Numerical Solutions of Non-Local Problem for fractional order differential equations

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Summary:

The paper focuses on the study of problem- related to the linear second order boundary value problem (BVP). Three generalization styles are discussed. In the first, the second order BVP with non-local condition of the Dirichlet type is studied. In the second, the fractional order BVP is considered. In the third, the fractional order BVP with non-local condition is treated. The finite difference representation of the BVP is employed to reduce the problem into a system of algebraic equations, in addition, the implicit- explicit treatment to the nonlocal problems is introduced. In the implicit- explicit treatment the computational work is reduced considerably.

Keywords: BVP, Nonlocal conditions, fractional order, finite difference, Implicit -Explicit treatment.

الحلول العددية لمسألة المعادلات التفاضلية غير المحلية من الرتب الكسرية

هويدا سالم منصور

الملخص:

يركز البحث على دراسة المسألة الحدية الخطية من الرتبة الثانية ويناقش البحث ثلاثة أساليب للتعميم. في الاسلوب الأول تم دراسة المسألة الحدية من الرتبة الثانية المرتبطة بالشروط غير المحلية من النوع ديرشليت، وفي الثاني تم دراسة المسألة من الرتبة الكسرية المرتبطة بالشروط الحدية من نوع ديرشليت، وفي الثالث تم دراسة المسألة الحدية من الرتبة الكسرية المرتبطة بالشروط غير المحلية .

وتم تحويل المسألة الحدية إلى نظام من المعادلات الجبرية باستخدام طريقة الفروق المنتهية. كما تم تقديم أسلوب الزحف التكراري الصريح عند حل المسألة الحدية غير محلية والذي ترتب عليه تقليل حجم الحسابات بطريقة ملموسة.

الكلمات المفتاحية: مسألة القيمة الحدية (BVP)، الشروط غير المحلية ، الرتبة الكسرية، الفروق المحدودة، المعالجة الضمنية الصريحة.

Abstract

The computational costs of the numerical solvability of the nonlocal two-point Dirichlet boundary value problem (NBVP) is investigated and reduced. The finite difference treatment of the BVP is used as implicit technique within the smallest available subdomain. A reduced consistent system of algebraic equations corresponding to a well-behaved standard BVP is generated within the smallest domain, $[0, \xi]$. A reduction of the computational work is introduced through reducing the size of the algebraic system. A marching explicit approach is reemployed to generate the solution in $(\xi, 1]$, the complement of the interval $[0, \xi]$ with respect to the overall domain. Application of the technique is illustrated through second order BVP's as well as a corresponding fractional order counterpart. Caputo fractional derivatives with their Grünwald approximation is considered. Properties of the discretized algebraic system are established for different values of the fractional order. Numerical experiments confirming the applicability of the treatment are introduced.

1. Introduction

A typical Dirichlet second order BVP can be written in the form, [1, 2, 3, 4]

$$F\big(t, u(t), u'(t), u''(t)\big) = 0, \quad a < t < b \,,$$
$$u(a) = r_1 \,, \ u(b) = r_2, \tag{1.a}$$

We consider the category of BVP which is linear in its highest order term (such problems sometimes are known as quasilinear BVP). Accordingly, (1.a) can be written in the form

$$u''(t) + f(t, u(t), u'(t),) = 0,$$
$$u(a) = r_1 \,, \ u(b) = r_2, \tag{1.b}$$

The existence and uniqueness of solutions to nonlocal two-point Dirichlet boundary value problem have been investigated by many authors, among them [5, 6]. Nonlocal boundary value problem appeared in many fields of science and engineering have been investigated by many authors, among them [7, 8, 9, 10]. Nonlocal conditions appear when the boundary data are not a valuable. In nonlocal Dirichlet boundary value problems (NBVP's), the value of the unknown solution is given within the interval under consideration at a point ξ, with $a < \xi < b$, not at the end points of the interval.

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The given boundary conditions in (1.a, b) are replaced by the conditions

$$u(a) = r_1, \ u(\xi) = r_2, \quad a < \xi < b \tag{1.c}$$

In general, nonlocal two-point Dirichlet boundary value problem includes a classical two-point Dirichlet boundary value problem defined on a small subdomain $[a, \xi]$.

Although, the finite difference is considered as the oldest numerical technique used in the approximation of differential equations it is still the most applicable method due its simplicity. The philosophy of the finite difference method is the replacement of the continuous domain by a discrete set of grid points as in figure (1) and the replacement of the derivatives appears in the differential equation by a corresponding difference representation at the grid points. Accordingly, a set of algebraic relations are defined at the grid points. Finite difference treatment for linear boundary value problems requires solution of structured large linear systems of algebraic equations.

