq}(\xi) = \frac{2}{pe^{\xi} + qe^{-\xi}}, \quad csch_{pq}(\xi) = \frac{2}{pe^{\xi} - qe^{-\xi}}$$

***Step 3:** The integer N calculated by balancing the capitals. Substituting Eq. (5) into Eq. (4) we obtain a system of algebraic equations in terms of $M^l(\xi)$where $(l = 0,1,2, \dots, N)$.*

***Step 4:** **S**olving these algebraic equations, we determine the values of the unknown parameters.*

4. Mathematical Analysis

In this section, we delved into the study of the space-time fractional cubic Nonlinear Fractional Schrödinger Equations (NLFSEs) with the aim of obtaining broader and more conventional exact wave solutions. To achieve this, we applied the Sardar sub-equation technique, a method that simplifies the solution process for fractional partial differential equations by transforming them into ordinary differential equations.

Furthermore, we conducted a comprehensive mathematical analysis of these wave solutions to gain a deeper understanding of their behavior and properties. By utilizing a fractional transformation, we converted the original Equation (1) into an ordinary differential equation that accounts for both the real and imaginary components of the solution. This transformation facilitated the derivation of analytical solutions that provide insights into the dynamics and characteristics of the system under study.

$$(v + a_1\,\theta)\varphi\emptyset' + \theta\varphi + (a_1 v - a_2)\varphi(\emptyset')^2 + (a_2 - a_1 v =)\varphi'' + a_3\varphi^3 0, \qquad (7)$$

$$(v - a_1\,\theta)\varphi' + 2(a_2 - a_1 v)\varphi'\emptyset' + (a_2 - a_1 v)\varphi\varphi'' = 0. \qquad (8)$$

Integrating Eq.(8) yields

$$\emptyset' = \frac{a_1\theta - v}{a_2 - a_1 v} + C\,\varphi^{-2}. \qquad (9)$$

Concurrently, the chirp is shown by

$$\delta\omega = -\left[\frac{a_1\theta - v}{a_2 - a_1 v} + C\,\varphi^{-2}\right]. \qquad (10)$$

Substituting into the real part we get

$$A_0\varphi^4 + A_1\varphi^2 + A_2 + A_3\varphi^3\varphi'' + a_3\varphi^6 = 0. \qquad (11)$$

Where $A_0 = \frac{(v+a_1\theta)^2-2v}{4(a_2-a_1v)} + \theta, A_1 = 2Cv, A_2 = C^2(a_1v - a_2), A_3 = (a_2 - a_1v)$, *and* C *is the integration constant.*

Now balancing between $\varphi^3\varphi''$ *and* φ^6 *the highest order derivatives and highest power of the nonlinear term in Eq. (11), we obtain* $N = 1$. *Therefore, the solution of Eq. (11) is of the form*

$$\varphi(\xi) = b_0 + b_1 M(\xi). \qquad (12)$$

Upon substituting Eq. (11) together with Eq. (6), we derive a polynomial expression in $M^l(\xi)$. *Setting this polynomial to zero, representing the coefficient power of* $M^l(\xi)$, *we formulate a system of algebraic equations that incorporate the variables* b_0, b_1, μ, *and* γ. *Solving this system of algebraic equations using Matlab provides the parameter values as follows:*

$$b_0 = b_0,\quad b_1 = b_1,\quad \mu = \mu,\ \gamma = \gamma,\ v = \frac{a_2C - a_3b_0^6}{2b_0^2 + a_1C},$$

$$\theta = -\frac{a_2}{a_1^2} + \frac{a_2C - a_3b_0^6}{4a_1b_0^2 + 2a_1^2C} \pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3} - \frac{a_2C - a_3b_0^6}{2b_0^2 + a_1C}\right)},$$

Where ω *is real if and only if* $C \geq -\frac{a_1^3a_3b_0^6+2a_2b_0^2}{a_1a_2-a_1^3a_2}$. *Substituting into (12) and the hypothesis of the auxiliary equation for different conditions, we establish the travelling wave solutions of (1) which are given as:*

Case 1: *When* $\gamma > 0$ *and* $\mu = 0$, *then the solutions are as follows:*

$$u_1 = \{b_0 \pm b_1\sqrt{-pq\gamma}\ \operatorname{sech}_{pq}(\sqrt{\gamma}\xi)\}e^{i\left(\emptyset(\xi)-\left(-\frac{a_2}{a_1^2}+\frac{a_2C-a_3b_0^6}{4a_1b_0^2+2a_1^2C}\pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3}-\frac{a_2C-a_3b_0^6}{2b_0^2+a_1C}\right)}\right)t\right)}, \qquad (13)$$

