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Stability in nonlinear integro-differential equations with functional delay on Time scale

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Abstract:

The main purpose of this work is to show the stability of the zero solution for a non-linear neutral differential equation on space called the Time scale. In order to achieve our purposes, we assume a time scale $\mathbb{T}$, that is unbounded above and below, and we employ the contractive mapping theorem by converting the neutral differential equation into an equivalent integral equation, which allows us to reach the desired result of the asymptotic stability of the zero solution on Time scale provided that $Q(t,0) = f(0) = 0$

Keywords: contractive mapping theorem, stability, nonlinear neutral, differential equation, integral equation.

الاستقرار في المعادلات التفاضلية الغير الخطية مع تأخير وظيفي فضاء Time Scale

عائشة امبية محمد الزهاوي قسم الرياضيات كلية العلوم جامعة بنغازي
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الملخص :

الغرض الرئيسي من هذا العمل هو إظهار استقرار الحل الصفري لمعادلة تفاضلية غير خطية ومحايدة على مجال زمني يُسمى المقياس الزمني. من أجل تحقيق أهدافنا، نفترض وجود مقياس زمني T غير محصور من الأعلى والأسفل، ونستخدم مبرهنة التطبيق التقليدي لتحويل المعادلة التفاضلية المحايدة إلى معادلة تكاملية مكافئة. وهذا يسمح لنا بالوصول إلى النتيجة المرغوبة في استقرار الحل الصفري على المقياس الزمني بشرط أن $Q(t,0) = f(0) = 0$.

الكلمات المفتاحية : نظرية الانكماش – الإستقرار – غير خطية – المعادلة التفاضلية – المعادلة التكاملية .

1. Introduction

The study of Stability of integro-differential equations with functional delays is one of the most important topics that researchers are interested in, due to its importance in many scientific fields for instance in biology, mechanics and economics. Researchers have been devoted to finding ways to study stability of delay differential equations one of these methods is fixed-point theory. The fixed-point method has become a powerful tool to show the stability of solutions compare it with Lyapunov function, where Lyapunov function has many difficulties, which are (if the delay is unbounded or if the differential equation in question has unbounded terms). Other than if the fixed-point method has been applied the most of these difficulties vanish, for more details we refer to [1, 2,3], as well as references in them.

In [4] the authors consider the nonlinear integro-differential equations with an infinite delay

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}\sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right) + \int_{-\infty}^{t}(D(t,s)f(x(s)) + h(s))ds \qquad (1.1),$$

and showed the existence and uniqueness of periodic solutions . Moreover, authors in [5] studyied the stability of Eq(1.1) by means of contraction mappings. Recently, in [6] authors investigated the nonlinear integro-differential equations with an infinite delay, for t∈T

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + \sum_{i=1}^{p} Q_i\left(t, x\big(t-g(t)\big)\right)^{\Delta} + \int_{-\infty}^{t}\Big(D(t,s)f\big(x(s)\big) + h(s)\Big)\Delta s \qquad (1.2),$$

by assuming $a(t)$ is a continuous real-valued function. Taking into consideration $Q_i: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $D: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, $x: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$ are continuous function, and to ensure periodicity the following assumption has been made $a(t)$, $g(t)$, $D(t,x)$, $Q(t,x)$ are periodic functions.

We are interested to study the stability of the zero solution on the equation Eq (1.2) on the space called Time scale by mutating Eq. (1.2) to an integral mapping equation appropriate for the contraction mapping theorems. Time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. The main point of this space is unifying the theory of difference equations with that of differential equations. Let $0 \in \mathbb{T}$ and $g: \mathbb{T} \to \mathbb{R}$, $id - g: \mathbb{T} \to \mathbb{T}$ is strictly increasing this leads that

$x(t - g(t))$ is well-defined over $\mathbb{T}$. The work is inspired and motivated by the works done by Ardjouni and Djoudi [7], for a wealth of reference material on the subject, we refer to [8, 9, 10], and the references in them.

The following sections 2 and 3 in this article is devoted to present the hypotheses that will be used in this study, introducing lemma that converts Eq. (1.2) to an essential equation as well as the final findings respectively.

2. Preliminaries

This section present the important notations which are related to the concepts of calculus on time scales for dynamic equations mostly all definitions, lemmas and theorems can be found in [11,12] . A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as

$$\sigma(t) = inf\{s \in \mathbb{T} : s > t\}$$
$$, and\ \rho(t) = sup\{s \in \mathbb{T} : s < t\}.$$

These operators allow elements in the time scale to be classified as follows. We say t is

i. right scattered if $\sigma(t) > t$,

ii. right dense if $\sigma(t) = t$,

iii. left scattered if $\rho(t) < t$,

iv. left dense if $\rho(t) = t$.