Our main objective is to introduce the solution of (1.b) subject to the nonlocal boundary condition (1.c) at low computational costs and moreover generalize this treatment to cover the fractional order cases, was appeared in [11]. This objective can be achieved through splitting the problem into two tracks. In the first track the included two-point Dirichlet boundary value problem defined on $[a, \xi]$ is solved through an implicit process by solving a reduced algebraic system. In the second track the differential equation (1.a) is considered over the domain $[\xi, b]$ and a reformulation of the finite difference scheme is introduced in an explicit form and the required initial data are taken from the first track.

There is no loss of generality to take the interval $[a, b]$ as the interval $[0, 1]$.

Thus, our target can be achieved through the treatment of a simple fractional order nonlocal Dirichlet boundary value problem (NBVP) of the form

$$-u^{(\alpha)}(t) + u(t) = f(t), \quad 0 < t < 1, \ 1 < \alpha \le 2 \tag{2.a}$$

$$u(0) = r_1, \ u(\xi) = r_2, 0 < \xi < 1 \tag{2.b}$$

The general outcome can be seen through the treatment of three simple problems.

The first sub problem is the nonlocal problem

$$-u''(t) + u(t) = f(t), \quad 0 < t < 1 \tag{3.a}$$

$$u(0) = r_1, \ u(\xi) = r_2, 0 < \xi < 1 \tag{3.b}$$

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There is no doubt that along the interval $[0, \xi]$ this problem is well posed and admits a unique solution, which can be obtained through an implicit process. The solutions obtained and the nonlocal boundary condition at the point ξ are used to generate the solution through an explicit process along the interval $[\xi, 1]$ as illustrated in example (1) case I

The second sub problem is the fractional order BVP

$$-u^{(\alpha)}(t) + u(t) = f(t), \quad 0 < t < 1, 1 < \alpha \leq 2 \tag{4.a}$$

$$u(0) = r_1, \quad u(1) = r_2 \tag{4.b}$$

The third sub problem is the fractional NBVP, [12, 13]

$$-u^{(\alpha)}(t) + u(t) = f(t), \quad 0 < t < 1, 1 < \alpha \leq 2 \tag{5.a}$$

$$u(0) = r_1, \quad u(\xi) = r_2, 0 < \xi < 1 \tag{5.b}$$

1.1 Fractional Calculus, [11, 14]

There are several definitions of fractional derivatives, we are interested in the Caputo and Grünwald-Letnikov definitions and on Grünwald-Letnikov discretization approach.

Caputo Fractional Derivatives

The Caputo fractional derivative is defined as

$$D_C^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{d^m u(\tau)}{d\tau^m} (t-\tau)^{m-\alpha-1} d\tau. \tag{6}$$

where, $m - 1 < \alpha < m$ and $m = [\alpha] + 1$ with $[\alpha]$ denotes the integer part of α.

Grünwald-Letnikov fractional derivative

The Grünwald-Letnikov fractional derivative of a function $u(t), t \in [a, b]$ is defined as

$$D_{GL}^\alpha u(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t^\alpha} \sum_{k=0}^{\left[\frac{t-a}{\Delta t}\right]} (-1)^k \frac{\Gamma(\alpha+1)}{k! \, \Gamma(\alpha-k+1)} u(t - k\Delta t), \alpha > 0 \tag{7}$$

It is a generalization of the classical derivatives. It is proved that the series in this definition is absolutely and uniformally convergent for each $\alpha > 0$

Shifted Grünwald-Letnikov Form

To increase the order of the accuracy and to introduce stable numerical schemes a shifted form of the Grünwald-Letnikov fractional derivative is introduced as

$$D_{SG}^\alpha u(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t^\alpha} \sum_{k=0}^{\left[\frac{t-\alpha}{\Delta t}\right]} (-1)^k \frac{\Gamma(\alpha+1)}{k! \, \Gamma(\alpha-k+1)} u(t - (k-1)\Delta t) \tag{8}$$

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The Grünwald-Letnikov (G-L) weights are defined as

$$g_k^\alpha = (-1)^k \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}, \quad \sum_{n=0}^\infty g_n^\alpha = 0. \tag{9}$$

Thus, $g_0^\alpha = 1, g_1^\alpha = -\alpha < 0, g_2^\alpha = \frac{\alpha\,(\alpha-1)}{2} > 0, 1 < \alpha \le 2$

It can be proved that the G-L weights satisfy

$$g_k^\alpha = \left(1 - \frac{\alpha+1}{k}\right) g_{k-1}^\alpha, \quad k = 1,2,3,\dots \tag{10}$$