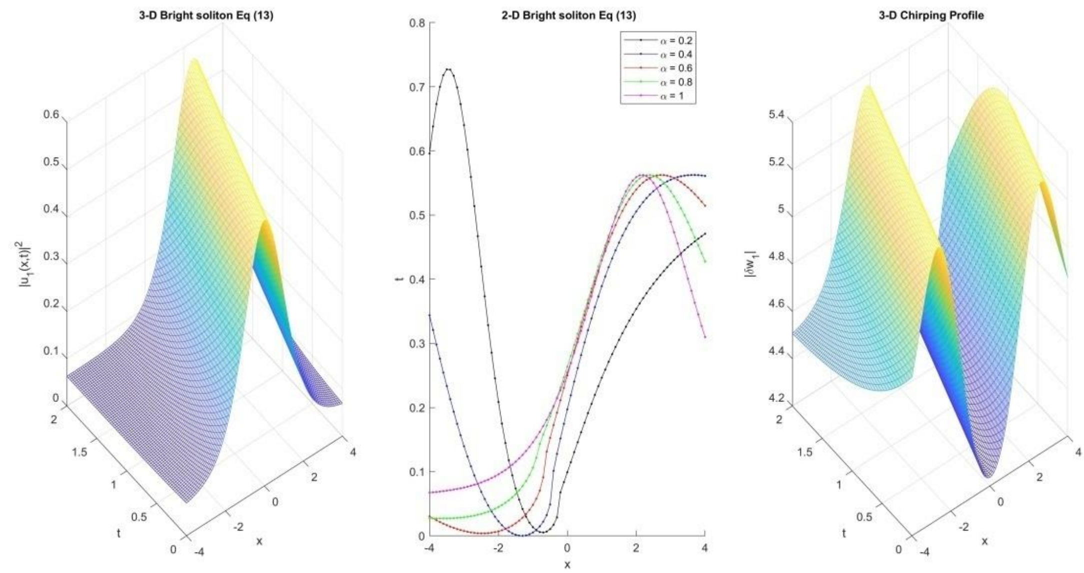

Figure 1: 3D, 2D and 3D chirped Bright soliton solutions of Equation (13) for $\mathbf{a_1 = 0.15, a_2 = 0.5, a_3 = 2, b_0 = 0.25, \gamma = 0.5, b_1 = 1, C = 0.5}$, and $\boldsymbol{\alpha = 1}$, with $\mathbf{x \in [-4, 4]}$ and $\mathbf{x \in [0, 2]}$.

$$\delta\omega_1 = -\left[\frac{a_1\omega - v}{a_2 - a_1v} + C\left(b_0 \pm b_1\sqrt{-pq\gamma}\ \operatorname{sech}_{pq}(\sqrt{\gamma}\xi)\right)^{-2}\right], \qquad (14)$$

$$u_2 = \{b_0 \pm b_1\sqrt{-pq\gamma}\ \operatorname{csch}_{pq}(\sqrt{\gamma}\xi)\}e^{i\left(\emptyset(\xi)-\left(-\frac{a_2}{a_1^2}+\frac{a_2C-a_3b_0^6}{4a_1b_0^2+2a_1^2C}\pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3}-\frac{a_2C-a_3b_0^6}{2b_0^2+a_1C}\right)}\right)t\right)}, \qquad (15)$$

$$\delta\omega_2 = -\left[\frac{a_1\omega - v}{a_2 - a_1v} + C\left(b_0 \pm b_1\sqrt{-pq\gamma}\ \operatorname{csch}_{pq}(\sqrt{\gamma}\xi)\right)^{-2}\right]. \qquad (16)$$

Case 2: *When* $\gamma < 0$ *and* $\mu = 0$, *we obtain*

$$u_3=\left\{b_0\pm b_1\sqrt{-pq\gamma}\ \sec_{pq}(\sqrt{-\gamma}\xi)\right\}e^{i\left(\emptyset(\xi)-\left(\frac{a_2}{a_1^2}+\frac{a_2C-a_3b_0^6}{4a_1b_0^2+2a_1^2C}\pm2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3}-\frac{a_2C-a_3b_0^6}{2b_0^2+a_1C}\right)}\right)t\right)}, \quad (17)$$