The graininess function μ: $\mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set inf $\emptyset$ = sup $\mathbb{T}$ and sup $\emptyset$ = inf $\mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f: \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted by $f^{\Delta}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon > 0$, there is a neighbourhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\Delta}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in T$. function: $\mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$.

We are now able to state some properties of the delta-derivative of f. Note that $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 2.1. [11]. Assume that, $f, g: \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.

i. $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$.

ii. $(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$.

iii. $(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t)$.

iv. $(fg)^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g^{\sigma}(t)$. (Theproduct rules)

v. If $g(t)g^{\sigma}(t) \neq 0$ then

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}.$$

The following two theorems deal with the composition of two functions.

Theorem 2.2 (Chain Rule) [11]. Assume, $v: \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale.

Let $w: \tilde{\mathbb{T}} \to R$. If $v^{\Delta}(t)$ and $w^{\tilde{\Delta}}(v(t))$ exist for $t \in \mathbb{T}^k$, then $(w \circ v)^{\Delta} = (w^{\tilde{\Delta}} \circ v)v^{\Delta}$.

In the sequel, we will need to differentiate and integrate functions of the form $f(t - g(t)) = f(v(t))$, where $v(t) := t - g(t)$.

Theorem 2.3 (Substitution) [11]. Assume $v: \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. If $f: \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and v is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,

$$\int_a^b f(t)v^{\Delta}(t)\Delta t = \int_{v(a)}^{v(b)} (f \circ v^{-1})(s)\,\tilde{\Delta} s$$

A function $p: \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $\in \mathbb{T}^k$. The set of all regressive rd-continuous functions $f: \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^{+}$ is given by

$$\mathcal{R}^+ = \{f \in \mathcal{R}: 1 + \mu(t)f(t) > 0 \quad for\ all\ t \in \mathbb{T}\}.$$

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t,s)m = exp\left(\int_s^t \frac{1}{\mu(z)} log(+\mu(z)p(z))\Delta z\right) \qquad (2.1)$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in T$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y\text{^}\Delta = p(t)y, y(s) = 1$. Other properties of the exponential function are given in the following lemma, **[11, Theorem 2.36]**.

Lemma 2.4. Let $p, q \in \mathcal{R}$. Then

$i. e_0(t,s) = 1\ and\ e_p(t,t) = 1,$

$ii. e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s),$

$iii. \frac{1}{e_p(t,s)} = e_{\ominus}(t,s),\ where \ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}\ ,$

$iv. e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)\ ,$

$v. e_p(t,s)e_p(s,r) = e_p(t,r)\ ,$

$vi. \left(\frac{1}{e_p(.,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(.,s)}\ .$

The notion of periodic time scales and the next two definitions are quoted from [7, 13].

Definition 2.5. We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$, such that, *if* $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p with this property is called the period of the time scale.

Example 2.6. The following time scales are periodic.

1. $\mathbb{T} = \cup_{i=-\infty}^{\infty}[2(i-1)h, 2ih], h > 0$ has period $p = 2h$.
2. $\mathbb{T} = hZ$ has period $p = h$.
3. $\mathbb{T} = \mathbb{R}$
4. $\mathbb{T} = \{t = k - q^m : k \in z, m \in N_0\}$, where $0 < q < 1$ has period $p = 1$.

Remark 2.7 [11]. All periodic time scales are unbounded above and below.

Definition 2.8. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f: \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that

$T = np, f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$and T is the smallest number such that $f(t \pm T) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f\ (t\ \pm\ T)\ =\ f(t)$ for all $t \in \mathbb{T}$.

Remark 2.9 [11]. If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies $\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period p.

Since we are searching for the asymptotic stability of the zero solution of Eq (1.2), it is natural to assume the following conditions, suppose that $Q(t,x)$ and$f(x)$ be two continuous functions. So, for E_1 ,E_2, $E_3$and E_4 are positive constants such that,

$$\sum_{i=1}^{p} |Q_i(t,x) - Q_i(t,y_i)| \leq E_1 \|x - y\|, \qquad (2.2)$$

And,

$$|f(x) - f(y)| \leq E_2 \|x - y\|. \qquad (2.3)$$

Also,

$$\int_{-\infty}^{t} |D(t,s)|\, ds \leq E_3 < \infty \ ,\ h(s) \leq E_4 \leq K E_4, \qquad (2.4),$$

where K is some positive constant. Let g be continuous with

$g(t) \geq\ 0$ for all $t \in\ \mathbb{T}$ such that $t \geq t_0$ **for some** $t_0 \in\ \mathbb{T}$ and that $Q(t\,,0) = f(0) = 0$.