$g_2^\alpha > g_3^\alpha > \cdots > g_n^\alpha; \ \lim_{n \to \infty} g_n^\alpha = 0;$

1.2 The Finite Difference Method, [2, 11, 14, 15]

In the finite difference method, the continuous domain [0, 1] is superimposed by a set of discrete points $\mathcal{P}_N = \{t_0, t_1, \cdots, t_N\} = \{t_i\}_{i=0}^N$, known as the finite difference grid as shown in figure (1)
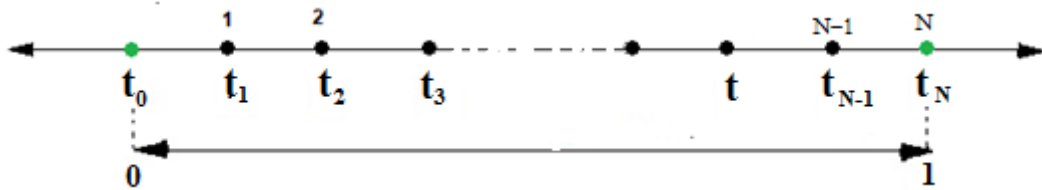

Figure 1 the grid imposed on the interval [0, 1]

$t_i = 0 + i\,h, \ i = 0,1,\dots,N$, with grid spacing $h = \frac{1-0}{N}$

It is natural to use the notation $u(t_i) = u_i$, the central difference approximation for the classical first order derivative $u'(t_i) = \frac{u_{i+1}-u_{i-1}}{2h} + O(h^2)$, the central difference approximation for the classical second order derivative $u''(t_i) = \frac{u_{i-1}-2u_i+u_{i+1}}{h^2} + O(h^2)$ and the Shifted Grünwald-Letnikov formula $D_{SG}^\alpha u(t_i) = \frac{1}{h^\alpha} \sum_{k=0}^{i+1} g_k^\alpha u_{i-k+1}, \ 1 < \alpha \le 2$, for the approximation of the fractional order derivatives, where the G-L weights are

$$g_k^\alpha = (-1)^k \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}.$$

It is generally accepted that every differential equation can be approximated by a corresponding finite difference scheme by replacing the derivative terms by their corresponding finite difference approximation at each grid point. Accordingly, equation (1.b) can be written in the discrete form

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$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + f\left(t_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right) = 0, \quad i = 1, \dots, N-1 \tag{11}$$

Accordingly, a system of algebraic equations is obtained the solution of the algebraic system gives approximation to the solution of the given boundary value problem.

Thus, the classical Dirichlet second order differential equation

$$-u''(t) + u(t) = f(t), \quad 0 < t < 1 \tag{12.a}$$

$$u(0) = r_1, \quad u(1) = r_2, \tag{12.b}$$

is approximated by the algebraic system,

$$-u_{i-1} + (2+h^2)u_i - u_{i+1} = h^2 f_i, \, i = 1, 2, \cdots, N-1 \tag{13}$$

It is well known that the finite difference scheme must pass through some tests of consistency, stability and convergence in order to give reliable results, was presented in [1, 2]. Another type of problems appears due to the nature of the associated boundary condition. It is well known that Derichlet linear boundary value problems without first order derivative term is well posed and the associated linear system can be solved efficiently. the finite difference equation (13) is written in matrix form as

$$\underbrace{\begin{pmatrix} 2+h^2 & -1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2+h^2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2+h^2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & -1 & 2+h^2 & -1 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & -1 & 2+h^2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2+h^2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ \vdots \\ u_{N-1} \end{pmatrix}}_{U} = \underbrace{\begin{pmatrix} h^2 f_1 - r_1 \\ h^2 f_2 \\ h^2 f_3 \\ h^2 f_4 \\ h^2 f_5 \\ \vdots \\ h^2 f_{N-1} \\ h^2 f_N - r_2 \end{pmatrix}}_{F} \tag{14}$$

The coefficient square matrix A, is of order $(N-1) \times (N-1)$, the unknown vector u and the right-hand side F are two $(N-1)$ columns vectors, the coefficient matrix, $A = (a_{ij})$ is strictly diagonally dominant, positive definite and tridiagonal.