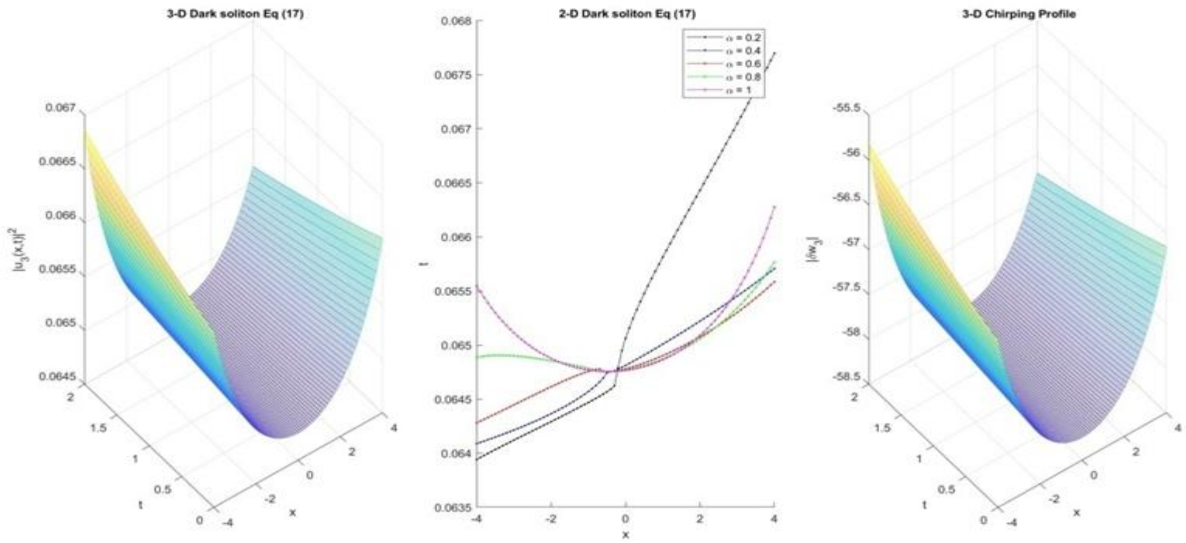

Figure 2: 3D, 2D and 3D chirped Dark soliton solutions of Equation (17) for $\mathbf{a_1=0.15}$, $\mathbf{a_2=0.05}$, $\mathbf{a_3=0.2}$, $\mathbf{b_0=0.25}$, $\boldsymbol{\gamma=-0.05}$, $\mathbf{b_1=0.02}$, $\mathbf{C=5}$, and $\boldsymbol{\alpha=1}$, with $\mathbf{x}\in[\mathbf{-4,4}]$ and $\mathbf{x}\in[\mathbf{0,2}]$.

$=\mathbf{1}$, with $\mathbf{x}\in[\mathbf{-4,4}]$ and $\mathbf{x}\in[\mathbf{0,2}]$.

$$\delta\omega_3=-\left[\frac{a_1\omega-v}{a_2-a_1v}+C\left(b_0\pm b_1\sqrt{-pq\gamma}\ \sec_{pq}(\sqrt{-\gamma}\xi)\right)^{-2}\right], \quad (18)$$

$$u_4=\left\{b_0\pm b_1\sqrt{-pq\gamma}\ \csc_{pq}(\sqrt{-\gamma}\xi)\right\}e^{i\left(\emptyset(\xi)-\left(\frac{a_2}{a_1^2}+\frac{a_2C-a_3b_0^6}{4a_1b_0^2+2a_1^2C}\pm2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3}-\frac{a_2C-a_3b_0^6}{2b_0^2+a_1C}\right)}\right)t\right)}, \quad (19)$$

$$\delta\omega_4=-\left[\frac{a_1\omega-v}{a_2-a_1v}+C\left(b_0\pm b_1\sqrt{-pq\gamma}\ \csc_{pq}(\sqrt{-\gamma}\xi)\right)^{-2}\right]. \quad (20)$$

Case 3: *When $\gamma<0$ and $\mu=\frac{\gamma^2}{4}$, we obtain*

$$u_5=\left\{b_0\pm b_1\sqrt{\frac{-\gamma}{2}}\tanh_{pq}\left(\sqrt{\frac{-\gamma}{2}}\xi\right)\right\}e^{i\left(\emptyset(\xi)-\left(\frac{a_2}{a_1^2}+\frac{a_2C-a_3b_0^6}{4a_1b_0^2+2a_1^2C}\pm2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3}-\frac{a_2C-a_3b_0^6}{2b_0^2+a_1C}\right)}\right)t\right)}, \quad (21)$$