Taking into consideration $\mathbb{T}$ is *a* time scale, that is unbounded above and below and that $0 \in\ \mathbb{T}$. We also indicate that $g : \mathbb{T} \rightarrow \mathbb{R}$ and that $id - g :\ \mathbb{T} \rightarrow\ \mathbb{T}$ *is* strictly increasing.

Now we introduce the next lemma to help convert Eq. (1.2) to an integral corresponding equation.

Lemma 2.1

Let $Q(t,x),\ D(t,s),\ a(t),\ f(t),\ x(t),\ g(t)\ \ and\ h(t)$ are defined as above, then $x(t)$ is a solution of *Eq* (1.1) if and only if

$$x(t) = \sum_{i=1}^{p} Q_i\left(t, x(t-g(t))\right) + \left[x(0) - \sum_{i=1}^{p} Q_i\left(0, x(0-g(0))\right)\right] e_{\ominus a}(t,0)\Delta u + \int_0^t \left[-a(u)\sum_{i=1}^{p} Q_i^{\sigma}\left(u, x(u-g(u))\right)\right] e_{\ominus a}(t,u)\Delta u \quad + \int_{-\infty}^{u}[D(u,s)f(x(s)) + h(s)]\,\Delta s]\, e_{\ominus a}(t,u)\Delta u \qquad (2.5)$$

The proof of lemma 2.1 is the same steps in [7].

4. Main Result

This section is primarily about the asymptotic stability of the zero solution of Eq (1.2). The methods employed in this section have been adapted from the paper of [7].

let $\psi : (-\infty, 0]_T \to \mathbb{R}$ *be a given* Δ- differentiable bounded initial fuction . We say $x(t) \coloneqq x(t, 0, \psi)$ is a solution of Eq (1.2) if $x(t) = \psi(t)$ for $t \le 0$ and satisfies Eq(1.2) for $t \ge 0$. We say the zero solution of Eq(1.2) is stable at t_0 if for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$,such that $[\psi : (-\infty, t_0]_T \to \mathbb{R}$ with $\|\psi\| < \delta\,]$implies $|x(t, t_0, \psi)| < \varepsilon$.

Let $\mathbb{C}_{rd} = \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R})$ be the spase of all rd-continuous function from $\mathbb{T} \to \mathbb{R}$ and define the set $\mho$ by

$$\mho = \left\{ \begin{matrix} \varphi \in \mathbb{C}_{rd}: \varphi(t) = \psi(t) \ if \ t \le 0, \\ \varphi(t) \to 0 \ \ as \ \ t\,\infty, and \ \ \varphi \ is \ bounded \end{matrix} \right\},$$

then $(\mho, \|\,.\,\|)$ is a complete metric space where, $\|\,.\,\|$is the supremum norm .We set the following conditions for the next theorem.

$$e_{\ominus a}(t, 0) \to 0 \ , \ as \ \ t \to \infty, \quad (3.1)$$

$$E_1 + \int_0^t [|a(u)|\, E_1 + E_2 E_3 + E_4]\; e_{\ominus a}(t, s)\Delta u \le \alpha < 1 \qquad t \ge 0, \qquad \alpha > 0, \quad (3.2)$$

$$t - g(t) \to \infty \ \ , as\ t \to \infty \qquad (3.3)$$

4.1. Theorem

If the inequalities (2.2) - (2.4) and the conditions (3.1) - (3.3) hold, then every solution $x(t,0,\psi)$ of Eq (1.2) with small continuous initial function $\psi(t)$ is bounded and approaches zero as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.

Proof. Define the mapping $P: \mho \to \mho$ *by* $(p\varphi)(t) = \psi(t)$ if $t \le 0$, and, if $t \ge 0$ .we have

$$(p\varphi)(t) = \sum_{i=1}^{p} Q_i\left(t, \varphi x(t-g(t))\right)$$

$$+\left[x(0) - \sum_{i=1}^{p} Q_i\left(0, x(0-g(0))\right)\right] e_{\ominus a}(t,0)$$

$$+ \int_0^t \left[-a(u)\sum_{i=1}^{p} Q_i^{\sigma}\left(u, \varphi(u-g(u))\right)\right] e_{\ominus a}(t,u)\Delta u$$

$$+\int_{-\infty}^{u} [D(u,s)f(\varphi(s)) + h(s)]\,\Delta s] e_{\ominus a}(t,u)\Delta u$$