2. Material and Methods

In nonlocal two-point BVP some of the boundary conditions are given within the overall domain as described in (1.c) and illustrated in figure (2).
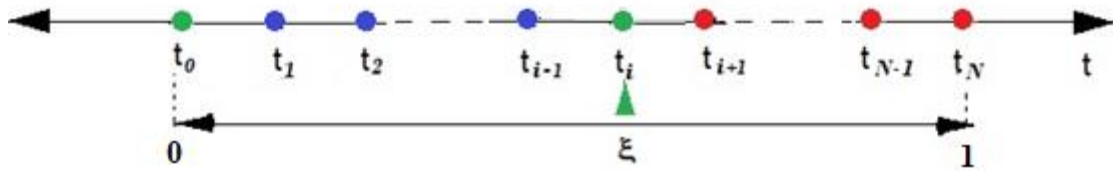
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Figure 2 a nonlocal boundary condition at the point ξ or the grid point t_i.

The grid space h is chosen such that the point ξ is a grid point (the case of non-grid point will be considered later in a subsequent work); thus we have a classical two-point BVP over the interval $[0, \xi]$. The given BVP is approximated by a corresponding finite difference scheme of the form.

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} + f\left(t_j, u_j, \frac{u_{j+1} - u_{j-1}}{2h}\right) = 0, \, j = 1, 2, \cdots, i - 1. \tag{15}$$

Accordingly, a reduced system of algebraic equations is obtained, the solution of the algebraic system gives approximation to the solution of the given boundary value problem over the interval $[0, \xi]$.

Our treatment depends on solving the problem over the interval $[0, \xi]$ as a typical standard BVP and use the difference scheme as a marching technique to generate the solution outside the interval $[0, \xi]$. Therefore, we introduce an explicit treatment over the interval $[\xi, 1]$.

2.1 The Implicit-Explicit treatment

The implicit track: In the implicit track an algebraic system of (i-1) equations is solved and the values $u_0, u_1, \cdots, u_i$ are obtained.

The explicit track: In the explicit track a rearrangement of the finite difference scheme is considered at each grid point in the interval $[\xi, 1]$, at the points $t_i, t_{i+1}, \cdots, t_{N-1}$.

$$u_{j+1} = H\left(t_j, u_j, \sum_{s=0}^{j} c_s \, u_s\right), \, j = i, i + 1, \cdots, N - 1. \tag{16}$$

Accordingly, the values $u_{i+1}, u_{i+2}, \cdots, u_N$ are determined sequentially with the same accuracy of the finite difference scheme and we will illustrate this track in the numerical examples.

2.2 Finite Difference Approximations of Fractional Order BVP

The fractional order analogy of the differential equation (3.a) is written in the form

$$-u^{\alpha}(t) + u(t) = f(t), 0 < t \leq 1, 1 < \alpha \leq 2 \tag{17}$$

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Where the fractional derivatives are understood in the Caputo sense and the boundary conditions as given in (1.b).

Finite Difference Approximations of Fractional Derivatives, as in the integer case the fractional order derivatives can be approximated by formulas which contain only function values at specific positions.

uses the shifted Grünwald approximation formula (8), we get

$$-\sum_{k=0}^{i+1} g_k^\alpha u_{i-k+1} + h^\alpha u_i = h^\alpha f_i, \quad i = 1, \ldots, N-1 \tag{18}$$

Equation (18) represents a linear system corresponding to the fractional order case, which can be written in matrix form as

$$
\begin{pmatrix}
-g_1^\alpha + h^\alpha & -g_0^\alpha & 0 & 0 & 0 & \cdots & \cdots & 0 \\
-g_2^\alpha & -g_1^\alpha + h^\alpha & -g_0^\alpha & 0 & 0 & \cdots & \cdots & 0 \\
-g_3^\alpha & -g_2^\alpha & -g_1^\alpha + h^\alpha & -g_0^\alpha & 0 & \cdots & \cdots & 0 \\
-g_4^\alpha & -g_3^\alpha & -g_2^\alpha & -g_1^\alpha + h^\alpha & -g_0^\alpha & 0 & \cdots & 0 \\
-g_5^\alpha & -g_4^\alpha & -g_3^\alpha & -g_2^\alpha & -g_1^\alpha + h^\alpha & -g_0^\alpha & \ddots & 0 \\
\ddots & -g_5^\alpha & -g_4^\alpha & -g_3^\alpha & -g_2^\alpha & -g_1^\alpha + h^\alpha & \ddots & 0 \\
-g_{N-2}^\alpha & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -g_0^\alpha \\
-g_{N-1}^\alpha & -g_{N-2}^\alpha & \ddots & -g_5^\alpha & -g_4^\alpha & -g_3^\alpha & -g_2^\alpha & -g_1^\alpha + h^\alpha
\end{pmatrix}
\underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ \vdots \\ u_{N-1} \end{pmatrix}}_{u}
=
\underbrace{\begin{pmatrix} h^\alpha f_1 - r_1 \\ h^\alpha f_2 \\ h^\alpha f_3 \\ h^\alpha f_4 \\ h^\alpha f_5 \\ \vdots \\ h^\alpha f_{N-1} \\ h^\alpha f_N - r_2 \end{pmatrix}}_{f}
\tag{19}
$$

The coefficient matrix A is of order $(N-1) \times (N-1)$.