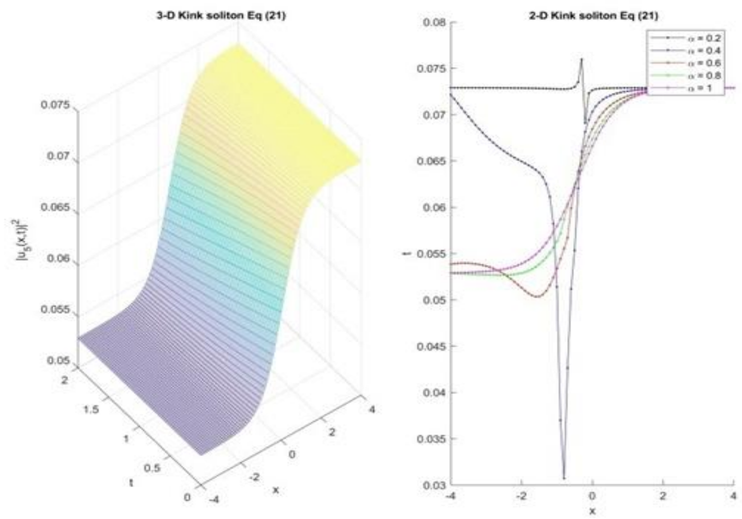

Figure 3: 3D, 2D and 3D chirped Kink soliton solutions of Equation (21) for $\mathbf{a_1 = 0.15}$, $\mathbf{a_2 = 0.05}$, $\mathbf{a_3 = 0.2}$, $\mathbf{b_0 = 0.25}$, $\boldsymbol{\gamma = -2}$, $\mathbf{b_1 = 0.02}$, $\mathbf{C = 5}$, and $\boldsymbol{\alpha = 1}$, with $\mathbf{x} \in [\mathbf{-4, 4}]$ and $\mathbf{x} \in [\mathbf{0, 2}]$.

$$\delta\omega_5 = -\left[\frac{a_1\omega - v}{a_2 - a_1 v} + C\left(b_0 \pm b_1\sqrt{\frac{-\gamma}{2}}\tanh_{pq}\left(\sqrt{\frac{-\gamma}{2}}\xi\right)\right)^{-2}\right], \tag{22}$$

$$u_6 = \left\{b_0 \pm b_1\sqrt{\frac{-\gamma}{2}}\coth_{pq}\left(\sqrt{\frac{-\gamma}{2}}\xi\right)\right\} e^{i\left(\emptyset(\xi) - \left(\frac{a_2}{a_1^2} + \frac{a_2C - a_3 b_0^6}{4a_1 b_0^2 + 2a_1^2 C} \pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3} - \frac{a_2 C - a_3 b_0^6}{2b_0^2 + a_1 C}\right)}\right)t\right)}, \tag{23}$$

$$\delta\omega_6 = -\left[\frac{a_1\omega - v}{a_2 - a_1 v} + C\left(b_0 \pm b_1\sqrt{\frac{-\gamma}{2}}\coth_{pq}\left(\sqrt{\frac{-\gamma}{2}}\xi\right)\right)^{-2}\right], \tag{24}$$

$$u_7 = \left\{b_0 \pm b_1\sqrt{\frac{-\gamma}{2}}\left(\tanh_{pq}(\sqrt{-2\gamma}\xi) \pm i\sqrt{pq}\,\mathrm{sech}_{pq}(\sqrt{-2\gamma}\xi)\right)\right\} e^{i\left(\emptyset(\xi) - \left(\frac{a_2}{a_1^2} + \frac{a_2C - a_3 b_0^6}{4a_1 b_0^2 + 2a_1^2 C} \pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3} - \frac{a_2 C - a_3 b_0^6}{2b_0^2 + a_1 C}\right)}\right)t\right)}, \tag{25}$$

$$\delta\omega_7 = -\left[\frac{a_1\omega - v}{a_2 - a_1 v} + C\left(b_0 \pm b_1\sqrt{\frac{-\gamma}{2}}\left(\tanh_{pq}(\sqrt{-2\gamma}\xi) \pm i\sqrt{pq}\,\mathrm{sech}_{pq}(\sqrt{-2\gamma}\xi)\right)\right)^{-2}\right], \tag{26}$$