It is clear that for $\varphi \in \mho$, $(p\varphi)(t)$ is continuous. Let $\varphi \in \mho$, with $\|\varphi\| \le K$, for some positive constant K. *Let* ψ be asmll given continuous initial function with $\|\psi\| < \delta$, $\delta > 0$. Then,

$$\|(P\varphi)(t)\| \le E_1 K + \int_0^t [|a(u)|E_1 K + E_2E_3K + KE_4] e_{\ominus a}(t,u)\Delta u\,,$$

$$\le (1+E_1)\delta + KE_1 + K\int_0^t [|a(u)|E_1 + E_2E_3 + E_4] e_{\ominus a}(t,u)\Delta u$$

$$\le (1+E_1)\delta + K\alpha,$$

Whish implies that, $\|(p\varphi)(t)\| \le K$, for the right δ. Thus, (3.4) implies $(p\varphi)(t)$ is bounded. Next, we show that $(p\varphi)(t) \to 0$ as $t \to \infty$

The second term on the right side of $(p\varphi)(t)$ tends to zero, by condition (3.1). In addition, the first term on the right side tends to zero, because of (3.3) and the fact that $\varphi \in \mho$. It is left to show that the integral term goes to zero as $t \to \infty$

Let $\varepsilon > 0$ be given and $\varphi \in \mho$ with $\|\varphi\| \le K$, $K > 0$. Then, there exists *a* $t_1 > 0$ so that for $t > t_1$,

$|\varphi(t-g(t)| < \varepsilon$. Due to condition (3.1), there exists a $t_2 > t_1$ such that $t > t_2$ implies that $e_{\ominus a}(t, t_1) < \frac{\varepsilon}{\alpha K}$.

Thus for $t > t_2$, we have

$$| \int_0^t [-a(u) \sum_{i=1}^p Q_i \left(u, \varphi(u-g(u))\right) + \int_{-\infty}^u [D(u,s) f(\varphi(s)) + h(s)] \Delta s] e_{\ominus a}(t,u) \Delta u |$$
$$\leq \int_0^{t_1} [|a(u)| E_1 K + E_2 E_3 K + E_4 K] e_{\ominus a}(t,u) \Delta u + \int_{t_1}^t [|a(u)| E_1 \epsilon + E_2 E_3 \epsilon + E_4 \epsilon] e_{\ominus a}(t,u) \Delta u$$

$$\leq K \int_0^{t_1} [|a(u)| E_1 + E_2 E_3 + E_4] e_{\ominus a}(t,u) \Delta u + \epsilon \int_{t_1}^t [|a(u)| E_1 + E_2 E_3 + E_4] e_{\ominus a}(t,u) \Delta u,$$

$$\leq$$
$$K e_{\ominus a}(t, t_1) \int_0^{t_1} [|a(u)| E_1 + E_2 E_3 + E_4] e_{\ominus a}(t,u) \Delta u + \alpha \epsilon,$$
$$\leq \alpha K e_{\ominus a}(t, t_1) + \alpha \epsilon, \quad \leq \epsilon + \alpha \epsilon .$$

Hence، $(p\varphi)(t) \to 0$ as $t \to \infty$. It remains to show that $(p\varphi)(t)$ is a contraction under the supremum norm.

Theorem 4.2.

Let α be a positive constant satisfying the inequality

$$E_1 + \int_0^t [|a(s)| E_1 + E_2 E_3] e_{\ominus a}(t,s) \Delta s \leq 1 \qquad (3.5),$$

then $(P\varphi)(t)$ is a contraction under the supremum norm.

Proof.

Let $\mathfrak{H}, \mathfrak{g} \in \mho$. Then

$$|(p\mathfrak{H})(t) - (p\mathfrak{g})(t)| \leq \left\{ E_1 + \int_0^t [|a(s)| E_1 + E_2 E_3] e_{\ominus a}(t,s) \Delta s \right\} \| \mathfrak{H} - \mathfrak{g} \| \leq \alpha \| \mathfrak{H} - \mathfrak{g} \| .$$

Thus, by contractive mapping theorem, $(p\varphi)(t)$ has a unique fixed point in $\mho$ which solves Eq (1.2), is bounded and tends to Zero as t tends to infinity. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing K by ε. That ends the proof.

Conclusion

In this paper, lemma 2.1 allows Eq (1.1) to be converted into an integrated equation. The integral equation was created to apply the concept of the contraction-map and ensure the stability of periodic solutions for nonlinear neutral differential of the first order with functional delay.

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