3. Error Estimates

In order to estimate the accuracy of the obtained solution define the global error estimate and local error estimate as follows

1. $U = [U_1, U_2, \ldots, U_n]^T$ denotes the approximate solution generated by some FD scheme with no round-off errors and $u = [u(t_1), u(t_2), \ldots, u(t_n)]$ is the exact solution at the grid points $t_1, t_2, \ldots, t_n$, then the global error vector is defined as $E = U - u$.

2. The local truncation error refers to the difference between the original differential equation and its FD approximation at a grid point.

$$T_i \equiv L_h w(t_i) - Lw(i), i = 1, \ldots, N,$$

where w is a smooth function on I.

It is interesting to note that increasing the nonlocal points to get closer to the local point (a) give better approximate solution, approaching the exact solution.

4. Numerical Examples

To illustrate the theoretical results described above two simple numerical examples are considered. The first is the fractional order differential equation with low degree polynomial solution in which all the three cases described (the integer case $\alpha = 2$, with nonlocal boundary – the fractional order case $1 < \alpha < 2$, and the fractional order with nonlocal boundary condition) different step size h for the grid are considered and different values of the fractional order α. The second example to illustrate that the treatment works well even when the solution is not polynomial, exponential behavior is considered and the error in all cases is within the proved theoretical attitudes (proportional to h^{α}). The second example is restricted to the integer case to guarantee h^2 accuracy.

Example (1) Consider the following differential equation

$$-u^{\alpha}(t) + 4u = -\frac{\Gamma(4)}{\Gamma(4-\alpha)}(t)^{3-\alpha} + 4t^3 \; , \; 0 < t < 1 \; , \; 1 < \alpha \leq 2 \qquad (20)$$

Subject to the boundary condition

$u(0) = 0$, and the nonlocal boundary conditions $\quad u(\xi) = \xi^3 ; 0 < \xi \leq 1$

it can be proved that the exact solution is $u(t) = t^3$,

this example is reduced to the standard classical case when $\alpha = 2$ and $\xi = 1$, the negative sign in the left-hand side to guarantee the positive definite of the differential operator as well as the coefficient matrix of the discretized system described in equation (13)

case I

The classical second order BVP with nonlocal boundary condition, $\alpha = 2$ and $\xi = 0.8$

the first track gives the finite deference approximation

$$\begin{pmatrix} 2.04 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2.04 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2.04 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2.04 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2.04 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2.04 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2.04 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = \begin{pmatrix} -0.00596 \\ 0.01168 \\ -0.01692 \\ -0.02144 \\ -0.025 \\ -0.02736 \\ 0.48372 \end{pmatrix}, \qquad (21)$$

It is well known that this tridiagonal system has a unique solution as given in table 1 below. The unknowns u_9 and u_{10} are determined by an explicit proses through the use of finite

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difference scheme at the points u_8 to obtain u_9 (the second track) and then at u_9 to obtain u_{10}

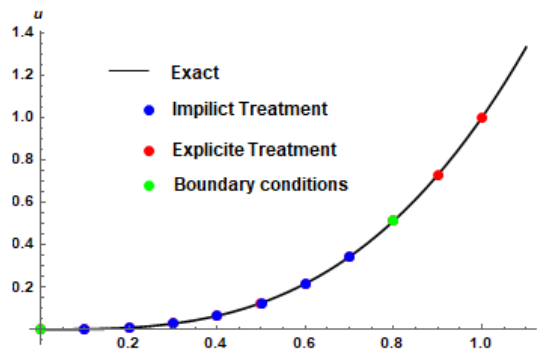| t_i	Exact	App.
0.0	0	0
0.1	0.001	0.001
0.2	0.008	0.008
0.3	0.027	0.027
0.4	0.064	0.064
0.5	0.125	0.125
0.6	0.216	0.216
0.7	0.343	0.343
0.8	0.512	0.512
0.9	0.729	0.729
1.0	1	1

Figure 3: $\alpha = 2, h = 0.1$

Table 1 the results of applying the implicit-explicit treatment

case II

The fractional order BVP with classical boundary conditions, $\alpha = 1.9$, $\xi = 1.0$ and, $\alpha = 1.8$, $\xi = 1.0$ (h= 0.1, h =0.02).