$$u_8 = \left\{b_0 \pm b_1\sqrt{\frac{-\gamma}{2}}\left(\coth_{pq}(\sqrt{-2\gamma}\xi) \pm \sqrt{pq}\,\mathrm{csch}_{pq}(\sqrt{-2\gamma}\xi)\right)\right\} e^{i\left(\emptyset(\xi) - \left(\frac{a_2}{a_1^2} + \frac{a_2C - a_3 b_0^6}{4a_1 b_0^2 + 2a_1^2 C} \pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3} - \frac{a_2 C - a_3 b_0^6}{2b_0^2 + a_1 C}\right)}\right)t\right)}, \tag{27}$$

$$\delta\omega_8 = -\left[\frac{a_1\omega - v}{a_2 - a_1 v} + C\left(b_0 \pm b_1\sqrt{\frac{-\gamma}{2}}\left(\coth_{pq}(\sqrt{-2\gamma}\xi) \pm \sqrt{pq}\,\mathrm{csch}_{pq}(\sqrt{-2\gamma}\xi)\right)\right)^{-2}\right], \tag{28}$$

$$u_9 = \left\{b_0 \pm b_1\sqrt{\frac{-\gamma}{8}}\left(\tanh_{pq}\left(\sqrt{\frac{-\gamma}{8}}\xi\right) + \coth_{pq}\left(\sqrt{\frac{-\gamma}{8}}\xi\right)\right)\right\} e^{i\left(\emptyset(\xi) - \left(\frac{a_2}{a_1^2} + \frac{a_2C - a_3 b_0^6}{4a_1 b_0^2 + 2a_1^2 C} \pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3} - \frac{a_2 C - a_3 b_0^6}{2b_0^2 + a_1 C}\right)}\right)t\right)}, \tag{29}$$

$$\delta\omega_9 = -\left[\frac{a_1\omega - v}{a_2 - a_1 v} + C\left(b_0 \pm b_1\sqrt{\frac{-\gamma}{8}}\left(\tanh_{pq}\left(\sqrt{\frac{-\gamma}{8}}\xi\right) + \coth_{pq}\left(\sqrt{\frac{-\gamma}{8}}\xi\right)\right)\right)^{-2}\right]. \tag{30}$$

Case 4: *When $\gamma > 0$ and $\mu = \frac{\gamma^2}{4}$, we obtain*

$$u_{10} = \left\{b_0 \pm b_1\sqrt{\frac{\gamma}{2}}\tan_{pq}\left(\sqrt{\frac{\gamma}{2}}\xi\right)\right\} e^{i\left(\emptyset(\xi) - \left(\frac{a_2}{a_1^2} + \frac{a_2C - a_3 b_0^6}{4a_1 b_0^2 + 2a_1^2 C} \pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3} - \frac{a_2 C - a_3 b_0^6}{2b_0^2 + a_1 C}\right)}\right)t\right)}, \tag{31}$$

$$\delta\omega_{10} = -\left[\frac{a_1\omega - v}{a_2 - a_1 v} + C\left(b_0 \pm b_1\sqrt{\frac{\gamma}{2}}\tan_{pq}\left(\sqrt{\frac{\gamma}{2}}\xi\right)\right)^{-2}\right], \tag{32}$$

$$u_{11} = \left\{b_0 \pm b_1\sqrt{\frac{\gamma}{2}}\cot_{pq}\left(\sqrt{\frac{\gamma}{2}}\xi\right)\right\} e^{i\left(\emptyset(\xi) - \left(\frac{a_2}{a_1^2} + \frac{a_2 C - a_3 b_0^6}{4a_1 b_0^2 + 2a_1^2 C} \pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3} - \frac{a_2 C - a_3 b_0^6}{2b_0^2 + a_1 C}\right)}\right)t\right)}, \tag{33}$$

$$\delta\omega_{11} = -\left[\frac{a_1\omega - v}{a_2 - a_1 v} + C\left(b_0 \pm b_1\sqrt{\frac{\gamma}{2}}\cot_{pq}\left(\sqrt{\frac{\gamma}{2}}\xi\right)\right)^{-2}\right], \tag{34}$$

$$u_{12} = \left\{b_0 \pm b_1\sqrt{\frac{\gamma}{2}}\left(\tan_{pq}(\sqrt{2\gamma}\xi) \pm \sqrt{pq}\,\sec_{pq}(\sqrt{2\gamma}\xi)\right)\right\} e^{i\left(\emptyset(\xi) - \left(\frac{a_2}{a_1^2} + \frac{a_2 C - a_3 b_0^6}{4a_1 b_0^2 + 2a_1^2 C} \pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3} - \frac{a_2 C - a_3 b_0^6}{2b_0^2 + a_1 C}\right)}\right)t\right)}, \tag{35}$$