$$
\begin{pmatrix}
1.95036 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.855 & 1.95036 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.0285 & -0.855 & 1.95036 & -1 & 0 & 0 & 0 & 0 & 0 \\
-0.007838 & -0.0285 & -0.855 & 1.95036 & -1 & 0 & 0 & 0 & 0 \\
-0.003292 & -0.007838 & -0.0285 & -0.855 & 1.95036 & -1 & 0 & 0 & 0 \\
-0.001701 & -0.003292 & -0.007838 & -0.0285 & -0.855 & 1.95036 & -1 & 0 & 0 \\
-0.000996 & -0.001701 & -0.003292 & -0.007838 & -0.0285 & -0.855 & 1.95036 & -1 & 0 \\
-0.000635 & -0.000996 & -0.001701 & -0.003292 & -0.007838 & -0.0285 & -0.855 & 1.95036 & -1 \\
-0.0004304 & -0.000635 & -0.000996 & -0.001701 & -0.003292 & -0.007838 & -0.0285 & -0.855 & 1.95036
\end{pmatrix}
\begin{pmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
-0.00568312 \\
-0.0118871 \\
-0.0178382 \\
-0.0231213 \\
-0.0273786 \\
-0.0302743 \\
-0.0314833 \\
-0.0306871 \\
0.972429
\end{pmatrix}
$$

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t_i	Exact	App.
0.0	0	0
0.1	0.001	0.00224214
0.2	0.008	0.0100561
0.3	0.027	0.0295831
0.4	0.064	0.0668738
0.5	0.125	0.127951
0.6	0.216	0.218823
0.7	0.343	0.345484
0.8	0.512	0.51392
0.9	0.729	0.730104
1.0	1	1

Figure 4: $\alpha = 1.9, \text{h} = 0.1$

Table 2: the results of exact and approximation solutions
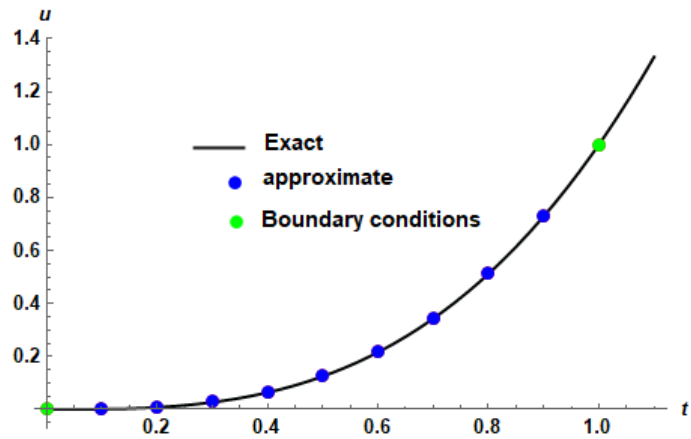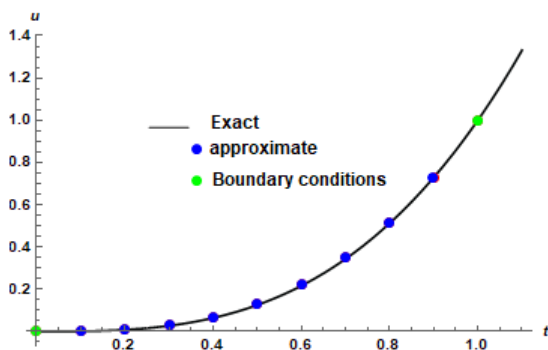
Figure 5: when $\alpha = 1.8, \text{h} = 0.1$

Figure 6: when $\alpha = 1.9, \text{h} = 0.02$

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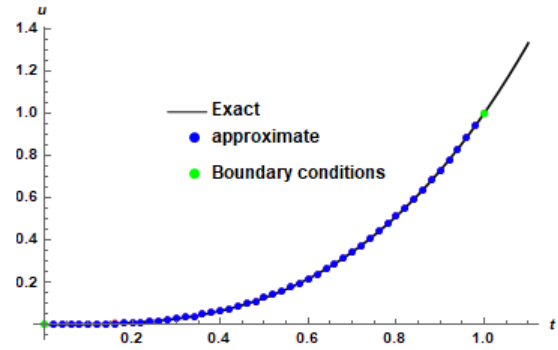
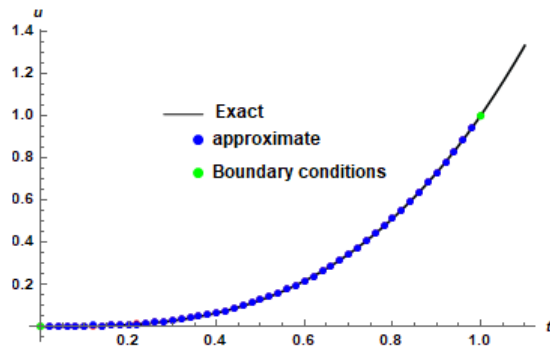العدد السابع والخمسون / مارس / 2022