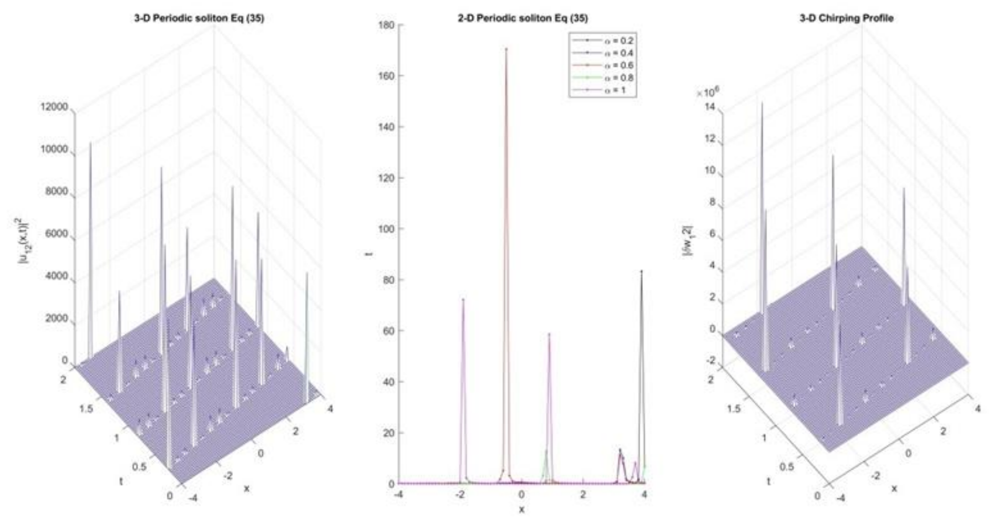

Figure 4: 3D, 2D and 3D chirped Periodic soliton solutions of Equation (35) for $\mathbf{a_1 = 0.15, a_2 = 1.5}$, $\mathbf{a_3 = 0.12, b_0 = 0.25, \gamma = 0.12,\ b_1 = 0.5, C = -1.5}$,and$\boldsymbol{\alpha} = \mathbf{1}$, with $\mathbf{x} \in [\mathbf{-4, 4}]$ and $\mathbf{x} \in [\mathbf{0, 2}]$.

$$\delta\omega_{12} = -\left[\frac{a_1\omega - v}{a_2 - a_1 v} + C\left(b_0 \pm b_1\sqrt{\frac{\gamma}{2}}\left(\tan_{pq}(\sqrt{2\gamma}\xi) \pm \sqrt{pq}\,\sec_{pq}(\sqrt{2\gamma}\xi)\right)\right)^{-2}\right], \tag{36}$$

$$u_{13} = \left\{b_0 \pm b_1\sqrt{\frac{\gamma}{2}}\left(\cot_{pq}(\sqrt{2\gamma}\xi) \pm \sqrt{pq}\,\csc_{pq}(\sqrt{2\gamma}\xi)\right)\right\} e^{i\left(\emptyset(\xi) - \left(\frac{a_2}{a_1^2} + \frac{a_2 C - a_3 b_0^6}{4a_1 b_0^2 + 2a_1^2 C} \pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3} - \frac{a_2 C - a_3 b_0^6}{2b_0^2 + a_1 C}\right)}\right)t\right)}, \tag{37}$$

$$\delta\omega_{13} = -\left[\frac{a_1\omega - v}{a_2 - a_1 v} + C\left(b_0 \pm b_1\sqrt{\frac{\gamma}{2}}\left(\cot_{pq}(\sqrt{2\gamma}\xi) \pm \sqrt{pq}\,\csc_{pq}(\sqrt{2\gamma}\xi)\right)\right)^{-2}\right], \tag{38}$$

$$u_{14} = \left\{b_0 \pm b_1\sqrt{\frac{\gamma}{8}}\left(\tan_{pq}\left(\sqrt{\frac{\gamma}{8}}\xi\right) + \cot_{pq}\left(\sqrt{\frac{\gamma}{8}}\xi\right)\right)\right\} e^{i\left(\emptyset(\xi) - \left(\frac{a_2}{a_1^2} + \frac{a_2 C - a_3 b_0^6}{4a_1 b_0^2 + 2a_1^2 C} \pm 2\sqrt{\frac{a_2}{a_1}\left(\frac{a_2}{a_1^3} - \frac{a_2 C - a_3 b_0^6}{2b_0^2 + a_1 C}\right)}\right)t\right)}, \tag{39}$$

$$\delta\omega_{14} = -\left[\frac{a_1\omega - v}{a_2 - a_1 v} + C\left(b_0 \pm b_1\sqrt{\frac{\gamma}{8}}\left(\tan_{pq}\left(\sqrt{\frac{\gamma}{8}}\xi\right) + \cot_{pq}\left(\sqrt{\frac{\gamma}{8}}\xi\right)\right)\right)^{-2}\right]. \quad (40)$$

The soliton solutions obtained in this study are diverse and novel, originating from the general solutions.

4. CONCLUSION

This article discusses the use of the Sardar sub-equation approach to produce novel optical and chirped optical solitons from the space-time fractional cubic nonlinear Schrödinger equation with Kerr law nonlinearity. The solutions show a wide range of behaviors within the system and are stated in terms of trigonometric and hyperbolic functions. A wide range of phenomena are displayed by the several varieties of optical and chirped optical solitons that have been found, including dark, bright, kink, and periodic. These solutions are shown in two and three-dimensional graphics. The results provide a thorough understanding of nonlinear dynamics and shed some light on phenomena related to condensed matter physics, quantum mechanics, and nonlinear optics.

5. REFERENCES

1. Agarwal A. Fiber-Optic Communication Systems.Wiley. 2010. https://onlinelibrary.wiley.com/doi/book/10.1002/9780470918524

2. Altwaty AA, Hassan SM, Baleanu D. Soliton and wave solutions to the extended Gerdjikov-Ivanov equation in DWDM system with auxiliary equation method. Math SciLett. 2020;9(3):57-63.

   https://doi.org/10.18576/msl/090302

3. Altwaty AA. Optical solitons in Fiber Bragg Gratings for the coupled form of the nonlinear (2+1)-dimensional Kundu-Mukherjee-Naskar equation via four powerful techniques. Results Phys. 2023;44:106205.

   https://doi.org/10.1016/j.rinp.2022.106205

4. Rappaport TS. Wireless communications: principles and practice. Pearson Education; 2019.

5. Oppenheim AV, Schafer RW, Buck JR. Discrete-time signal processing. Pearson; 2020.

6. Gonzalez RC, Woods RE. Digital image processing. Pearson; 2017.

7. Saha Ray S, New Techniques on Fractional Reduced Differential Transform Method. In: Nonlinear Differential Equations in Physics. Springer, Singapore 2020.

   https://doi.org/10.1007/978-981-15-1656-6_7

8. Fushun L, Yuqiang F.The modified generalized Kudryashov method for nonlinear space–time fractional partial differential equations of Schrödinger type.Results in PhysicsVolume 53, October 2023, 106914.

   https://doi.org/10.1016/j.rinp.2023.106914

9. Mustafa Ekici.Exact Solutions to Some Nonlinear Time-Fractional Evolution Equations Using the Generalized Kudryashov Method in Mathematical Physics.Symmetry 2023, 15(10), 1961.

   https://doi.org/10.3390/sym15101961

10. Rana TS. A Novel Approach for Solving Nonlinear Time Fractional Fisher Partial Differential Equations.Journal of Advances in Mathematics Vol 22 (2023).

    https://doi.org/10.24297/jam.v22i.9558

11. Jocemar QC, Giuliano G. La G, Ervin KL.A finite difference method for a class of nonlinear fractional advection–diffusion equations. Partial Differential Equations in Applied Mathematics.Volume 4, December 2021, 100090.

    https://doi.org/10.1016/j.padiff.2021.100090

12. Azin A, Habibirad A, Baghani O.Legendre-finite difference method for solving fractional nonlinear Sobolev equation with Caputo derivative.Journal of Computational Science.Volume 74, December 2023, 102177.

    https://doi.org/10.1016/j.jocs.2023.102177

13. Muhammad BH, Marek K. Fractional Spectral and Fractional Finite Element Methods: A Comprehensive Review and Future Prospects. Archives of Computational Methods in Engineering.2024.