Figure 7: when $\alpha = 1.8$, $h = 0.02$

case III

The fractional order BVP with nonlocal boundary condition, $\alpha = 1.9$, $\xi = 0.8$ and, $\alpha = 1.8$, $\xi = 0.8$ (h= 0.1, h =0.02).

$$
\begin{pmatrix}
1.95036 & -1 & 0 & 0 & 0 & 0 & 0 \\
-0.855 & 1.95036 & -1 & 0 & 0 & 0 & 0 \\
-0.0285 & -0.855 & 1.95036 & -1 & 0 & 0 & 0 \\
-0.0078375 & -0.0285 & -0.855 & 1.95036 & -1 & 0 & 0 \\
-0.0032918 & -0.007837 & -0.0285 & -0.855 & 1.95036 & -1 & 0 \\
-0.0017007 & -0.003292 & -0.007838 & -0.0285 & -0.855 & 1.95036 & -1 \\
-0.0009961 & -0.001701 & -0.003292 & -0.007838 & -0.0285 & -0.855 & 1.95036
\end{pmatrix}
\begin{pmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7
\end{pmatrix} =
$$

$$
\begin{pmatrix}
-0.00568312 \\
-0.0118871 \\
-0.0178382 \\
-0.0231213 \\
-0.0273786 \\
-0.0302743 \\
0.480517
\end{pmatrix}
$$

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t_i	Exact	App.
0.0	0	0
0.1	0.001	0.00206328
0.2	0.008	0.00970725
0.3	0.027	0.0290556
0.4	0.064	0.0661485
0.5	0.125	0.126999
0.6	0.216	0.217604
0.7	0.343	0.343947
0.8	0.512	0.512
0.9	0.729	0.72772
1.0	1	0.997049

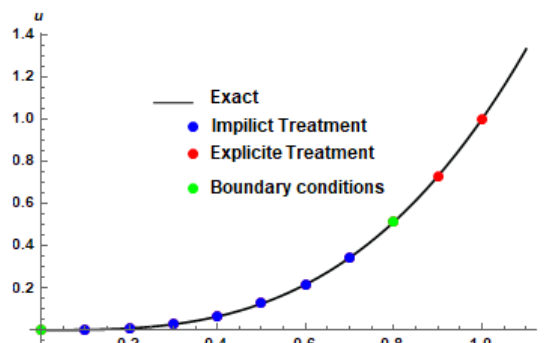**Table 3 the results of applying the impilict-explicite treatment**
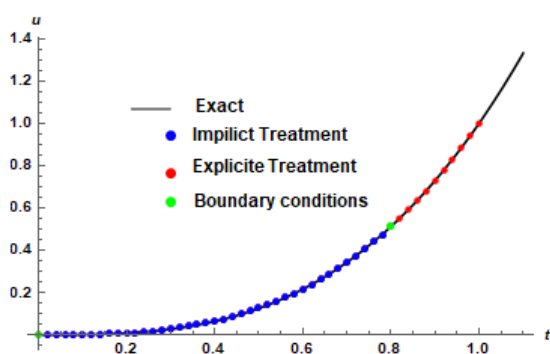

Figure 8: when $\alpha = 1.9, \text{h} = 0.1$
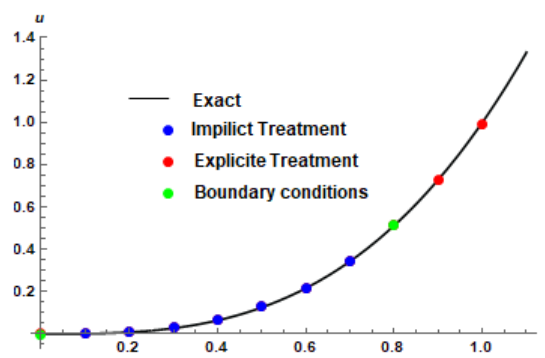

Figure 9: when $\alpha = 1.8, \text{h} = 0.02$

Figure 10: when $\alpha = 1.8, \text{h} = 0.1$

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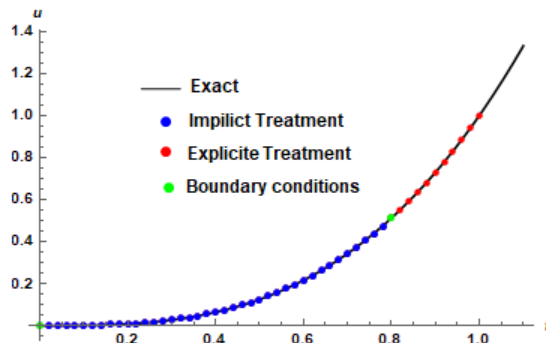
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Figure 11: when $\alpha = 1.9, \mathrm{h} = 0.02$