    https://doi.org/10.1007/s11831-024-10083-w

14. Malatip A, Wansophark N,Dechaumphair P. Fractional four-step finite element method for analysis of thermally coupled fluid-solid interaction problems. Applied Mathematics and Mechanics.Volume 33, pages 99–116,2012.

    https://doi.org/10.1007/s10483-012-1536-9

15. Lakhlifa S , Tania AL, Ishak H. Conformable finite element method for conformable fractional partial differential equations.AIMS Mathematics. 2023, Volume 8, Issue 12: 28858-28877.

    https://doi.org/10.3934/math.2023147

16. Changtao S, Jie S. A Hybrid Spectral Element Method for Fractional Two-Point Boundary Value Problems.Numerical Mathematics: Theory, Methods and Applications, Volume 10 , Issue 2 , May 2017 , pp. 437 – 464.

    https://doi.org/10.4208/nmtma.2017.s11

17. Baoli Y, Yang L, Hong L, Zhimin Z. Finite Element Methods Based on Two Families of Second-Order Numerical Formulas for the Fractional Cable Model with Smooth Solutions. J SciComput 84, 2 (2020).

    https://doi.org/10.1007/s10915-020-01258-1

18. Bruch, E.K. Introduction to the Boundary Element Method. In: The Boundary Element Method for Groundwater Flow. Lecture Notes in Engineering, vol 70. Springer, Berlin, Heidelberg, 1991.

https://doi.org/10.1007/978-3-642-84577-2_3

19. Liu Q, Zhuang P, Liu F. et al. An investigation of radial basis functions for fractional derivatives and their applications. ComputMech 65, 475–486 (2020).

https://doi.org/10.1007/s00466-019-01779-z

20. Tseng CC, Lee SL, "Design of fractional delay FIR filter using radial basis function, IEEE International Symposium on Circuits and Systems, Taipei, Taiwan, 2009, pp. 485-488,

https://doi.org/10.1109/ISCAS.2009.5117791.

21. Hussain M, Haq S. Radial Basis Functions Collocation Method for Numerical Solution of Coupled Burgers' and Korteweg-de Vries Equations of Fractional Order. Iran J SciTechnol Trans Sci 45, 707–725 (2021).

https://doi.org/10.1007/s40995-021-01065-9

22. Shaiq MS, Iqbal Z, Mohyud-Din ST. Homotopy Analysis Method for Time-Fractional Wave-Like Equations. Comput Math Model 24, 592–603 (2013).

https://doi.org/10.1007/s10598-013-9201-2

23. Saratha SR, Bagyalakshmi M, SaiSundara Krishnan G. Fractional generalisedhomotopy analysis method for solving nonlinear fractional differential equations. Comp. Appl. Math. 39, 112 (2020).

https://doi.org/10.1007/s40314-020-1133-9

24. Shah,NA, El-Zahar ER, Aljoufi MD. et al. An efficient approach for solution of fractional-order Helmholtz equations.Adv Differ Equ 2021, 14 (2021).

https://doi.org/10.1186/s13662-020-03167-x

25. Nadeem M, He JH, The homotopy perturbation method for fractional differential equations: part 2, two-scale transform, International Journal of Numerical Methods for Heat & Fluid Flow,2022, Vol. 32 No. 2, pp. 559-567.

https://doi.org/10.1108/HFF-01-2021-0030

26. Tasnim F, Akbar MA, Osman MS. The Extended Direct Algebraic Method for Extracting Analytical Solitons Solutions to the Cubic Nonlinear Schrödinger Equation Involving Beta Derivatives in Space and Time. JAMA. 2023.

https://doi.org/10.3390/fractalfract7060426

27. Gang-Zhou W, Li-Jun Y, Yue-Yue W, Fractional optical solitons of the space-time fractional nonlinear Schrödinger equation, Optik Volume 207, April 2020, 164405.

https://doi.org/10.1016/j.ijleo.2020.164405

28. Atangana A, Alqahtani RT. Modelling the Spread of River Blindness Disease via the Caputo Fractional Derivative and the Beta-Derivative. Entropy. 2016;18:40.

https://doi.org/10.3390/e18020040

29. Sharma VK. Chirped soliton-like solutions of generalized nonlinear Schrödinger equation for pulse propagation in negative index material embedded into a Kerr medium. Indian J Phys. 2016;90:1271-1276.

https://doi.org/10.1007/s12648-016-0840-y