Example (2) Consider the following differential equation

$$-u''(t) + 4u(t) = -\frac{2\,\Gamma(4)}{\Gamma(4-\alpha)}(t)^{3-\alpha} - 0.5(t)^{2-\alpha}E_{1,3-\alpha}(-t) + 8t^3 + 2e^{-t}, \ 0 < t < 1 \quad (22)$$

with nonlocal boundary condition, $\mathrm{u}(\xi) = 2\xi^3 + 0.5e^{-\xi}, \alpha = 2$ and $\xi = 0.8$

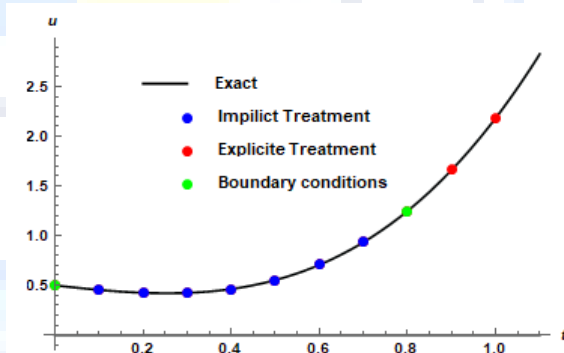| t_i	Exact	App.
0.0	0.5	0.5
0.1	0.454419	0.454428
0.2	0.425365	0.42538
0.3	0.424409	0.424427
0.4	0.46316	0.463178
0.5	0.553265	0.553281
0.6	0.706406	0.706418
0.7	0.934293	0.9343
0.8	1.24866	1.24866
0.9	1.66128	1.66128
1.0	2.18394	2.18392

Table 4 the results of applying

the impilict-explicite treatment

Figure 12: when $\alpha = 2, \mathrm{h} = 0.1$

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5.Discussion

The two-point boundary value problems are considered in many publications theoretically or numerically, [1, 2]. The shooting combined with Runge-Kutta techniques and the finite difference methods are standard numerical methods for solving BVP's, [1, 2,15]. The finite difference method is known as global method while the shooting technique depends on solving initial value problem through a marching prosses. Recently, fractional order boundary value problems with nonlocal boundary conditions appears in many theoretical works, [9, 10]. The problem of nonlocal boundary value problems even in the standard second order case appeared in many engineering and physical applications, [5, 6, 7]. Although, the finite difference method is considered as one of the simplest and straightforward methods that can treat linear BVP's, the large size of associated linear algebraic system is a problem in itself especially in the fractional order (lower triangular part of the coefficient matrix is full as in [14, 15]. It is interesting to note that the computational work required in solving algebraic system with elimination techniques is proportional to n^3, [2] where n is the size of the coefficient matrix thus reducing the size of the system is the most effective part (reducing the size from $n = 10$ to $n = 8$ is equivalent to reduce $n^3 = 1000$ to $n^3 = 512$ as in case h =0.1and reducing the size from $n = 50$ to $n = 40$ is equivalent to reduce $n^3 = 125000$ to $n^3 = 64000$ as in case h =0.02 in the numerical examples).

The finite difference method is classified as global method in the sense that it requires solution of algebraic systems related to the overall domain.

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6. Conclusion

The finite difference is an efficient method for solving BVP of fractional order as in the case of the classical second order differential equation, [1,15]. The algebraic system for fractional order is structured system with fixed values along the diagonals with $\alpha + h^{\alpha}$ value along the main diagonal and (-1) along the supper diagonal and g_{k+1}^{α} along the kth sub-diagonal.

The finite difference method works efficiently for problems with nonlocal boundary conditions the step size h must be decreased as the fractional order decrease due to the error term contains the factor h^{α}.

A finite difference treatment for linear nonlocal boundary value problems can be used as global method over the bounded domain defined by the first boundary condition (at $t = 0$) to the first nonlocal boundary and a marching technique over the rest of the domain.

The computational work is decreased considerably in the Impilict-Explicite treatment introduced due to the decrease in the dimension of the algebraic system in the implicit track and the low computational costs in the explicit track, it is just function evaluation.

The use of the shifted Grünwald for the fractional order derivatives makes the integer case special case, [14] also the algebraic systems tends to the one obtained in the standard integer case as the order α tends to 2.

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